“Refractive Index Map Reconstruction in Optical Deflectometry Using Total-Variation Regularization”

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1. Introduction
Sparsity in Wavefields

- World Before Sparsity:
  - Human Readable Sensing (up to an orthogonal transform, e.g., Fourier)
  - Nyquist sampling (wrt bandlimitedness)

Examples: Photography, First MRIs, Interferometry, ...

Optimized
Paradigm shift allowed by Sparsity:
Computer “Readable” Sensing + Prior information (sparsity)

Examples:
Seismology, Radio-Interferometry, Compressed Sensing, MRI,
Computed Photography, ...
(Many in this Workshop!)
2. Optical Deflectometry
Optical Computed Tomography Classification:

- Absorption Tomography (X-ray tomography)
  (Check Session 12, Tue 23)
- Phase Tomography (using interferometry)
- and Deflection Tomography ...
Deflectometric Framework:

‣ **Problem:**
Reconstructing the refractive index map of transparent materials from light deflection measurements under multiple orientations.

‣ **Interests:**
  ‣ (transparent) object surface topology
  ‣ default detection in glass, crystal growth study

‣ **Advantages of deflectometry:**
  ‣ Insensitive to vibrations (vs. interferometry)
  ‣ Less sensitive to dispersive medium
  ‣ Precision: up to 10nm flatness deviation on 50mm FOV
    (e.g., on window glass)
Mathematical Model:

\[
\Delta(\tau, \theta) \approx \int_{\mathbb{R}^2} \left( \nabla n(x) \cdot p_\theta \right) \delta(\tau - x \cdot p_\theta) \, d^2x
\]

(first order approximation)
Experimentally: Phase-Shifting Schlieren

“Coding light deviation in intensity variations”

Incoherent Backlight Source

LCD based programmable spatial filter:

\[ m(x) = \sin\left(\frac{2\pi}{\Lambda} x\right) \]
Experimentally: Phase-Shifting Schlieren

\[ \Delta x = f \tan \alpha \]

\[ m(x) = \sin\left(\frac{2\pi}{\Lambda} x\right) \]

Intensity change

Object

Phase-Shifting Algorithm
3. Reconstruction Methods
Continuous Facts:

Sensing Model:

\[ \Delta(\tau, \theta) \simeq \int_{\mathbb{R}^2} (\nabla n(x) \cdot p_\theta) \, \delta(\tau - x \cdot p_\theta) \, d^2x \]

Deflectometric Central Slice Theorem:

\[ z(\omega, \theta) := \int_{\mathbb{R}} \Delta(\tau, \theta) \, e^{-i\tau\omega} \, d\tau = i\omega \hat{n}(\omega p_\theta) \]

\[ \Rightarrow z(\omega, \theta) = i\omega \hat{n}(\omega p_\theta) \]

with \( \hat{n} \) the 2-D FFT of \( n \).
Refractive Index Prior

- Heterogenous transparent materials with slowly varying refractive index separated by sharp interfaces (e.g., optical fibers)

The perfect “cartoon shape” model (BV, TV, ...)

“Sparse” gradient, i.e., small Total Variation norm

\[ \| n \|_{TV} = \int_{\mathbb{R}} \| \nabla n(x) \| \, dx \]
Discretization of Measurements

- Weighting: Given $z = \mathcal{F}_\tau(\Delta(\cdot, \theta))$

$$y(\omega, \theta) := \frac{z(\omega, \theta) + \varepsilon(\omega, \theta)}{i\omega} = \hat{n}(\omega p_\theta) + \eta(\omega, \theta).$$

with $\varepsilon$ additive white Gaussian noise (in $\tau$ and $\omega$)

- Polar to Cartesian interpolation:

Given $z = \mathcal{F}_\tau(\Delta(\cdot, \theta))$ (noise error)

$$(\omega, \theta) \sim \omega p_\theta \rightarrow (k_x, k_y)$$

$y(\omega, \theta) \rightarrow \tilde{y}(k_x, k_y)$

Controlling error:
Cartesian grid resolution
(could be NFFT)

Experimental Coordinate System

$\begin{array}{c}
k_x \\
\uparrow \\
k_y
dots
\end{array}$
Discretization of Measurements

- Recording interpolated frequency pixel locations:

\[ \tilde{y} = \begin{pmatrix} \tilde{y}(k_1) \\ \vdots \\ \tilde{y}(k_{M/2}) \end{pmatrix} \in \mathbb{C}^{M/2} \simeq \mathbb{R}^M \]

\[ \mathcal{K} = \{k_1, \ldots, k_{M/2}\} \subset \mathbb{R}^2 \]

- Final Discrete Inverse Problem: Tomography (but “colored”)

\[ \tilde{y} = SFn + \eta \]

\[ \in \mathbb{R}^N \]

Interp. + meas.

Noise (colored)

Ill-posed: \( M \leq N \)
Filtered Back Projection

- In our notation: FBP amounts to solve

\[
\arg \min_{u \in \mathbb{R}^N} \|u\|_2 \quad \text{s.t.} \quad \tilde{y} = SFu
\]

\[\Rightarrow n^* = (SF)^\dagger \tilde{y} = F^* S^* \tilde{y}\]

since \(SS^* = FF^* = \text{Id}\)
Regularized Whitened Reconstruction

- Using TV prior and whitening noise: **TV-L2**

\[
\arg\min_{\mathbf{u} \in \mathbb{R}^N} \|\mathbf{u}\|_{TV} \quad \text{s.t.} \quad \|\mathbf{W}(\tilde{\mathbf{y}} - S\mathbf{F}\mathbf{u})\|_2 \leq \epsilon
\]

with \(\mathbf{W} = \text{diag}(\sigma(\eta_1), \cdots, \sigma(\eta_{M/2}))^{-1}\)

- Estimating \(\mathbf{W}\):
  - Montecarlo on Normal Gaussian
  - Estimating std. dev. from \(\Delta(\tau, \theta)\)

- Solving the reconstruction:
  - Proximal methods
  - Iterative Chambolle-Pock algorithm
4. Results
Synthetic Images

- Generating realistic maps:

- Using Image Reconstruction Toolbox (IRT\(^1\)) for computing \(\Delta(\tau, \theta)\)

  Varying \(\theta\), Fixed \(\tau\)

- Adding noise and reconstructing FBP & TV-L2

\(^1\) [http://www.eecs.umich.edu/~fessler/code/](http://www.eecs.umich.edu/~fessler/code/)
Synthetic Images

- No meas. noise
- $M/N = 10\%$

4.94 dB

10.37 dB
Synthetic Images

- Meas. noise: 11.51dB
- $M/N = 10\%$
Synthetic Images

- Measurement noise = 20 dB  (another image)
Experimental Data

- bundle of 10 fibers immersed in an optical fluid
- working on one z-slice (2-D problem)
  \[ \# \tau = 696 \]
- Origin of \( \tau \) must be calibrated!
- Noise is estimated on \( \Delta(\tau, \theta) \)
Experimental Data

- $\#\theta = 60, M/N \simeq 30\%$
Experimental Data

- $\#\theta = 60, M/N \approx 30\%$
Do not forget calibration!

- If misaligned $\tau$ origin FBP and TV-L2 will look something like this:
5. Conclusion
Conclusion and Perspectives

- Recent sparsity-driven reconstruction methods can be applied to Optical Deflectometry.
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- Same framework valid for: Phase-Contrast X-Ray Tomography (& no first order approx!)
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- Recent sparsity-driven reconstruction methods can be applied to Optical Deflectometry.
- Outside of easy synthetic Cartezian worlds, hard time with gridding/interpolation/noise/calibration/...
- Same framework valid for: Phase-Contrast X-Ray Tomography (& no first order approx!)
- Future Improvements:
  - Integrating positivity constraint
  - Using NFFT and reduce interpolation noise
  - Testing other reconstruction (Dantzig Fidelity ?)

\[ \| \Phi^* (\tilde{y} - \Phi n) \|_\infty \leq \rho \]

forces spatially stationary noise
Thank you!