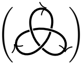
Supercategorification and Odd Khovanov Homology Part 1

Léo Schelstraete

13 october 2020

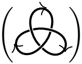
1 Khovanov homology

Jones polynomial

$J\Big($  $\Big) = -q^{-9} + q^{-5} + q^{-3} + q^{-1}$

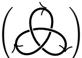
1 Khovanov homology

Khovanov homology

$Kh\Big($  $\Big) =$

i	-3	-2	-1	0
Kh_i	$\mathbb{Q}[-9]$	$\mathbb{Q}[-5]$	0	$\mathbb{Q}[-3] \oplus \mathbb{Q}[-1]$

Jones polynomial

$J\Big($  $\Big) = -q^{-9} + q^{-5} + q^{-3} + q^{-1}$

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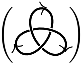
$Kh\Big($  $\Big) =$

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$\mathrm{qdim}\,(Kh_{-3}) = q^{-9}$

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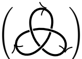
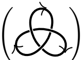
$J\Big($  $\Big) = -q^{-9} + q^{-5} + q^{-3} + q^{-1}$

Khovanov homology

$Kh\Big($  $\Big) =$

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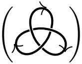
$(-1)^{-3}q^{-9}$ $\quad$ $(-1)^{-2}q^{-5}$ $\quad$ $(-1)^{0}(q^{-3}+q^{-1})$

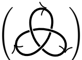
Jones polynomial

$J\Big($  $\Big) = -q^{-9} + q^{-5} + q^{-3} + q^{-1}$

Khovanov homology

$\sum_i (-1)^i \mathrm{qdim}\,(Kh_i)$

Jones polynomial

Kh() =

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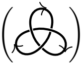
J() $= -q^{-9} + q^{-5} + q^{-3} + q^{-1}$

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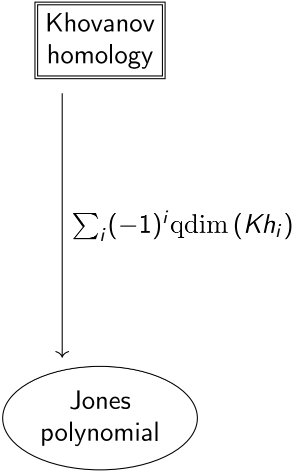

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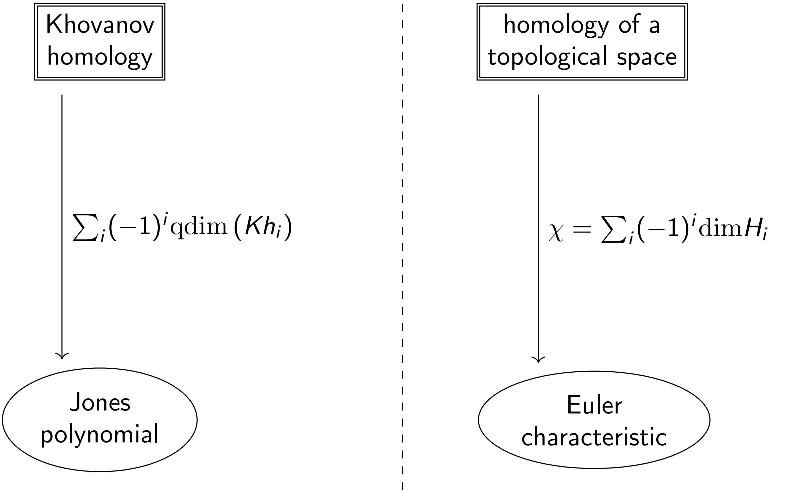
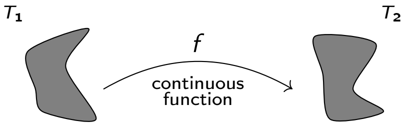


2 Categorification

Topological spaces



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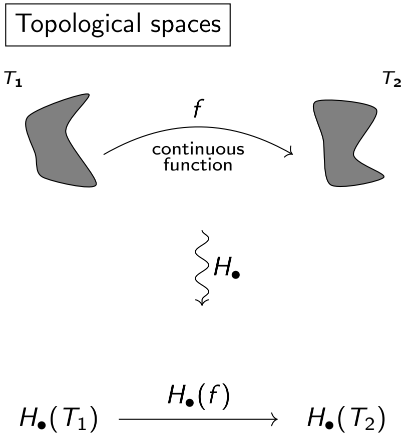

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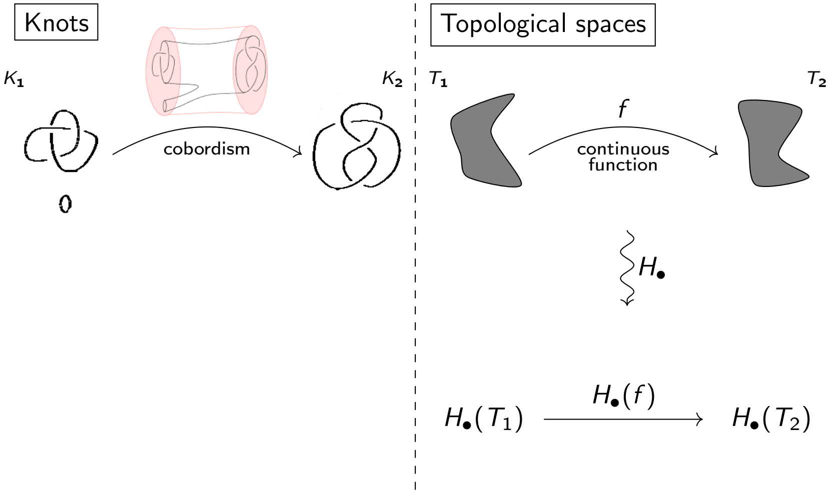
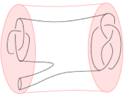


2 Categorification

Knots

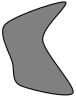
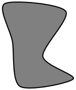


K_1  $\xrightarrow{\text{cobordism}}$ K_2 

$\rightsquigarrow Kh$

$$Kh(K_1) \xrightarrow{Kh(f)} Kh(K_2)$$

Topological spaces

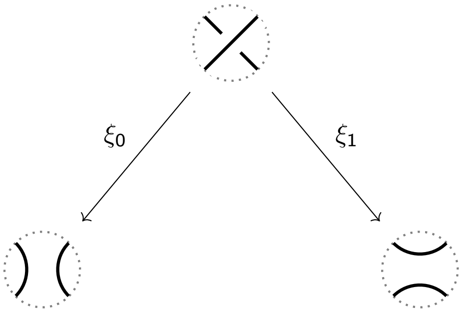
T_1  $\xrightarrow[\text{continuous function}]{f}$ T_2 

$\rightsquigarrow H_\bullet$

$$H_\bullet(T_1) \xrightarrow{H_\bullet(f)} H_\bullet(T_2)$$

$\boxed{1}+\boxed{2}$ Construction of Khovanov homology

Kauffman state sum of Jones polynomial:



resolution for K: $\xi \in \{0,1\}^{\#\text{crossings}}$, that is a choice of resolution ξ_0 or ξ_1 for each crossing.

$$V(K) = \sum_{\xi} (-1)^{\#\{\xi_1 \text{ in } \xi\}} (q + q^{-1})^{\#\{\text{circles in } \xi\}}$$

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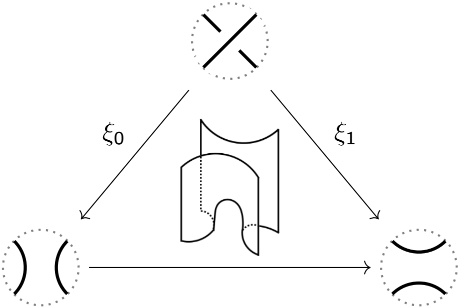

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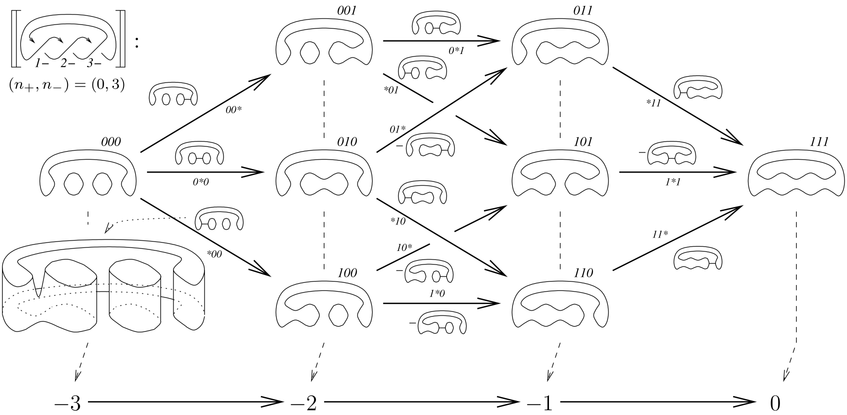
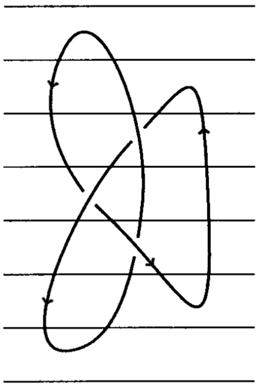
1+2 Construction of Khovanov homology



taken from Bar-Natan, "Khovanov's homology for tangles and cobordisms"

2′ The slice (or tangle) strategy: classical case



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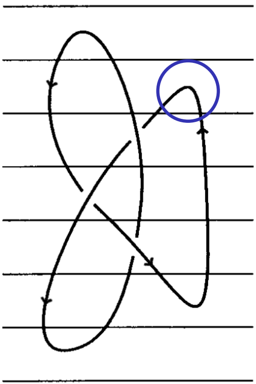

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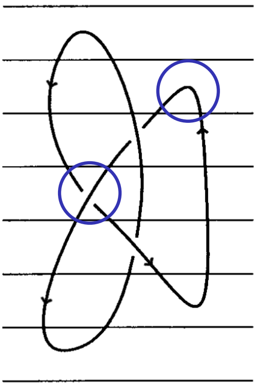

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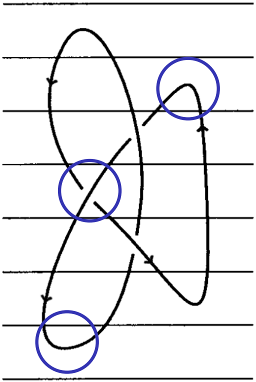

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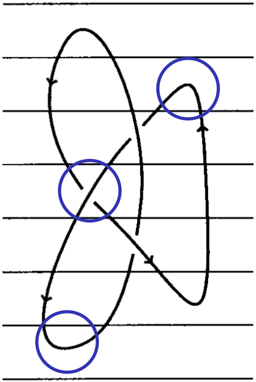

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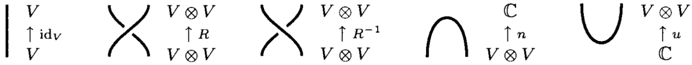

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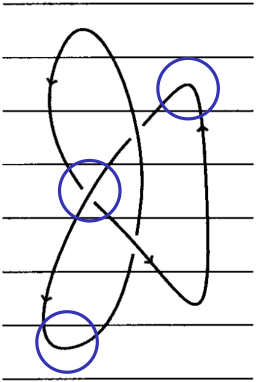

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$\rightsquigarrow$

$$
\begin{array}{c}
\mathbb{C} \\
\uparrow n \\
V \otimes V \\
\uparrow \mathrm{id}_V \otimes \mathrm{id}_V \otimes n \\
V \otimes V \otimes V \otimes V \\
\uparrow \mathrm{id}_V \otimes R^{-1} \otimes \mathrm{id}_V \\
V \otimes V \otimes V \otimes V \\
\uparrow R \otimes \mathrm{id}_V \otimes \mathrm{id}_V \\
V \otimes V \otimes V \otimes V \\
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V \otimes V \otimes V \otimes V \\
\uparrow \mathrm{id}_V \otimes \mathrm{id}_V \otimes u \\
V \otimes V \\
\uparrow u \\
\mathbb{C}
\end{array}
$$

$V \xleftarrow{\mathrm{id}_V} V$: $\begin{array}{c} V \\ \uparrow \mathrm{id}_V \\ V \end{array}$

$\begin{array}{c} V \otimes V \\ \uparrow R \\ V \otimes V \end{array}$

$\begin{array}{c} V \otimes V \\ \uparrow R^{-1} \\ V \otimes V \end{array}$

$\begin{array}{c} \mathbb{C} \\ \uparrow n \\ V \otimes V \end{array}$

$\begin{array}{c} V \otimes V \\ \uparrow u \\ \mathbb{C} \end{array}$

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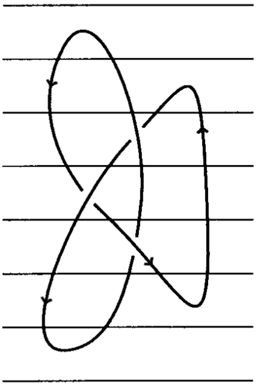

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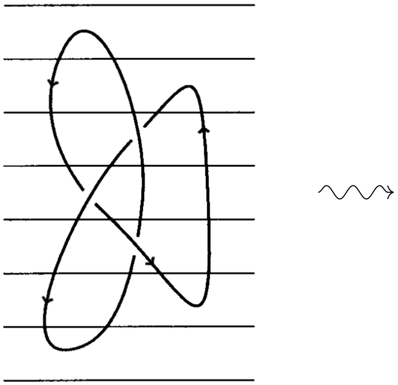

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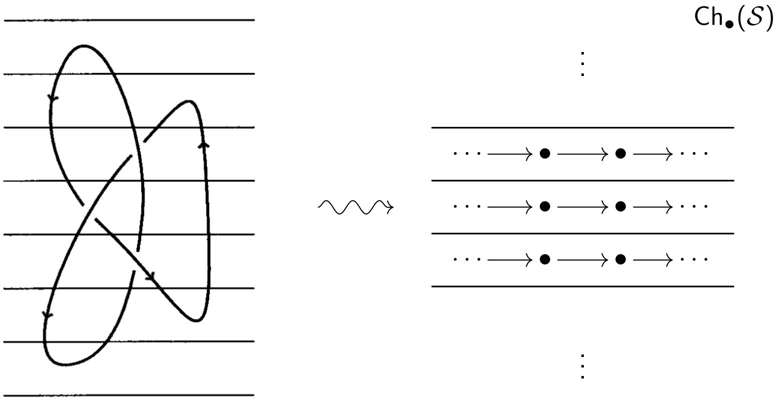

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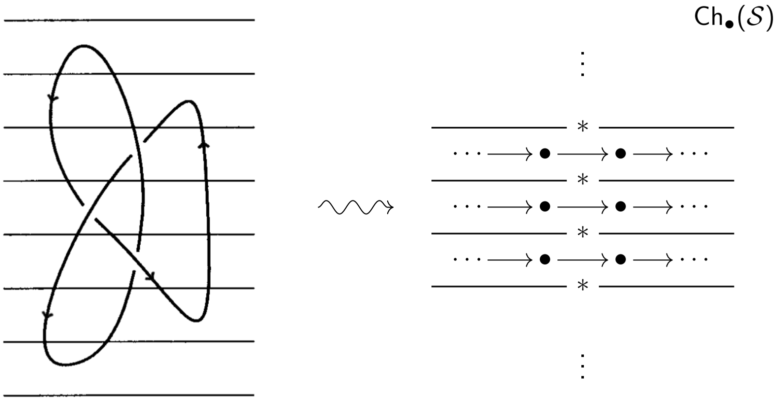

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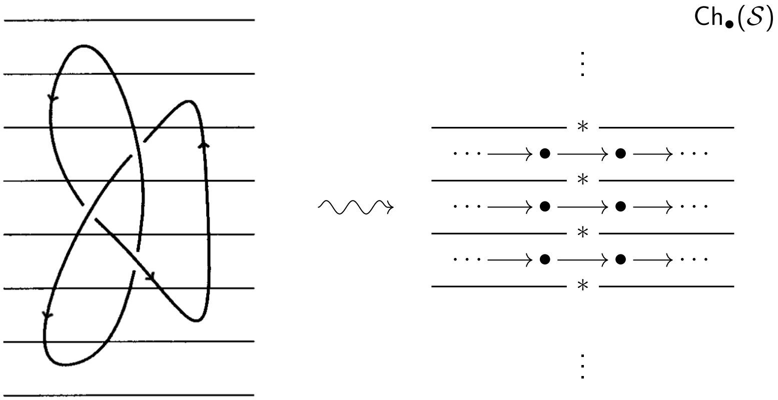
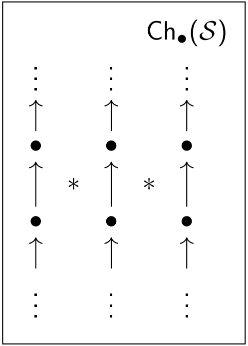


homology is an invariant $\Leftrightarrow$ homotopy class independent of the diagram

2′ 2-categories



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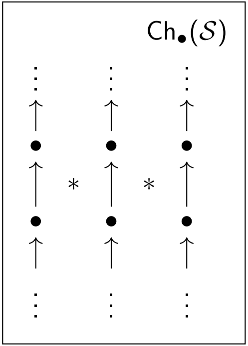
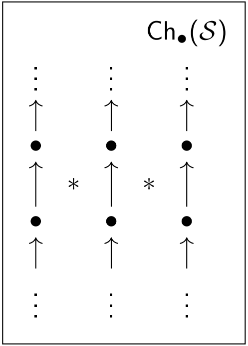


$\Rightarrow$ $\mathcal{S}$ is a 2-category



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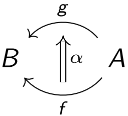

2-morphism

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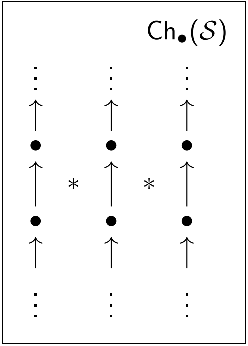

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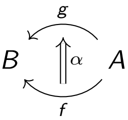

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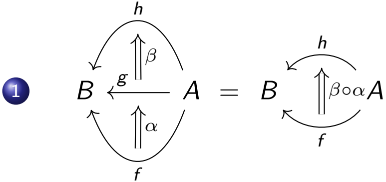
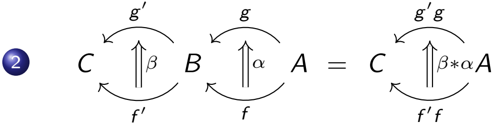
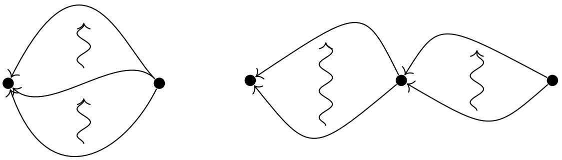
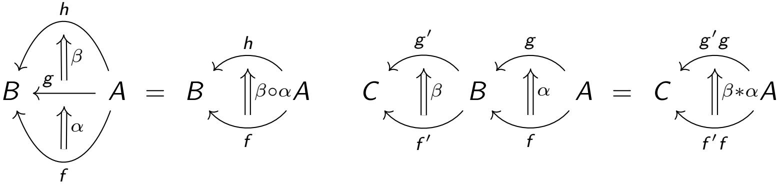




1. homotopies:



2. natural transformations:



$2'$ Defining the invariant

$$\cap \longmapsto [\![\; * \;]\!] \qquad\qquad \cup \longmapsto [\![\; * \;]\!]$$

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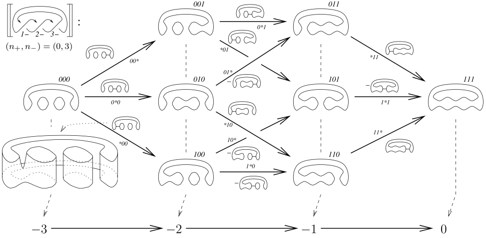
1. The complex is a **cube** of dimension *n*, where *n* is the **number of crossings** $\Rightarrow$ similar to **Khovanov homology**!



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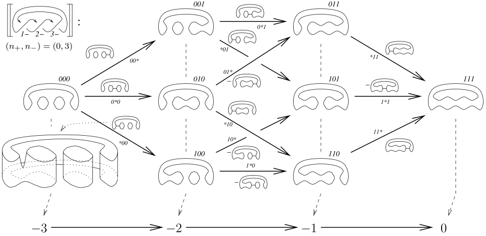

2. But we used the **"slice strategy"**, similarly to **quantum algebras**, and $\mathcal{S}$ is purely **algebraic**.

Conclusion

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3. What is odd Khovanov homology? And how to adapt this construction to it (**superstructures**)? See you after the break!

Supercategorification and Odd Khovanov Homology Part 2

Léo Schelstraete

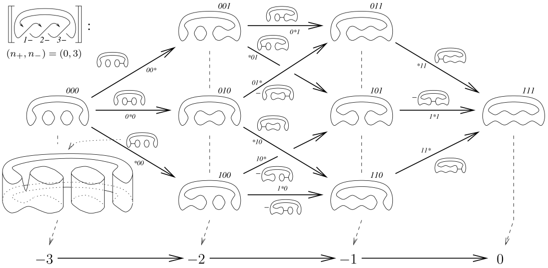
13 october 2020

TOP Property of odd Khovanov homology

"super" $=$?

TOP Property of odd Khovanov homology

“super” = ?



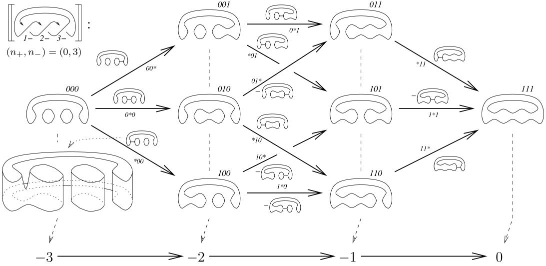
symmetric algebras

↓

even Khovanov homology

TOP Property of odd Khovanov homology

“super” $=$?



~~symmetric~~
exterior
algebras

↓

odd
Khovanov
homology

TOP Property of odd Khovanov homology

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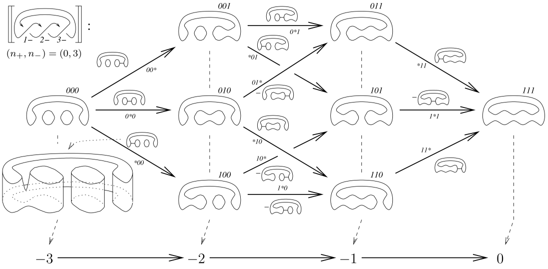

$$x \wedge y = (-1)^{|x||y|} y \wedge x$$

~~symmetric~~
exterior
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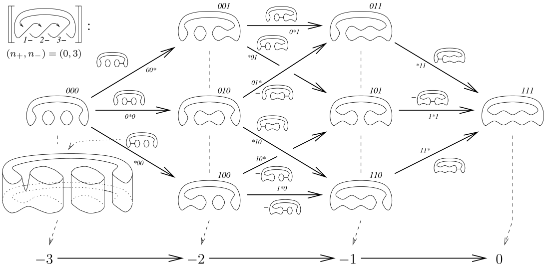
↓

odd
Khovanov
homology

TOP Property of odd Khovanov homology

"super" = parity



$$x \wedge y = (-1)^{|x||y|} y \wedge x$$

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$\boxed{ALG}$ superstructures: superspaces

A **superspace** V is a $\mathbb{Z}/2\mathbb{Z}$-graded vector space:

- even and odd vectors:

$$V = V_0 \oplus V_1 \qquad |v| := \text{grading of } v \text{ (0 or 1)}$$

- $End(V, V)$ inherits a superspace structure:

$$\text{even maps} := \text{maps preserving the parity}$$
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- $End(V, V)$ inherits a superspace structure:

$$\begin{aligned} \text{even maps} &:= \text{maps preserving the parity} \\ \text{odd maps} &:= \text{maps exchanging the parity} \end{aligned}$$

- **super tensor product**:

$$(V \otimes W)_0 = V_0 \otimes W_0 \oplus V_1 \otimes W_1 \text{ and } (V \otimes W)_1 = V_0 \otimes W_1 \oplus V_1 \otimes W_0$$

$$(f \otimes g)(v \otimes w) := (-1)^{|g||v|} f(v) \otimes g(w)$$

  - **super interchange law** (compatibility law between composition and tensor product):

$$(f \otimes g) \circ (h \otimes k) = (-1)^{|g||h|} (f \circ h) \otimes (g \circ k)$$

$\boxed{ALG}$ superstructures: supercategories

A **supercategory** is a category where:

- each Hom-set is a superspace
- composition induces an even map:

$$|f \circ g| = |f| + |g|$$

- a **superfunctor** is a functor preserving parity

$\boxed{ALG}$ superstructures: supercategories

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A **monoidal supercategory** is a supercategory...

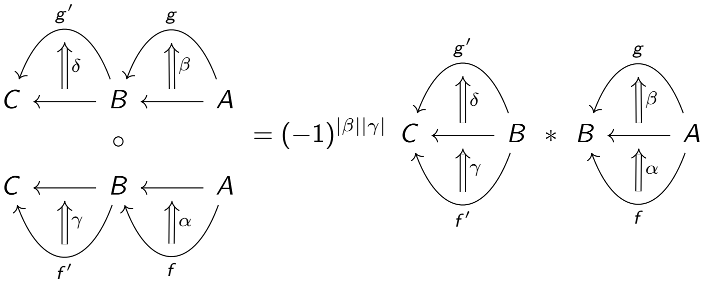
- ...like a monoidal category (category with a "product" like a tensor product)...
- ...but with the *super* interchange law:

$$(f \otimes g) \circ (h \otimes k) = (-1)^{|g||h|}(f \circ h) \otimes (g \circ k)$$

$\boxed{ALG}$ superstructures: 2-supercategories

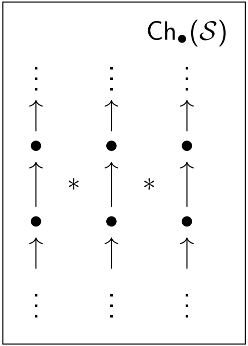
A **2-supercategory** is

- a 2-category whose 2-morphisms have a parity...
- ...and compatibility between horizontal and vertical product is given by the super interchange law:



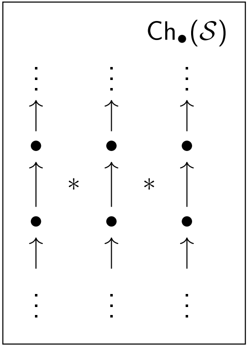
$$(\delta * \beta) \circ (\gamma * \alpha) = (-1)^{|\beta||\gamma|} (\delta \circ \gamma) * (\beta \circ \alpha)$$

TOP Product of complexes



$\mathcal{S}$ is now a 2-*super*category

TOP Product of complexes



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$\Rightarrow$ Product of complexes $A_\bullet * B_\bullet$?

$A_\bullet * B_\bullet$ must preserve homotopy classes

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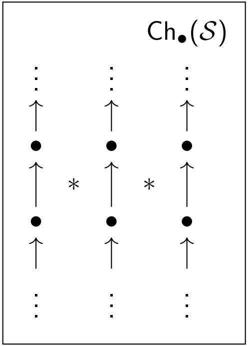

$\mathcal{S}$ is now a 2-*super*category

$\Rightarrow$ Product of complexes $A_\bullet * B_\bullet$?

$A_\bullet * B_\bullet$ must preserve homotopy classes

$\Rightarrow$ Need of a definition applicable to 2-supercategories!

ALG Product of complexes

TASK: define horizontal product of complexes such that homotopy classes are preserved.

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Remark

monoid	$\leftrightarrow$	one-object category
monoidal category	$\leftrightarrow$	one-object 2-category
monoidal supercategory	$\leftrightarrow$	one-object 2-supercategory

ALG Product of complexes

TASK: define horizontal product of complexes such that homotopy classes are preserved.

Remark

$$\begin{array}{ccc} \text{monoid} & \leftrightarrow & \text{one-object category} \\ \text{monoidal category} & \leftrightarrow & \text{one-object 2-category} \\ \text{monoidal supercategory} & \leftrightarrow & \text{one-object 2-supercategory} \end{array}$$

TASK2: define *tensor* product of complexes such that...

ALG Product of complexes

$$\begin{array}{c} A^{(0)} \xrightarrow{\alpha} A^{(1)} \\ \otimes \\ B^{(0)} \xrightarrow{\beta} B^{(1)} \end{array} \quad = \quad \begin{array}{ccc} A^{(0)} \otimes B^{(0)} & \xrightarrow{\alpha\otimes 1} & A^{(1)} \otimes B^{(0)} \\ \downarrow {\scriptstyle 1\otimes\beta} & & \downarrow {\scriptstyle 1\otimes\beta} \\ A^{(0)} \otimes B^{(1)} & \xrightarrow{\alpha\otimes 1} & A^{(1)} \otimes B^{(1)} \end{array}$$

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$$B^{(0)} \xrightarrow{\beta} B^{(1)}$$

$$(\alpha \otimes 1) \circ (1 \otimes \beta) = (1 \otimes \beta) \circ (\alpha \otimes 1) \quad \text{(commutative)}$$

ALG Product of complexes

$$
\begin{array}{c}
A^{(0)} \xrightarrow{\alpha} A^{(1)} \\[2em]
\otimes \\[2em]
B^{(0)} \xrightarrow{\beta} B^{(1)}
\end{array}
\quad = \quad
\begin{array}{ccc}
A^{(0)} \otimes B^{(0)} & \xrightarrow{\alpha \otimes 1} & A^{(1)} \otimes B^{(0)} \\
\Big\downarrow{\scriptstyle 1\otimes\beta} & & - \Big\downarrow{\scriptstyle 1\otimes\beta} \\
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Assume α and β are odd:

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$\Rightarrow$ super Koszul rule?

$\boxed{ALG}$ Product of complexes

Theorem

The super Koszul rule:

1. exists for **homogeneous** complexes (complexes whose differentials are either even or odd)
2. is unique, at least for cubes (all choices of signs result in isomorphic complexes)
3. preserves homotopy classes: given complexes $A_\bullet$, $B_\bullet$, $C_\bullet$ and $D_\bullet$, if

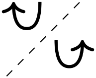
$$A_\bullet \simeq B_\bullet \qquad \text{and} \qquad C_\bullet \simeq D_\bullet$$

   Then

$$A_\bullet \otimes C_\bullet \simeq B_\bullet \otimes D_\bullet$$

TOP Definition of the invariant

 $\longmapsto$

 $\longmapsto$

 $\longmapsto$

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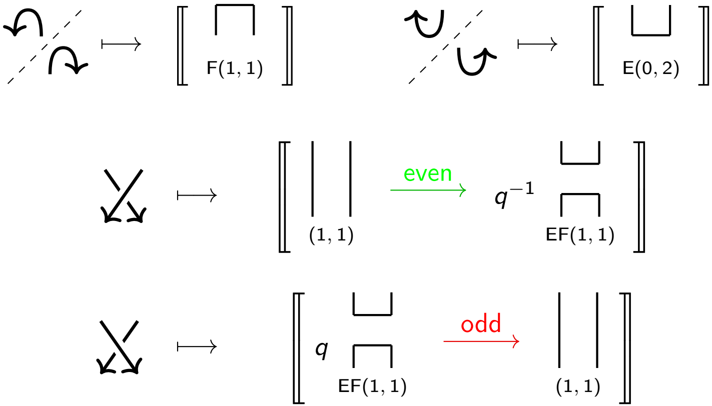
TOP Definition of the invariant



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$$Kh_{\mathrm{odd}} = Kh'_{\mathrm{odd}} \oplus Kh'_{\mathrm{odd}}.$$

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3. Application? Proof of functoriality