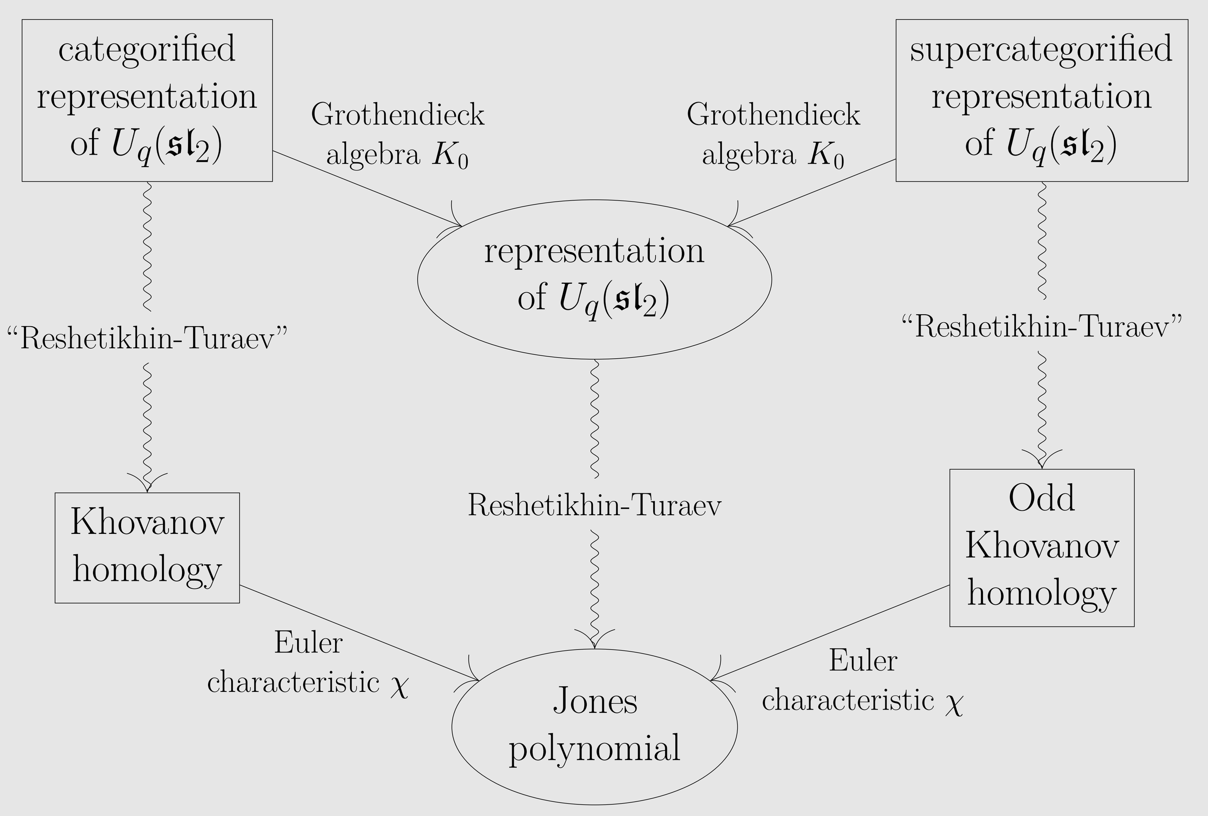
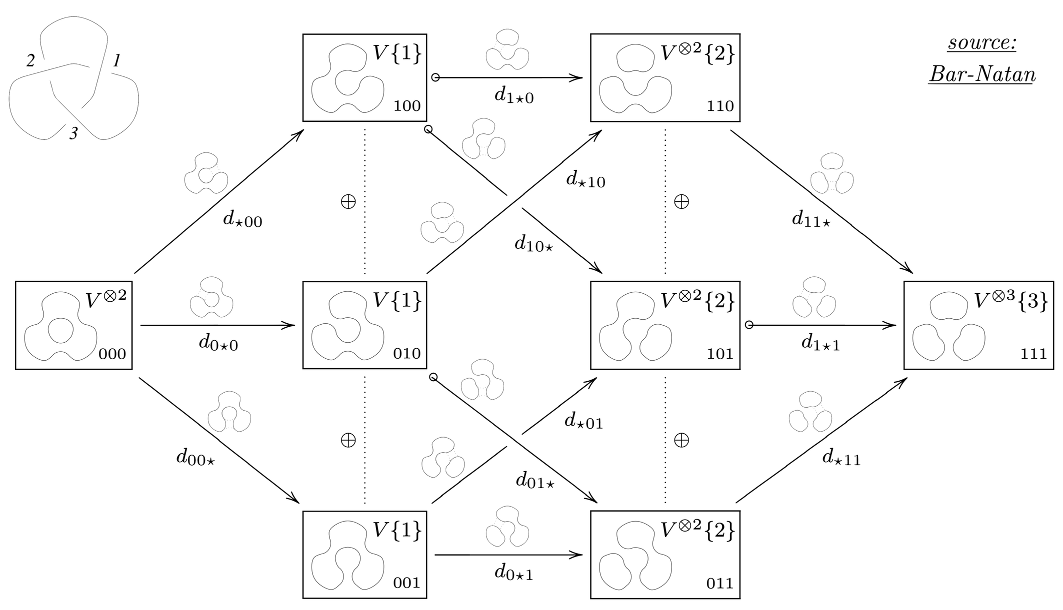
Odd Khovanov Homology and Supercategorification

Léo Schelstraete
UCLouvain

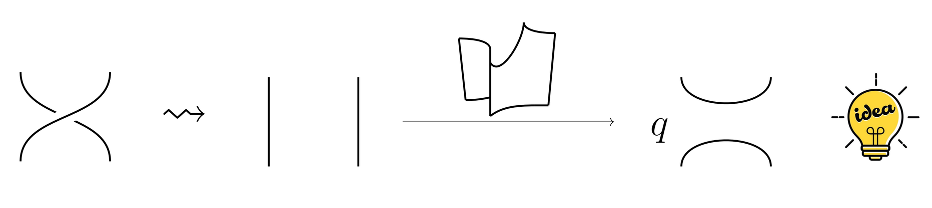
ABSTRACT. The Jones polynomial admits two generalisations. On one hand, *Khovanov homology* is an invariant of links which *categorify* the Jones polynomial. On the other hand, the Jones polynomial can be derived from the representation theory of $U_q(\mathfrak{sl}_2)$, through the *Reshetikhin-Turaev construction*. It has been shown that the two approaches can be combined: one can categorify the representation theory of $U_q(\mathfrak{sl}_2)$ to derive Khovanov homology. We present here similar results for *odd Khovanov homology*, an invariant of links close but distinct from Khovanov homology. Along the way, we develop some homological results in the context of *superstructures*.





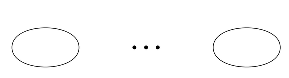
Khovanov homology

The Jones polynomial is a polynomial invariant of links computed using the Kauffman bracket. *Khovanov homology* is a homological invariant of links which recovers the Jones polynomial through its Euler characteristic. The main idea of Khovanov's construction is to **view the Kauffman bracket as a cobordism map** between the 0-resolution and to 1-resolution of the crossing.

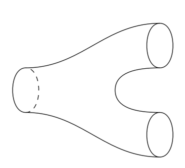


categorified Kauffman bracket

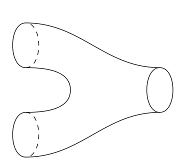
Carrying on this idea for each crossing of a given link, one gets a *cube of resolutions* (see figure below-left). Finally, **1** *applying a TQFT into* ***symmetric algebras***:



$$\mapsto \quad A_n = \frac{\Bbbk[x_1,\dots,x_n]}{x_1^2 = \ldots = x_n^2 = 0}$$



$$\mapsto \quad \begin{bmatrix} \Bbbk[x]/x^2 \to \Bbbk[x_1,x_2]/x_1^2,x_2^2 \\ 1 \mapsto x_1 + x_2 \\ x \mapsto x_1x_2 \end{bmatrix}$$



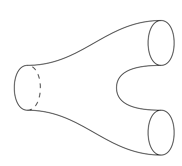
$$\mapsto \quad \begin{bmatrix} \Bbbk[x_1,x_2]/x_1^2,x_2^2 \to \Bbbk[x]/x^2 \\ 1 \mapsto 1 \\ x_1, x_2 \mapsto x \end{bmatrix}$$

and **2** *fixing the sign using the* ***Koszul rule***, one turns the cube of resolutions into an actual chain complex and gets an explicitly computable homology.

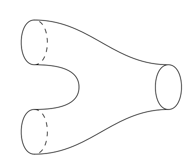
Odd Khovanov homology

Odd Khovanov homology is a homological invariant of links, similar to Khovanov homology, but distinct (each distinguishes links the other cannot). The difference lies in the final two steps. One starts again with the cube of resolutions, but then **1** *apply a "projective" TQFT into* ***exterior algebras***:

$$\mapsto \quad \Lambda_n = \frac{\Bbbk[x_1,\dots,x_n]}{x_ix_j = -x_jx_i, \forall i,j}$$



$$\mapsto \quad \begin{bmatrix} \Lambda(x_1,x_2) \to \Lambda(x) \\ 1 \mapsto x_1 - x_2 \\ x \mapsto x_1 \wedge x_2 \end{bmatrix}$$



$$\mapsto \quad \begin{bmatrix} \Lambda(x) \to \Lambda(x_1,x_2) \\ 1 \mapsto 1 \\ x_1, x_2 \mapsto x \end{bmatrix}$$

and **2** *fix the sign using the* ***super Koszul rule***. This again turns the cube of resolutions into an actual chain complex and provides an explicitly computable homology.

The TQFT in step **1** is only defined up to a sign (thus the denomination "projective"). Indeed, it turns some commuting squares into anti-commuting squares. This makes step **2** more subtle, hence the need for a "super" Koszul rule. See the paragraph "Supercategorification" for more details.

References: *Not Even Khovanov Homology* (Vaz 2020); my master thesis (S. 2020) for partial results; complete results to be published soon.

Categorification

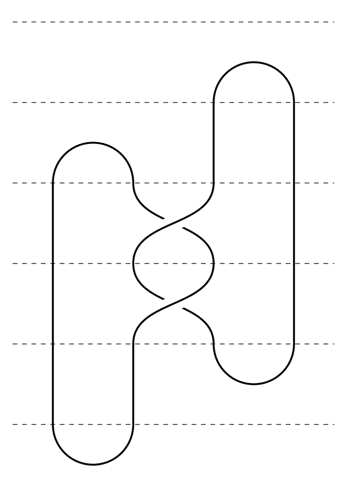
Categorification is the idea of "enhancing" an object with additional categorical structure (typically, morphisms between "elements" of the object). As topologists, the hope is that additional categorical structure implies stronger, but also functorial invariants.

additional (categorical) algebraic structure ⇝ *stronger (functorial) topological invariants*

For example, **Khovanov homology may be considered as a categorification of Jones polynomial**. Note that the Euler characteristic of a chain complex (here, the Jones polynomial) only depends on (the dimension of) the vertices, while the homology of the chain complex (here, Khovanov homology) also depends on its differentials.

Reshetikhin-Turaev construction

A *quantum algebra* (also called *quantum group*) is a kind of "deformation" $U_q(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$. The *Reshetikhin-Turaev construction* is **a general method to construct invariants of tangles using the representation theory of quantum algebras**. It assigns to each elementary tangle a linear map (drawn from intertwiners of $U_q(\mathfrak{g})$-representations), and extend to generic tangles by composition. To categorify, one should assign to each elementary tangle a *chain complex*, and extend to generic tangles by tensor product of chain complexes.



closed tangle

$\mapsto$

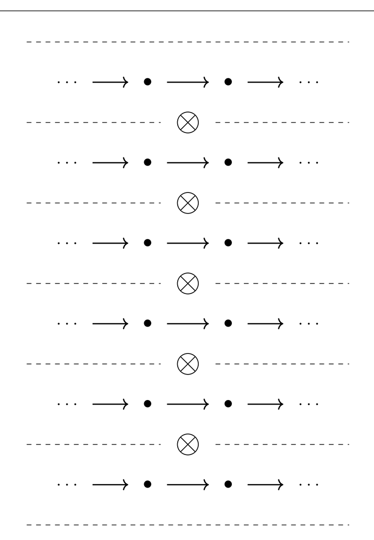
`Reshetikhin-Turaev`

$$\begin{array}{c} \mathbb{C}(q) \\ \uparrow n \\ \mathbb{C}(q) \otimes V \otimes V \\ \uparrow n \\ V \otimes V \otimes V \otimes V \\ \uparrow R \\ V \otimes V \otimes V \otimes V \\ \uparrow R \\ V \otimes V \otimes V \otimes V \\ \uparrow u \\ V \otimes V \otimes \mathbb{C}(q) \\ \uparrow u \\ \mathbb{C}(q) \end{array}$$

polynomial

or

`categorified Reshetikhin-Turaev`



chain complex

Supercategorification

A **supercategory** is a category whose morphism spaces are $\mathbb{Z}/2\mathbb{Z}$-graded vector spaces (vector spaces equiped with a notion of *parity*). A **monoidal supercategory** is a supercategory with a "monoidal-like" structure, in the sense that the interchange law is twisted by the $\mathbb{Z}/2\mathbb{Z}$-grading:

$$(f \circ g) \otimes (h \circ k) = (-1)^{|g||h|}(f \otimes h) \circ (g \otimes k),$$

where $|\cdot|$ denotes the parity of the morphisms. One can analagously define the notion of **2-supercategory**.

To construct homological invariants using superstructures, one needs a notion of tensor product of chain complexes that is coherent with homotopy classes. In the non-super setting, one would use the *Koszul rule*. If $(A^\bullet, \alpha^\bullet)$ and $(B^\bullet, \beta^\bullet)$ are chain complexes:

$$\alpha^\bullet \otimes \beta^\bullet|_{A^i \otimes B^j} = \alpha^i \otimes \mathrm{id} + (-1)^j \,\mathrm{id} \otimes \beta^j \qquad \textit{Koszul rule}$$

This ensures that $(\alpha^\bullet \otimes \beta^\bullet)^2 = 0$ and $(A^\bullet \otimes B^\bullet, \alpha^\bullet \otimes \beta^\bullet)$ is a *chain* complex. In the super setting, the Koszul rule does not work, as signs can appear due to the above twisted interchange law. This calls for a super Koszul rule, satisfying enough properties to construct topological invariants.

Theorem (S. 2020). Let $(\mathcal{C}, \otimes)$ be a monoidal supercategory. If one restricts to bounded chain complexes which factor into homogeneous differentials, then:

- there exists **a super Koszul rule**...
- ...which is **essentially unique for cubes**...
- ...and is **coherent with homotopy classes**:

$$A^\bullet \simeq B^\bullet \quad \text{and} \quad C^\bullet \simeq D^\bullet \quad \Rightarrow \quad A^\bullet \otimes B^\bullet \simeq C^\bullet \otimes D^\bullet.$$

The previous result allows the definition of tangle invariants "à la Reshetikhin-Turaev" in the context of superstructures. The following result shows that it can be used to give a **representation theoretic construction of odd Khovanov homology**.

Theorem (Vaz 2020; S. and Vaz 2022).

- There exists a categorification of some representation of $U_q(\mathfrak{sl}_2)$ into a 2-supercategory.
- One can construct an invariant of oriented tangles using this 2-supercategory.
- This invariant coincide with odd Khovanov homology in the case of links.