UCLouvain
Institut de recherche
en mathématique et physique

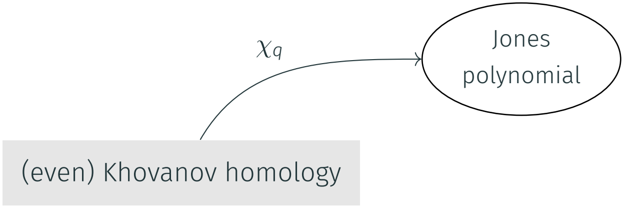
Odd Khovanov homology *via* higher representation theory

Léo Schelstraete (joint work with Pedro Vaz)

January 5th, 2024

Joint Mathematics Meetings 2024

A biased overview on odd Khovanov homology



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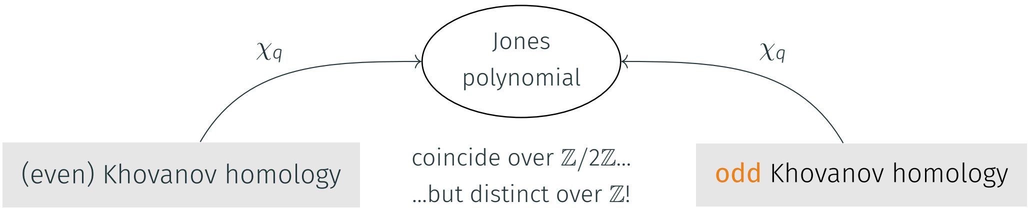

A biased overview on odd Khovanov homology

Jones polynomial

χ_q (even) Khovanov homology → Jones polynomial ← χ_q odd Khovanov homology

(even) Khovanov homology

coincide over $\mathbb{Z}/2\mathbb{Z}$...
...but distinct over $\mathbb{Z}$!

odd Khovanov homology

spectral sequence **over $\mathbb{Z}/2\mathbb{Z}$**
to Heegaard–Floer homology...

...conjectured to extend **over $\mathbb{Z}$**
using odd Khovanov homology!

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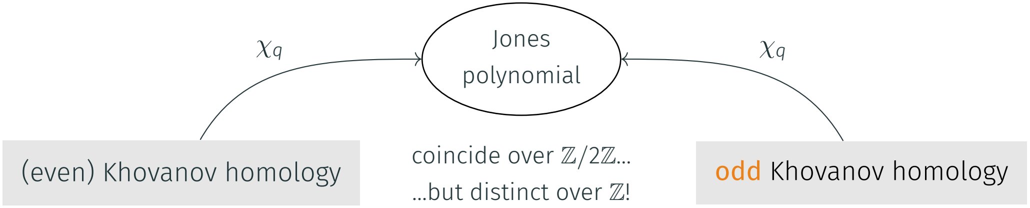

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related to the **complex** variety $\mathbb{C}P^1$

related to the **real** variety $\mathbb{R}P^1$

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χ_q | χ_q

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arises from the supercategorified
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χ_q ← (even) Khovanov homology

χ_q ← odd Khovanov homology ①

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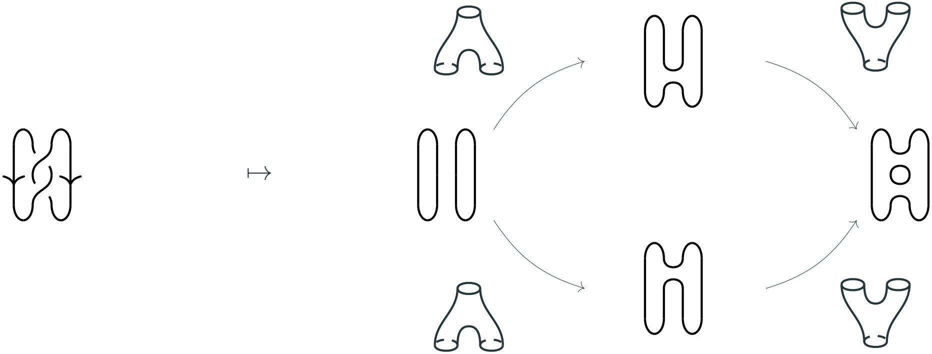
Odd Khovanov homology

Step 1: the hypercube of resolutions



Step 2: a projective TQFT

(even) Khovanov homology

$$\mathcal{F}\Big(\underbrace{\bigcirc \amalg \ldots \amalg \bigcirc}_{n}\Big) = \mathbb{Z}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$$

(*commutative*)

odd Khovanov homology

$$\mathcal{OF}\Big(\underbrace{\bigcirc \amalg \ldots \amalg \bigcirc}_{n}\Big) = \Lambda(x_1, \ldots, x_n)$$

(*anti-commutative*)

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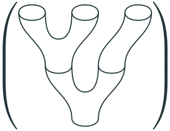
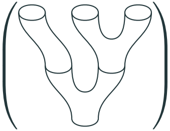
TQFT

$\mathcal{F}$  $=\mathcal{F}$ 

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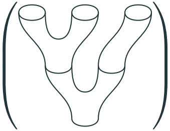
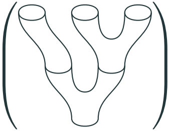
projective TQFT

$\mathcal{OF}$  $= -\,\mathcal{OF}$ 

Step 3: fixing the signs

Remark

To get a complex, we need that all squares anti-commute.

(EVEN) KHOVANOV HOMOLOGY	ODD KHOVANOV HOMOLOGY
$\mathcal{F}$ is functorial	$\mathcal{OF}$ is functorial up to signs

Step 3: fixing the signs

Remark

To get a complex, we need that all squares anti-commute.

(even) Khovanov homology

$\mathcal{F}$ is functorial

$\Downarrow$

commutativity is **easy** to control

signs can be fixed **uniformly**

odd Khovanov homology

$\mathcal{OF}$ is functorial up to signs

$\Downarrow$

commutativity is **hard** to control

require **ad-hoc** fixes

Theorem (Ozsváth–Rasmussen–Szabó 2007)

Yet, signs can be fixed, such that the resulting homology is a link invariant.

The categorified representation theory of $U_q(\mathfrak{gl}_2)$

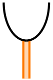
Representation theory of $U_q(\mathfrak{gl}_2)$

Elementary $U_q(\mathfrak{gl}_2)$-representations:

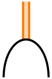
- V_1 = the q-**regular** representation $\quad U_q(\mathfrak{gl}_2) \curvearrowright \mathbb{C}(q) \times \mathbb{C}(q)$
- V_2 = the q-**determinant** representation $\quad U_q(\mathfrak{gl}_2) \curvearrowright \mathbb{C}(q)$

Elementary $U_q(\mathfrak{gl}_2)$-intertwiners (i.e. equivariant maps):

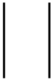
$$V_2 \xrightarrow{\ \epsilon\ } V_1 \otimes V_1 \quad \sim$$



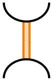
$$V_1 \otimes V_1 \xrightarrow{\ \eta\ } V_2 \quad \sim$$



$$\Rightarrow \quad V_1 \otimes V_1 \xrightarrow{\ \mathrm{id} - q\, \epsilon\eta\ } V_1 \otimes V_1 \quad \sim$$



$- q$



$=:$

Representation theory of $U_q(\mathfrak{gl}_2)$

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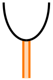

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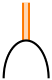

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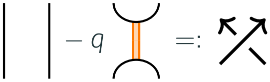
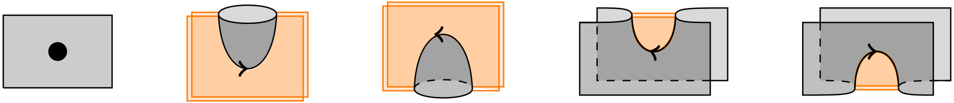


Fact

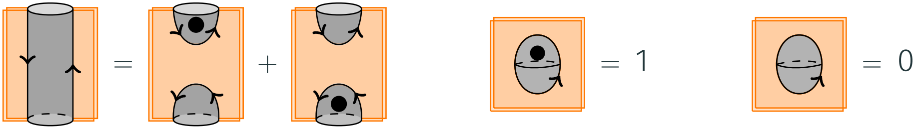
This construction recovers the Jones polynomial.

Foams

Foams are certain singular surfaces with *1-facets* (black) and *2-facets* (orange), with the following local behaviours:



and subject to local relations such as:

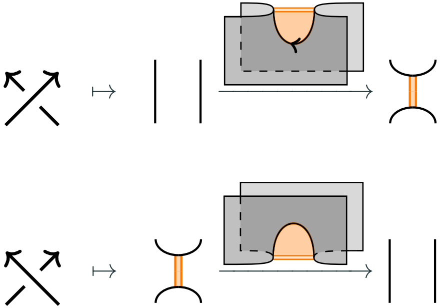


Fact

This defines a linear 2-category $\mathcal{F}oam$ that categorifies the rep. theory of $U_q(\mathfrak{gl}_2)$.

Recovering Khovanov homology

Thanks to the linear 2-category $\mathcal{F}oam$, we get an **invariant** of tangles, **defined as a tensor product** of complexes:

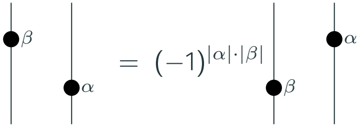


Theorem (Blanchet 2014)

This invariant recovers Khovanov homology when restricted to links.

Odd Khovanov homology and supercategorification

A **2-supercategory** is a categorical structure akin to a 2-category, except that **morphisms have parities** and **the interchange law is twisted**:



A 2-supercategory of foams

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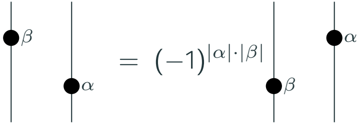

Theorem (S.–Vaz 2023)

There exists a **2-supercategory of foams** $\mathcal{OF}oam$ that categorifies the rep. theory of $U_q(\mathfrak{gl}_2)$.

A local definition of odd Khovanov homology

Theorem (S. 2020)

In any 2-supercategory, the **tensor product** of complexes **exists** and **preserves homotopy type**:

$$A^\bullet \simeq B^\bullet \text{ and } B^\bullet \simeq D^\bullet \quad \Rightarrow \quad A^\bullet \otimes C^\bullet \simeq B^\bullet \otimes D^\bullet.$$

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Theorem (S.–Vaz 2023)

Using the above, one can define an **invariant of tangles** that **coincides with odd Khovanov homology** when restricted to links.

Conclusion

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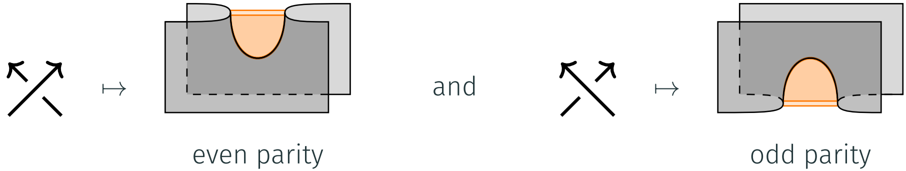
Take-home message

Odd Khovanov homology arises from a supercategorification of the rep. theory of $U_q(\mathfrak{gl}_2)$. *Corollary*: **local** (an controllable) **definition** of odd Khovanov homology.



Further directions:

- Can this provides a **functorial** definition of odd Khovanov homology?
- Are there other **odd link homologies**?
- Connections with **Lie superalgebras**? (e.g. $\mathfrak{osp}_{1|2}$)