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Odd Khovanov homology and higher representation theory

Léo Schelstraete

23th February, 2023

Winter Braids XII

Knot Theory

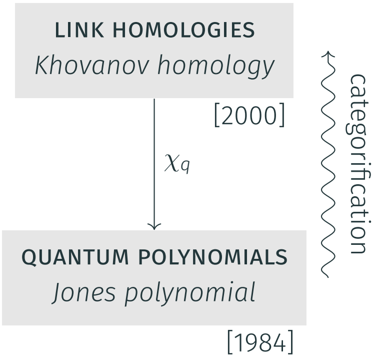
QUANTUM POLYNOMIALS

Jones polynomial

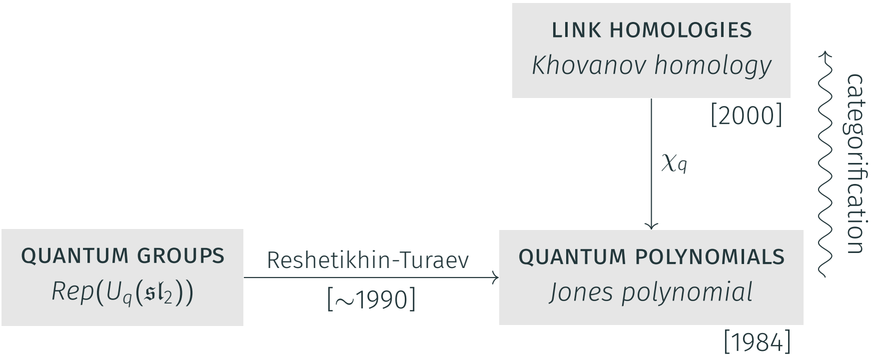
[1984]

Overview

Knot Theory

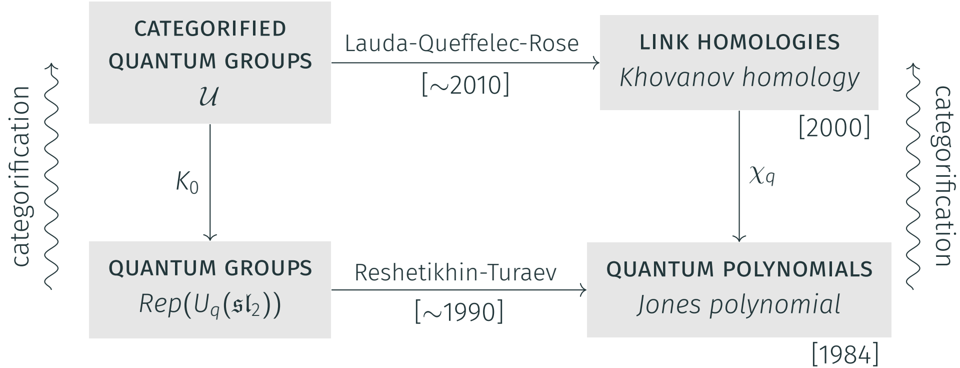


Representation Theory Knot Theory



Overview

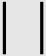
Representation Theory — Knot Theory



Khovanov homology

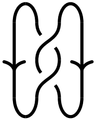
or the topological side of the story

Definition of the Jones polynomial

(1)  $\mapsto$  $-$  q

(2)  $\amalg D \quad \mapsto \quad (q + q^{-1})\langle D \rangle$

(for any diagram D)

$L =$ 

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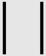
(1)  $\mapsto$  $-$  q

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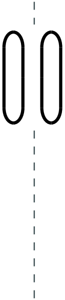
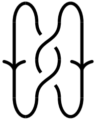


$L =$ 

$J(L) \;:=\; (q+q^{-1})^2$

Definition of the Jones polynomial

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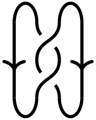
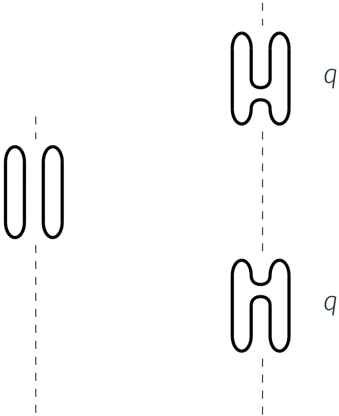
$L =$ 



$$J(L) := (q + q^{-1})^2 - 2(q + q^{-1})q$$

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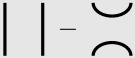
(1)  $\mapsto$  q

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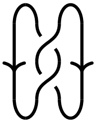
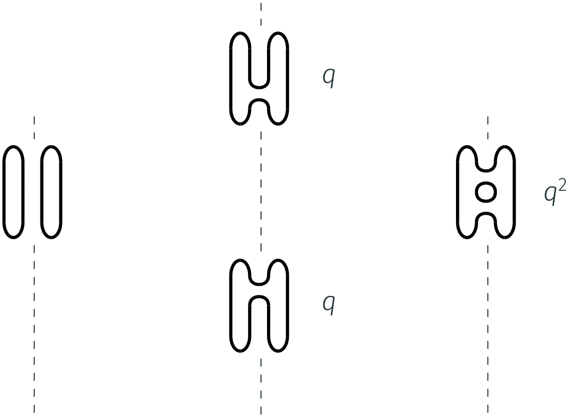
$L =$ 



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Definition of Khovanov homology

(1)  $[1]$

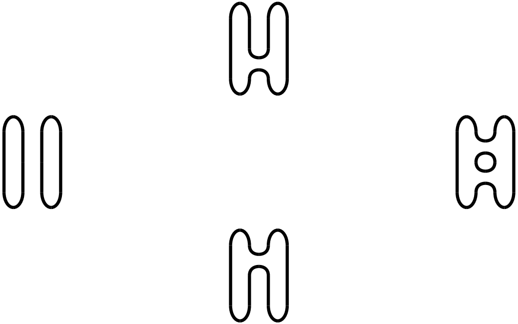
(2) $\mathcal{F}\colon (2Cob, \amalg) \to (Vec, \otimes)$

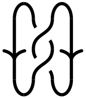
$$\bigcirc \mapsto A := \mathbb{Z}[x]/x^2$$

(3) add signs $\rightsquigarrow$ Koszul rule

(4) **grading** $\Rightarrow$ $\mathrm{gdim}(A) = q + q^{-1}$

$$A = \mathbb{Z} \oplus \mathbb{Z}x \cong \mathbb{Z}[-1] \oplus \mathbb{Z}[1]$$



$L =$ 

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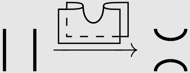
(1)  $\mapsto$  [1]

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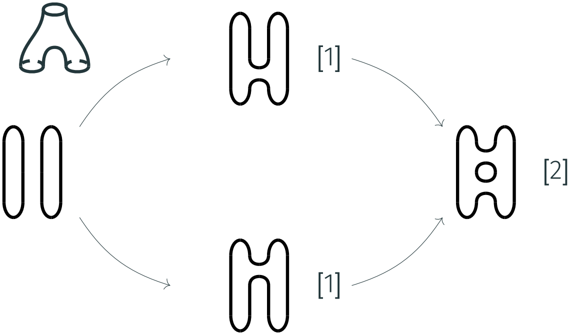
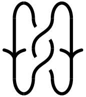


$L =$ 

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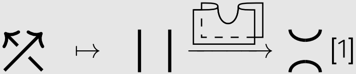
(1) 

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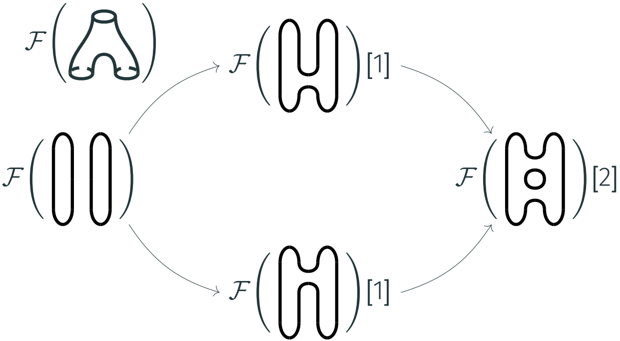
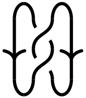


$L =$ 

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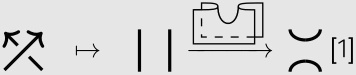
(1) 

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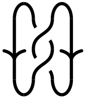
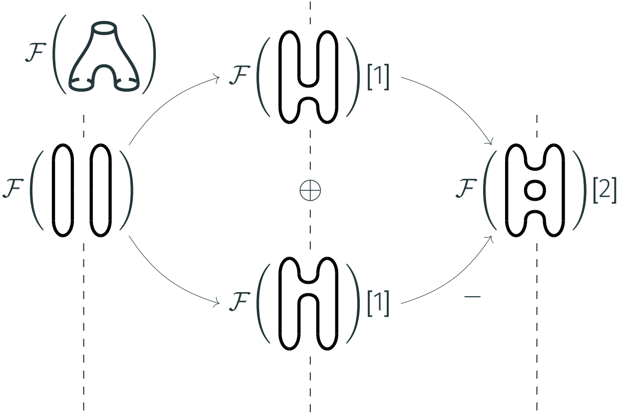
$L =$ 



$$A \otimes A \longrightarrow (A \oplus A)[1] \longrightarrow (A \otimes A)[2]$$

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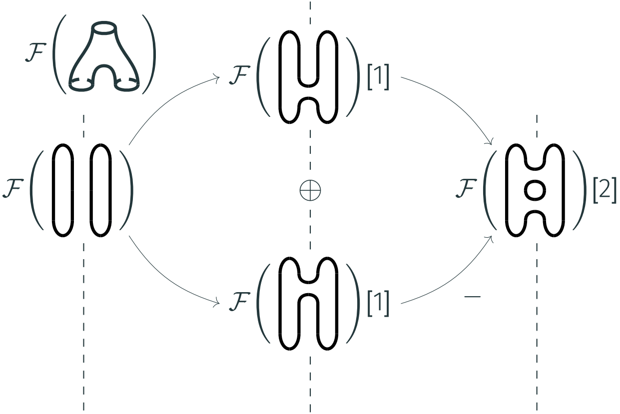
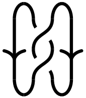


$L =$ 

$$\mathrm{Kh}^\bullet(L) := H^\bullet\Big(A \otimes A \longrightarrow (A \oplus A)[1] \longrightarrow (A \otimes A)[2] \Big)$$

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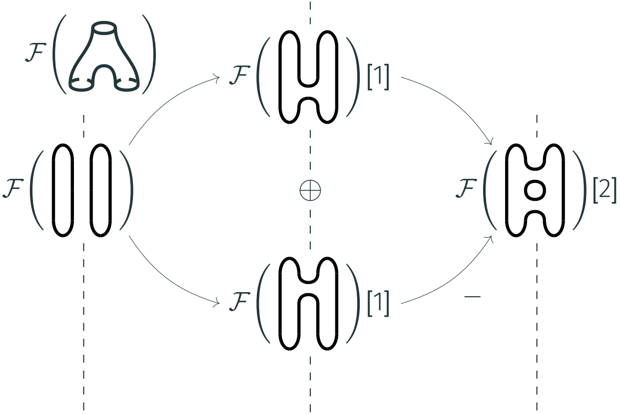
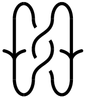


$L =$ 

χ_q

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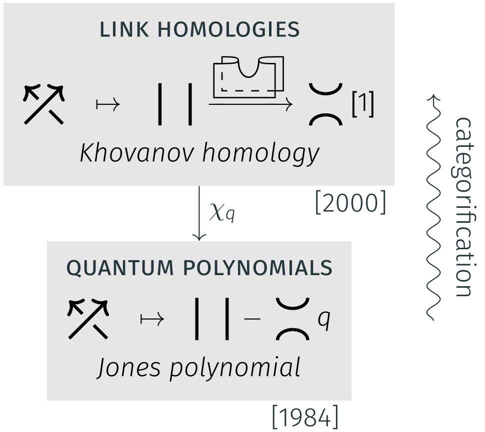
The principle of categorification

Why care?
functorial invariant ⇒ 4D-information!
(e.g. applications to sliceness detections)

THE PRINCIPLE OF CATEGORIFICATION
Given some objects, can we (meaningfully) "enrich" (categorify) them with morphisms?

Knot Theory



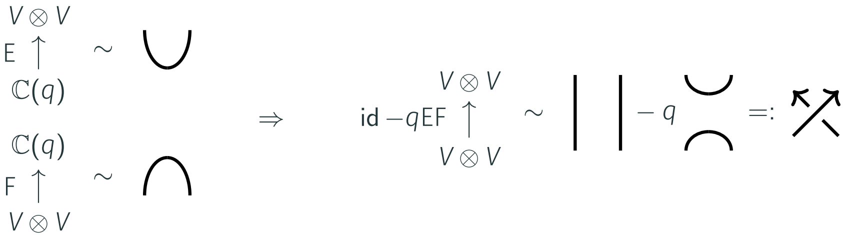
Categorifying $U_q(\mathfrak{sl}_2)$-representation theory

or the algebraic side of the story

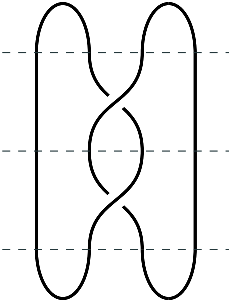
Elementary $U_q(\mathfrak{sl}_2)$-representations:

- $\mathbb{C}(q)$ = the **trivial** representation $U_q(\mathfrak{sl}_2) \curvearrowright \mathbb{C}(q)$
- V = the q-**regular** representation $U_q(\mathfrak{sl}_2) \curvearrowright \mathbb{C}(q) \times \mathbb{C}(q)$

Elementary $U_q(\mathfrak{sl}_2)$-intertwiners (i.e. equivariant maps):



Reshetikhin-Turaev construction: Jones from $U_q(\mathfrak{sl}_2)$-representation theory



$$\mapsto$$

$$\begin{array}{ccccccccc}
\mathbb{C}(q) & & \otimes & & \mathbb{C}(q) & & \cong & \mathbb{C}(q) \\
& & \uparrow \mathrm{F} \otimes \mathrm{F} & & & & & \uparrow \\
V \otimes V & & \otimes & & V \otimes V & & & \\
& & \uparrow \mathrm{id} \otimes (\mathrm{id} - q\mathrm{EF}) \otimes \mathrm{id} & & & & & \\
V \otimes V & & \otimes & & V \otimes V & & & \\
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V \otimes V & & \otimes & & V \otimes V & & & \\
& & \uparrow \mathrm{E} \otimes \mathrm{E} & & & & & \\
\mathbb{C}(q) & & \otimes & & \mathbb{C}(q) & & \cong & \mathbb{C}(q)
\end{array}$$

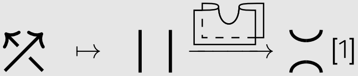
Theorem

This construction recovers the Jones polynomial.

Khovanov homology from higher $U_q(\mathfrak{sl}_2)$-representation theory

Representation Theory — Knot Theory

LINK HOMOLOGIES



Khovanov homology

χ_q

categorification

QUANTUM GROUPS

 $\mapsto$ $\mathrm{id} - qEF$

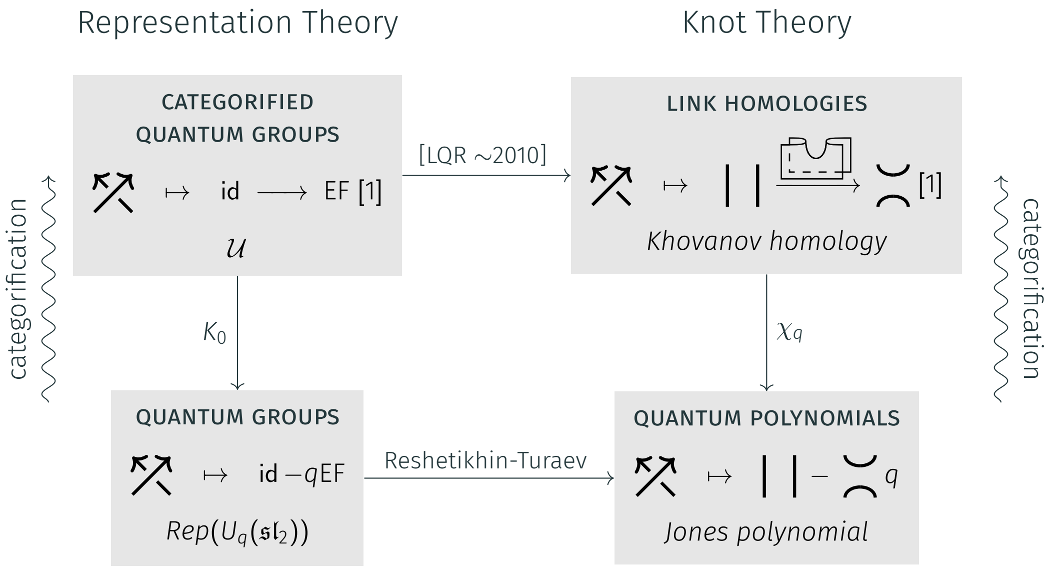
Rep$(U_q(\mathfrak{sl}_2))$

Reshetikhin-Turaev

QUANTUM POLYNOMIALS

Jones polynomial

Khovanov homology from higher $U_q(\mathfrak{sl}_2)$-representation theory



Odd Khovanov homology and supercategorification

A new world?

Odd Khovanov homology, another categorification of Jones

Theorem (Ozsváth-Rasmussen-Szabó 2007)

(even) Khovanov homology $\mathrm{Kh}(L) \xrightarrow{\chi_q} J(L) \xleftarrow{\chi_q} \mathrm{OKh}(L)$ **odd** Khovanov homology

There exists another categorification of the Jones!

FACTS: $\mathrm{Kh}(L)$ and $\mathrm{OKh}(L)$ **coincide over** $\mathbb{Z}/2\mathbb{Z}$, but they are **distinct over** $\mathbb{Z}$

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Question: a construction of odd Khovanov homology
from higher representation theory? YES! [Vaz-S. 22+]

Comparing even and odd Khovanov homology

(Even) Khovanov homology

$$\mathcal{F}\Big(\underbrace{\bigcirc \amalg \ldots \amalg \bigcirc}_{n}\Big) = \mathbb{Z}[x_1, \ldots, x_n]/(x_1^2, \ldots, x_n^2)$$

(*commutative*)

odd Khovanov homology

$$\mathcal{OF}\Big(\underbrace{\bigcirc \amalg \ldots \amalg \bigcirc}_{n}\Big) = \Lambda(x_1, \ldots, x_n)$$

(*anti-commutative*)

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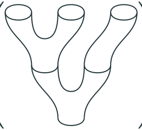
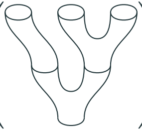
TQFT

$\mathcal{F}\Big($  $\Big) = \mathcal{F}\Big($  $\Big)$

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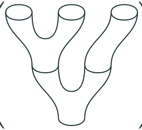
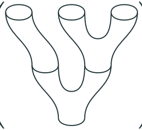
"almost" TQFT

$\mathcal{OF}\Big($  $\Big) = -\mathcal{OF}\Big($  $\Big)$

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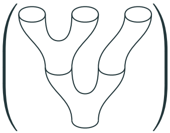
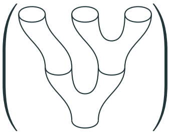
TQFT

$\mathcal{F}$  $= \mathcal{F}$ 

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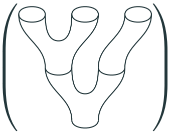
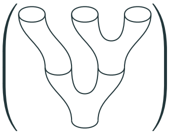
$\mathcal{OF}$  $= -\mathcal{OF}$ 

Q: how to capture this kind of behaviour?

A: use parities, that is, 2-supercategories.

Odd Khovanov homology from higher representation theory

Theorem (S.-Vaz 2022)

There exists a 2-supercategory $\mathcal{OU}$, from which one can define a **tangle invariant**.

Odd Khovanov homology from higher representation theory

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Work in progress (S.-Vaz 2023)

This invariant (very likely) **recovers odd Khovanov homology**.

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FURTHER DIRECTIONS:

- Is odd Khovanov homology **functorial**? (If so, in which sense?)
- What about **other homological invariants**? ($\mathfrak{sl}_n$-homologies, triply-graded homology)

  Do they also admit an "odd" version?

  Could we discover them through supercategorified quantum groups? (or analogues?)

Appendix

Khovanov homology from higher $U_q(\mathfrak{sl}_2)$-representation theory

Theorem (Lauda-Queffelec-Rose ∼2010)*

(i) There exists a 2-category $\mathcal{U}$ such that

$$K_0(\mathcal{U}) \cong Rep(U_q(\mathfrak{sl}_2)).$$

(ii) There exists a **tangle invariant**, defined with tensor product of complexes in $\mathcal{U}$.

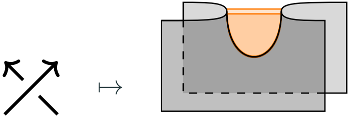
(iii) This tangle invariant **recovers Khovanov homology** in the case of links.

**notably building on work by Cautis-Kamnitzer-Morrison and Khovanov-Lauda*

Main features of the construction

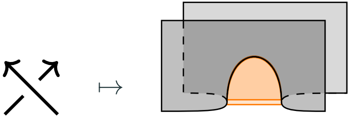
(1) **Need to extend** the construction **to tangle cobordisms**:
$\Rightarrow$ cannot put parities on merge and splits!

(2) Solution: use $\mathfrak{gl}_2$-foams $\Rightarrow$ two different saddles:



even parity

and



odd parity

(3) Difficulty: **different parities as original definition** $\Rightarrow$ hard to compare

2-supercategories

Definition

In a **2-supercategory**, 2-morphisms

- have *parities* (i.e. $\mathbb{Z}/2\mathbb{Z}$-grading),
- admit both *vertical* ($\circ$) and *horizontal* ($*$) compositions...
- ...but with a *twisted coherence law*:

$$(f \circ g) * (h \circ k) = (-1)^{|g||h|}(f * h) \circ (g * k)$$

super interchange law

Remark: a 2-supercategory is (in general) *not* a (strict) 2-category!

Proposition (S. 2020)

In any 2-supercategory, the tensor product of complexes exists and preserves homotopy type:

$$A^\bullet \simeq B^\bullet \text{ and } B^\bullet \simeq D^\bullet \quad \Rightarrow \quad A^\bullet * C^\bullet \simeq B^\bullet * D^\bullet.$$