

MILP-Based Algorithm for the Global Solution of Dynamic Economic Dispatch Problems with Valve-Point Effects

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Abstract—The Dynamic Economic Dispatch (DED) problem consists in satisfying a certain demand for electric power among scheduled generating units over a certain interval of time while satisfying the operating constraints of these units. The consideration of the valve-point effect (VPE) makes the problem more practical but also more challenging due to the non-linear and non-smooth constraints that are required for representing the model. We present a method, based on a sequence of piecewise linear approximations, which produces a feasible solution along with a lower bound to the global solution. In this way, this deterministic approach can trade off the speed which characterizes certain heuristics that are usually used to solve the DED-VPE for a better solution and insights about the problem. The method is applied to a widely used case study and provides a lower solution objective than the best known solution to date.

I. INTRODUCTION

The Economic Dispatch (ED) problem is an important optimization problem in short-term power system planning. It consists in the optimal dispatch of power among scheduled electricity generation facilities in order to meet the system load at a minimal cost. Commonly, the fuel cost functions have been modeled as a smooth quadratic function in the ED problem. Unfortunately, such a model does not reflect the valve-point effect (VPE) which significantly affects the output of facilities that are characterized by a non-smooth and non-convex cost function. The latter characteristics of the cost function prevent us from using traditional derivative-based optimization techniques for solving the problem.

In the past decades, a plethora of methods have been developed in order to address this problem, including neural networks [1], simulated annealing (SA) [2], genetic algorithms (GA) [3], evolutionary programming [4], differential evolution (DE) [5] and particle swarm optimization (PSO) [6]. A more exhaustive list of these methods and other hybrid combinations can be found in [7]. Most of the aforementioned techniques are heuristics and, if they often give a fast and reasonable solution, they lack guarantees with respect to the returned solution.

On the other side, deterministic mathematical programming-based optimization techniques have the advantage of providing information about the distance of the solution from optimality. Such methods have been developed by piecewise linearization of the VPE-term [8] and of the entire objective function [9], leading to respectively a mixed integer quadratic programming (MIQP) and a mixed integer linear programming (MILP) problem. More recently in [10], an adaptive MIQP method has been proposed with the significant feature of providing a global solution of the static ED. This method relies on a sequence of under-approximations for which the sequence of optimal solutions eventually converges to the global solution of the original problem. Early developments of the adaptive approach can be found in the technical report [11].

Here, we follow this adaptive approach. Our contributions are to (i) apply this approach on the dynamic ED and (ii) to investigate the benefit of using a MILP formulation.

The remainder of this paper is organized as follows. In Section II, the full problem is introduced and the VPE is described. The linear version of the adaptive method is presented in Section III. Then the application on a 10-generator case study over 24 hours is performed in Section IV. Finally, conclusions are drawn in Section V.

II. PROBLEM STATEMENT

This section outlines the problem of interest, namely the dynamic economic dispatch (DED) problem. This problem consists in minimizing the fuel costs of the thermal power units throughout a certain time period subject to operational constraints. The objective is defined as the sum of the individual cost functions over each time step and is therefore separable,

$$ f(p) = \sum_{i=1, t=1}^{n, T} f_t(p_i), $$

where $f$ is the total cost function [\$/h], $f_t$ the fuel cost associated with generator $i$ at time $t$ and $p$ is the stacked production vector of each individual generator production $p_{it}$ [MW]. A common model of the fuel cost functions including the VPE is the sum of a smooth quadratic part and a non-smooth rectified sine, i.e.,

$$ f_{it}(p) = f_i(p) = a_ip^2 + b_ip + c_i + d_i \left| \sin e_i(p - p_i^{\text{min}}) \right|; $$
with appropriate parameters $a_i, b_i, c_i, d_i, e_i$. The impact of the VPE, namely the non-smooth and high multimodal nature of the problem, is underlined in Figure 1. The model must of course enforce power balance which couples the optimization problem,
\[
\sum_{i=1}^{n} p_{it} = P_t^D + p^L(p_t),
\]
with $P_t^D$ and $p^L(p_t)$ respectively the loud demand at time $t$ and the transmission lost in MW. The latter is not included in the method presented here and the assumption $p^L(p_t) = 0$ is made throughout the rest of this paper. This extension is left for future research.

We consider spinning upward reserve, i.e., extra generating capacity in case of contingencies,
\[
\sum_{i=1}^{n} s_{it} \geq S_t, \quad \sum_{i=1}^{n} s_{it} \leq R_{it}^U,
\]
with $s_{it}$ the extra capacity that generator $i$ must be able to provide and $S_t$ the total spinning upward reserve required at time $t$.

Finally, the optimization problem is subject to operational constraints such as the admissible range of power production,
\[
P_i^{\text{min}} \leq p_{it},
\]
and ramp constraints,
\[
-R_i^D \leq p_{it} - p_{i(t-1)} \leq R_i^U,
\]
with $P_i^{\text{min}}$ and $P_i^{\text{max}}$ the minimum and maximum acceptable range of power production and $R_i^D, R_i^U$ the downward and upward ramp rate limit of the $i$-th generator.

Note that, with the exception of power loss, the method described in the following section can be easily extended to more complicated models that account for downward spinning reserve, multiple fuels and prohibited operating zones.

### III. METHOD DESCRIPTION

This section is devoted to the characterization of an algorithm for the solution of the DED. The method consists of a sequence of piecewise linear approximations tackled by a MILP solver.

#### A. Surrogate problem

For all $i$ and $t$, let $X_{it}$ be a set of points $X_{it1} < X_{it2} < \cdots < X_{itn_{it}^{\text{knot}}}$, called knots, from which we construct a piecewise linear approximation $g_{it}$ of $f_{it}$. We create the original set of knots of unit $i$ equally for each time $t$ as the union of two subsets. The first subset is the set of kink points, which are the points where the cost function is non-smooth. The second subset is the set of local maxima of the rectified sine. Hence for every unit $i$ and time $t$, the set of initial knots is equal to
\[
X_{itj} = P_i^{\text{min}} + \frac{(j-1)\pi}{2c_i}, \quad j = 1 \ldots n_{it}^{\text{knot}},
\]
with $n_{it}^{\text{knot}} = 1 + [(P_i^{\text{max}} - P_i^{\text{min}})2e_i]/\pi$. We then construct the surrogate approximation $g_{it}$ through a binary formulation, i.e.,
\[
g_{it}(p_{it}) := \begin{cases} 
\sum_{j=1}^{n_{it}^{\text{knot}}} \alpha_{itj}\xi_{itj} + \eta_{itj}\beta_{itj}, \\
\sum_{j=1}^{n_{it}^{\text{knot}}-1} \xi_{itj} = p_{it}, \\
\sum_{j=1}^{n_{it}^{\text{knot}}-1} \eta_{itj} = 1, \\
X_{itj}\eta_{itj} \leq \xi_{itj} \leq X_{itj+1}\eta_{itj}, \\
\eta_{itj} \text{ binary},
\end{cases}
\]
with $\alpha_{itj}$ and $\beta_{itj}$ are the slope and vertical intercept of the linear pieces; see [10] for details of the concept. The binary variables $\eta$ act as switches which select the different pieces. Following (1), we also define $g(p) := \sum_{i=1}^{n} \sum_{t=1}^{T} g_{it}(p_{it})$. The surrogate optimization problem becomes
\[
\begin{align*}
\min_{\eta_{itj}, \xi_{itj}, p_{it}, s_{it}} & \sum_{i=1}^{n} \sum_{t=1}^{T} n_{it}^{\text{knot}} \alpha_{itj}\xi_{itj} + \beta_{itj}, \\
\text{subject to} & \sum_{i=1}^{n} p_{it} = P_t^D, \\
& \sum_{i=1}^{n} s_{it} \geq S_t, \\
& s_{it} \leq R_{it}^U, \\
& p_{it} + s_{it} \leq P_i^{\text{max}}, \\
& -R_i^D \leq p_{it} - p_{i(t-1)} \leq R_i^U, \\
& \sum_{j=1}^{n_{it}^{\text{knot}}-1} \xi_{itj} = p_{it}, \\
& \sum_{j=1}^{n_{it}^{\text{knot}}-1} \eta_{itj} = 1, \\
& X_{itj}\eta_{itj} \leq \xi_{itj} \leq X_{itj+1}\eta_{itj}, \\
& \eta_{itj} \in \{0,1\}.
\end{align*}
\]

#### B. Knot Update Mechanism and Algorithm Statement

Assume a solution of the surrogate problem has been obtained. If the gap between the true and surrogate objective function evaluated at this point is too large, an increase in the number of knots should be contemplated. Indeed, the approximation will be enhanced and the same is true for the surrogate solution. In [9], the adopted approach was to increase the knot sampling over the entire allowable range. However, this increases the number of knots and therefore the number of binary variables exponentially. In this work, we follow the knot update mechanism from [10], i.e., the previous surrogate solution is added to the knot list. As a consequence, in the new iteration, the surrogate solution differs from the old one and convergence is guaranteed.
The proposed APLA algorithm (Algorithm 1) is very similar to Algorithm 1 in [10].

**Algorithm 1** APLA: Adaptive piecewise-linear approximation

1. Set tolerance parameter $\delta_{tol}$
2. for $i := 1 \ldots n$
   3. for $t := 1 \ldots T$
   4. Choose ordered set of knots $(X_{it})$ including kink points
   5. $G_{it} \leftarrow f_i(X_{it})$
   6. end for
   7. end for
8. while $\delta^k > \delta_{tol}$
   9. $p^k \leftarrow \text{optimal solution of MILP surrogate problem (10), obtained with MILP solver with tolerance } \gamma$
   10. $\delta^k \leftarrow \min_{i=1 \ldots k} f(p^k) - g(p^k)$
   11. for $i := 1 \ldots n$,
      12. if $\min_{j=1 \ldots n} \left| p^k_{it} - X_{it} \right| > 0$ then
      13. $X_{it} \leftarrow \text{insert}(X_{it}, p^k_{it})$ \triangleright insert at same index as previous line
      14. $n_{it}^{\text{knot}} \leftarrow n_{it}^{\text{knot}} + 1$
      15. end if
   16. end for
   17. end while
18. **return** $\arg \min_{i=1 \ldots k} f(p^i)$

**C. Bound to optimal solution**

At step $k$ of Algorithm 1, the objective function evaluated at the optimal solution $p^*$ can be bounded as follows,

$$f(p^k) - \delta^k - \gamma^k - \epsilon^k \leq f(p^*) \leq f(p^k),$$

where $\delta^k$ is the gap between the objective and surrogate function at point $p^k$ as computed in line 11 of Algorithm 1, $\gamma^k$ the tolerance used to solve the problem, and $\epsilon^k$ represents the over-approximation error. Let us explain more precisely each of these bounds and how it evolves as $k$ increases.

**a) The over-approximation error ($\epsilon^k$)**: Algorithm 1 is based on a sequence of piecewise linear approximations. In contrast with the APQUA method of [10], the approximation is not guaranteed to be an under-approximation. It can be seen that the approximation will be an under-approximation if the true cost function is concave in each segment delimited by the initial knots. However, this concavity assumption is not satisfied in our case because the curvature of the rectified sine vanishes at the kink points (see bottom right magnification in Figure 1). More precisely, function $f_{it}$ is convex on $X_{it} + \frac{1}{e_i} \arcsin \left( \frac{2a_i}{d_i e_i} \right) \times [-1, 1]$. In the example studied in Section IV, these convex parts are very small: about 0.5% of the whole domain. The error $\epsilon^k$ can be computed as

$$\epsilon^k = \max_{p \in \mathcal{G}} \left( g^k(p) - f(p) \right),$$

$$= \sum_{i=1, t=1}^{n, T} \max_{p_{it} \in [p_{it}^\text{min}, p_{it}^\text{max}]} \left( g^k(p_{it}) - f(p_{it}) \right).$$

Note that this last equation can be easily calculated at every iteration since it can be viewed as $n \times T \times n_{\text{knot}}$ decoupled optimization problems of a single variable. Besides, as we never remove points, $(\epsilon^k)_{k \in \mathbb{N}}$ can be bounded above; we obtain for all $k = 1, 2, \ldots$

$$\epsilon^k \leq \epsilon_{\text{max}} := \sup_{g \in \mathcal{G}} \max_{p \in \mathcal{P}} \left( g(p) - f(p) \right),$$

where $\mathcal{G}$ is the set of all piecewise interpolations of $f$ such that the kink points belong to the set of knots that defines the interpolation. In other words, $\mathcal{G}$ is the set of functions to which Algorithm 1 has access in order to approximate $f$. For the case study investigated in Section IV, we get $\epsilon_{\text{max}} = 0.32$ which is very small with respect to the other bounds.

**b) The gap between the objective and surrogate function ($\delta^k$)**: Following [10], we show that $(\delta^k)_{k \in \mathbb{N}}$ converges to 0.

**Theorem 1.** $\lim_{k \to \infty} \delta^k = 0$

**Proof.** We first show that $f$ and $g^k$, $k = 0, 1, 2, \ldots$ are Lipschitz continuous on the feasible set.

For each unit $i$ and time $t$, we have $|f_{it}(p + \Delta) - f_{it}(p)| \leq (2a_i P_{it}^\text{max} + b_i + d_i e_i) \Delta := K_i \Delta$ with $K_i$ the so-called Lipschitz constant. Summing up on each unit, $K := T \times \sum_{i=1}^{n} K_i$ is a valid Lipschitz constant for $f$. Since $g_{it}^k$ is a continuous piecewise interpolation of $f_{it}$, it is also Lipschitz continuous and $K_i$ (resp. $K$) is a valid Lipschitz constant for $g_{it}^k$ (resp. $g_{it}^k$). Let $(p^k)_{k \in \mathbb{N}}$ be the sequence of optimal solutions of the surrogate problem associated with function $g^k$, we then obtain

$$\delta^k = f(p^k) - g^k(p^k),$$

$$= f(p^k) - g^k(p_{it}^k) + g^k(p_{it}^k) - g^k(p_{it}^k),$$

$$= f(p^k) - f(p_{it}^k) + g^k(p_{it}^k) - g^k(p_{it}^k),$$

$$\leq 2K \left| \left| p^k - p_{it}^k \right| \right|,$$

where the 3rd line comes from the knot updating criterion and the last line from the Lipschitz continuity.

Suppose for contradiction that $(\delta^k)_{k \in \mathbb{N}}$ does not converge to 0. Then there is $\delta^* > 0$ and an infinite subsequence $(\delta^k_j)_{j \in \mathbb{N}}$ such that $|\delta^k_j| > \delta^*$ for all $j$. Then, given any $j$, we have that for all $J > j$, $\|p^m_J - p_{it}^k\| \geq \delta^*/(2K)$. This implies that the subsequence $(p^m_J)_{j \in \mathbb{N}}$ is unbounded, a contradiction with the admissible range constraints. □

The bottom left magnification in Figure 1 depicts an example of $\delta^k$, note that the $i$ and $t$ indices have been omitted. Finally, due to the definition of $\delta^k$, it immediately follows from Theorem 1 that

$$\lim_{k \to \infty} \delta^k \leq 0.$$

(15)
c) The solver tolerance ($\gamma^k$): The sequence $(\gamma^k)_{k \in \mathbb{N}}$ is not monotonic and is bounded below by $\gamma f(p^*)$ with $\gamma$ the solver relative tolerance gap to the global solution of the surrogate problem.

![Graph](image)

Fig. 1. Outline of the true $f$ and surrogate $g^k$ functions. The bottom left magnification allows a better visualization while the bottom right one shows a tiny convex zone around a kink point.

IV. Test Case Study

In this section, a 10-unit DED without losses over $T = 24$ hours is studied. The data set used for the case study can be found in [12] and the spinning reserve is set at 5% of the demand. The optimization is performed on a computer with an Intel-i7 CPU and 16 GB of RAM. Gurobi 8.0.0 [13] has been used with a relative gap tolerance of $\gamma = 0.25\%$ and the model has been coded in AMPL [14]. Note that a feasible solution stays feasible for the surrogate problem at every iteration. Hence, in order to benefit from the previous iterations, the MILP solver is fed with the best known solution of the true problem as an initial incumbent. Algorithm 1 is also slightly improved by asking, in line 9, the solver to return the 50 best incumbent solutions instead of the sole best one. Then, the best candidate solution becomes the minimum with respect to the true objective value of this set of incumbent solutions, provided that it is smaller than the actual best candidate.

After 9 iterations and 902 seconds, a solution with $\delta = 1.42$ is found with objective 1016276 [8]. This is an improvement over the previous best solution in the literature with objective 1016311 [8] [7]. Neglecting $\epsilon_{\max}$, the final relative optimality gap is

$$\frac{|f(p^*) - f(p)|}{f(p)} = \gamma + \frac{\delta}{f(p)} = 0.265\%. \quad (16)$$

The power dispatch among the generating units is provided in Table I.

V. Conclusion

In this paper, we have presented a deterministic method which accounts for valve-point effect in the dynamic economic dispatch problem. The method relies on a succession of piecewise linear approximations of the cost function which defines a surrogate problem. The method is adaptive in the sense that the previously computed best candidate minimum of the surrogate problem is added to the knots used to define the approximation. Doing so, the method converges to a solution which is optimal within the mixed-integer solver accuracy, plus an $\epsilon$ term. The latter can be bounded a priori and turns out to be negligible in problem instances found in the literature. The result of our case study confirms this distinctive feature by slightly improving the best known solution to the considered problem. However, as each surrogate problem remains challenging, the computation time of our approach is higher – approximately 10 times – than the previous best results from the literature. Also, the method is not limited to the dynamic economic dispatch problem, as any optimization problem with separable objective can be treated in a similar way.
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