The Minimum Rank Problem of a Graph

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Motivation

- A simple undirected graph G defines a family $\mathcal{Q}(G)$ of matrices : $\mathcal{Q}(G)=\{A\in\mathbb{R}^{|G|\times|G|}|A=A^T,$ for $i\neq j, a_{ij}\neq 0\Leftrightarrow\{i,j\}\in E(G)\}$
- The Inverse Eigenvalue Problem of a Graph consists in finding the possible spectra of the matrices in $\mathcal{Q}(G)$.
- A first step is to determine the maximum possible multiplicity of an eigenvalue of a matrix in $\mathcal{Q}(G)$.
- Computing this maximum is equivalent to determine the minimum rank of G .

The Minimum Rank Problem

- If G is a simple directed graph, $\mathcal{Q}(G)=\{A\in\mathbb{R}^{|G|\times|G|}|$ for $i\neq j, a_{ij}\neq 0\Leftrightarrow(i,j)\in E(G)\}$
- If G is a loop directed graph, $\mathcal{Q}(G)=\{A\in\mathbb{R}^{|G|\times|G|}|a_{ij}\neq 0\Leftrightarrow(i,j)\in E(G)\}$
- If G is a loop undirected graph, $\mathcal{Q}(G)=\{A\in\mathbb{R}^{|G|\times|G|}|A=A^T, a_{ij}\neq 0\Leftrightarrow\{i,j\}\in E(G)\}$
- The minimum rank of a graph G is : $\mathrm{mr}(G)=\min\{\text{rank}(A)|A\in\mathcal{Q}(G)\}$
- The minimum rank problem was resolved in [3] but only for a tree of any type.
- Hogben proposed in [3] a lower bound for the minimum rank of a graph of any type.

Contribution

- Improvement of Hogben's lower bound

Ideas :

- A hypergraph H_G is associated to a graph G. Any vertex of H_G is white.
- A color change rule (CCR) is defined on H_G : this rule colors in black some of its white vertices.
- The generating set number $Z(H_G)$ is the minimum number of vertices of H_G we have to color in black so that after applying the CCR to H_G all of its vertices are black.
- $|G|-Z(H_G)\leq \mathrm{mr}(G)$

A color change rule on a hypergraph

Consider a hypergraph H whose any vertex is either black or white.

The color change rule is applied to each hyperedge e of H until no more color change is possible : suppose the size of e is n. If exactly $n-1$ vertices of e are black, then the last one becomes black too.

A generating set $\mathcal{Z}$ is a white vertex set of H such that if all the vertices of $\mathcal{Z}$ are colored in black, after applying the CCR no more vertex is white.

The generating set number of H is :

$$Z(H)=\min\{|\mathcal{Z}|:\mathcal{Z}\text{ is a generating set of }H\}$$

where $|\mathcal{Z}|$ denotes the number of vertices in $\mathcal{Z}$.

Matrices and hypergraphs

Denote P the zero-nonzero pattern of a matrix of order N .
The following hypergraph H_P is associated to P :

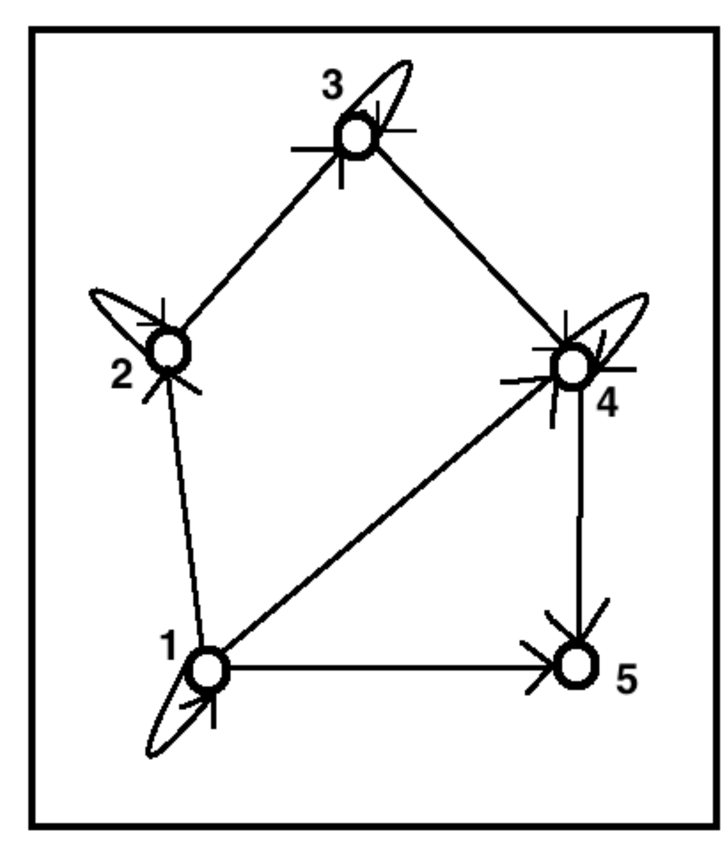
- H_P contains N (white) vertices (numbered from 1 to N)
- If the nonzero entries in a row of P lie in columns $\{i_1,\dots,i_l\}$, then $\{i_1,\dots,i_l\}$ is a hyperedge of H_P .

Proposition 1 : For all matrix $A\in\mathbb{R}^{N\times N}$ whose zero-nonzero pattern is P,

$$\dim\ker A\leq Z(H_P)$$

Example

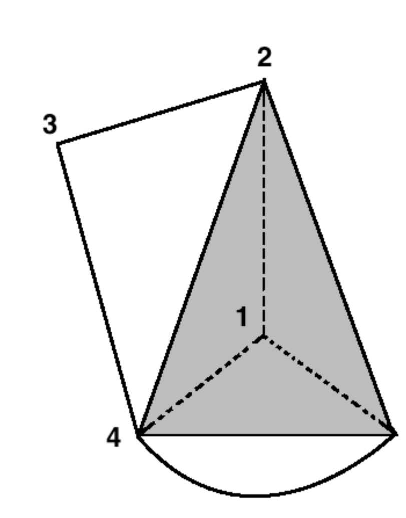
Thanks to Proposition 1, it is easy to compute a lower bound for the minimum rank of the following loop directed graph G :



The matrices in $\mathcal{Q}(G)$ have the following pattern P :

$$\begin{pmatrix}\star&\star&0&\star&\star\\0&\star&\star&0&0\\0&0&\star&\star&0\\0&0&0&\star&\star\\0&0&0&0&0\end{pmatrix}$$

The hypergraph H_P is then :



We compute easily that $Z(H_P)=1$.

Consequently, thanks to proposition 1,

$$5-1=4\leq \mathrm{mr}(G)$$

Moreover, in this case, we conclude : $\mathrm{mr}(G)=4.$

References

[1] L. Hogben, **Spectral graph theory and the inverse eigenvalue problem of a graph**, Electronic Journal of Linear Algebra, 14: 12-31, 2005.

[2] S. Fallat, L. Hogben, **The minimum rank of symmetric matrices described by a graph : a survey**, *Linear Algebra and Applications*, 426 : 558-582, 2007.

[3] L. Hogben, **Minimum rank problem**, *Linear Algebra and its applications*, 432(8), p.1961, Apr.2010.