Strong controllability of networked systems and minimum rank of a graph: what can we compute ?

Maguy Trefois Jean-Charles Delvenne

Department of Applied Mathematics
Université catholique de Louvain (UCL)
Louvain-La-Neuve, Belgium

1 Motivation

The controllability of a networked system is currently an important field of research [1, 2]. The weak and strong structural controllability provide us respectively with a lower bound and an upper bound for the minimum number of nodes which have to be directly controlled by the outside controller in order to have full control over the network. In [1], it has been proved that the weak structural controllability can be studied from a maximum matching on the network. Instead, to study the strong structural controllability, the notion of maximum constrained matching (introduced in [3]) in a bipartite graph is needed [2].

In [4], it has been proved that the zero forcing number [5] of a directed graph (allowing loops), introduced to study the minimum rank of the graph, is equivalent to a maximum constrained matching in a bipartite graph associated with the given directed graph. But, unfortunately, computing a maximum constrained matching in any bipartite graph is NP-hard. We show that for some structures of graphs however (namely different sorts of trees), it is possible to compute the minimum rank exactly. This allows to study the strong controllability of systems organized according to these sorts of trees.

2 Introduction

The minimum rank of a graph is a graph parameter which plays a role in many problems like the inverse eigenvalue problem of a graph, the biclique partition of the edges of a graph or the characterization of the singular graphs (see [6] for descriptions and interesting references). The definition of the minimum rank of a graph depends on the type of the graph. In this paper, we will focus on the directed graphs. We distinguish two types of directed graphs: the **loop directed graphs**, which are graphs allowing loops, and the **simple directed graphs**, which are graphs prohibiting the loops. The definition of the minimum rank is slightly different according to the case.

Definition 1. *Any loop directed graph $G = (V(G), E(G))$ defines a matrix set*

$$\mathcal{Q}_{ld}(G) := \{A \in \mathbb{R}^{|G|\times|G|} : a_{ij} \neq 0 \text{ if and only if } (i, j) \in E(G)\}.$$

The ***minimum rank*** *of G is the minimum possible rank for a matrix in $\mathcal{Q}_{ld}(G)$.*

The minimum rank of a simple directed graph is similarly defined. The only difference lies in the fact that, in the case of a simple directed graph, the diagonal entries of the matrices the graph defines are completely free: they can be nonzero even though there is no loop on the nodes. The formal definition is given below.

Definition 2. *Any simple directed graph $G = (V(G), E(G))$ defines a matrix set*

$$\mathcal{Q}_{sd}(G) := \{A \in \mathbb{R}^{|G|\times|G|} : \text{ for any } i \neq j, a_{ij} \neq 0 \text{ if and only if } (i, j) \in E(G)\}.$$

The ***minimum rank*** *of G is the minimum possible rank for a matrix in $\mathcal{Q}_{sd}(G)$.*

Nowadays, computing, even approximately, the minimum rank of any directed graph is still a challenging problem. In [4], it has been proved that the minimum rank of any loop oriented tree (defined in the following section) can be computed in linear time thanks to an elimination process. In the following, we generalize this elimination process and we prove that this generalization allows to compute in linear time the minimum rank of any simple oriented tree (defined in Section 3). Moreover, the results in [2, 4] show the close relation between the strong structural controllability of a networked system and the minimum rank of a loop directed graph. In particular, the undamped loop directed graphs (defined in Section 4) are of interest in the study of the controllability of quantum systems [7]. In Section 4, we show that the elimination process defined in Section 2 also allows to compute the minimum rank of any undamped loop directed tree.

3 The minimum rank of any loop oriented tree

In this section, we define the notion of loop oriented tree and we remind the elimination process which allows to compute in linear time the minimum rank of any loop oriented tree.

Definition 3.

- *A* ***loop oriented graph*** $G = (V(G), E(G))$ *is a directed graph with no antiparallel edges, that is for any* $i \neq j$*, if* $(i, j) \in E(G)$ *then* $(j, i) \notin E(G)$*. In such a graph, loops are permitted.*
- *If* G *is a directed graph, its* ***associated undirected graph*** $\hat{G}$ *is the graph having the same vertex set, and* $\{i, j\}$ *is an edge in* $\hat{G}$ *when at least one of* $(i, j), (j, i)$ *is an egde in* G*.*
- *Any undirected graph which is connected and without cycle of length greater than one is called a* ***loop tree****.*
- *A* ***loop oriented tree*** *is a loop oriented graph such that its associated undirected graph is a loop tree.*

The elimination process computing the minimum rank of any loop oriented tree is based on Proposition 1 proved in [5].

A **zero-nonzero pattern** is a matrix with only zeros and stars. An element of a zero-nonzero pattern is said to be nonzero if it is a star entry.

A **realization** A of a zero-nonzero pattern P is a real matrix such that any entry a_{ij} is nonzero if and only if p_{ij} is a star entry.

The minimum rank of a zero-nonzero pattern is the minimum possible rank for a realization of this pattern.

Proposition 1. *Let* P *be a zero-nonzero pattern and* p_{st} *be a star entry of* P *such that either row* s *or column* t *or both have exactly one nonzero entry. Then,*

$$mr(P) = mr(P_0(s|t)) + 1,$$

where $P_0(s|t)$ *is the pattern obtained from* P *by setting row* s *and column* t *to zero.*

In [4], the following theorem has been proved.

Theorem 1. *Let* T *be a loop oriented tree and* P *be the zero-nonzero pattern associated with* T *(that is,* p_{ij} *is a star if and only if* (i, j) *is an edge in* T*). The minimum rank of* T *can be computed by applying repeatedly the elimination process stated in Proposition 1 to* P*.*

In the next section, we generalize the previous elimination process and we prove that the minimum rank of any simple oriented tree (defined in the next section) can be computed thanks to this generalized elimination process.

4 The minimum rank of any simple oriented tree

In this section, we present a generalization of the elimination process, presented in the previous section, in order to compute the minimum rank of any simple oriented tree.

Definition 4.

- *A* ***partial zero-nonzero pattern*** *P is a matrix whose each entry is either a star, or a question mark, or a zero.*

- *A* ***realization*** *A of a partial zero-nonzero pattern P is a real matrix such that for any entry (i, j), if p_{ij} is a star (resp. zero), then a_{ij} is nonzero (resp. zero).*

- *The* ***minimum rank of a partial zero-nonzero pattern*** *is the minimum possible rank for a realization of this pattern.*

An element of a partial zero-nonzero pattern is said to be nonzero if it is a star or a question mark.

Since the question mark entries can take any real value (zero or nonzero), the following proposition is straightforward.

Proposition 2. *Let P be a partial zero-nonzero pattern and p_{st} be a question mark entry of P such that row s (or column t) has exactly one nonzero entry. Then,*

$$mr(P) = mr(P(s|.)) \quad (or = mr(P(.|t))),$$

where $P(s|.)$ (resp. $P(.|t)$) is the pattern obtained from P by setting row s (or column t) to zero.

Notice that Proposition 1 still holds in the case of a partial zero-nonzero pattern.

The rest of this section is devoted to the proof that any simple oriented tree (defined below) can be computed in linear time thanks to Propositions 1 and 2. So, Proposition 1 combined with Proposition 2 provides a generalization of the elimination process, applicable to any partial zero-nonzero pattern.

Any simple directed graph G defines a partial zero-nonzero pattern whose any off-diagonal entry (i, j) is a star if and only if there is an edge from node i to node j in G and all the diagonal entries are question marks. The other entries are zeros.

The minimum rank of a simple directed graph is by definition the minimum rank of the partial zero-nonzero pattern it defines.

Definition 5. *A* ***simple oriented tree*** *is a simple oriented graph such that its associated undirected graph is a tree.*

The notions of root, leaf, relation father/child in a simple oriented tree are similar to these notions in an undirected tree, regardless of the direction on the edges.

Lemma 1. *Let T be a rooted simple oriented tree, let node i be a leaf of T and P be the partial zero-nonzero pattern associated with T. Then, either row i has exactly one nonzero entry, which is a question mark, and column i has exactly two nonzero entries: a star and a question mark ($\star$), or the same result holds by permuting row i and column i ($\triangle$).*

Proof. There are two possible situations:

- either leaf i has no out-neighbor. In this case, i is the out-neighbor of its father and situation ($\star$) occurs,

- or leaf i has its father as unique out-neighbor and situation ($\triangle$) holds.

□

Corollary 1. *Let T be a rooted simple oriented tree, let node i be a leaf of T and P be the partial zero-nonzero pattern of T. Then, row i and column i will be set to zero by the generalized elimination process.*

Thanks to the previous corollary, the proof of the following theorem is similar to the proof of Theorem 4.6 in [4].

Theorem 2. *Let T be a rooted simple oriented tree and P be the partial zero-nonzero pattern of T. Then, after applying repeatedly the generalized elimination process to P until no more change is possible, pattern P is zero.*

Corollary 2. *The minimum rank of any simple oriented tree can be computed in linear time thanks to the generalized elimination process.*

5 The minimum rank of any undamped loop directed tree

In this section, we define the undamped loop directed graphs and we show that the elimination process defined in Section 2 can be used to compute the minimum rank of any undamped loop directed tree.

Definition 6. *An* ***undamped loop directed graph*** *is a loop directed graph without loops.*

So, the zero-nonzero pattern associated with an undamped loop directed graph has only zeros along its diagonal.

The undamped loop directed graphs are of interest in the study of the controllability of a quantum system [7].

Below, we prove that the minimum rank of any undamped loop directed tree can be computed thanks to the elimination process, previously defined.

Definition 7. *A* ***loop directed tree*** *is a directed graph so that its associated undirected graph is a loop tree.*

Notice that a loop directed tree can have antiparallel edges, whereas a loop oriented tree cannot.

As in the previous section, the notions of root, leaf, relation father/child in a loop directed tree are similar to these notions in an undirected tree, regardless of the direction on the edges.

Lemma 2. *Let T be a rooted undamped loop directed tree, node i be a leaf of T and P be the zero-nonzero pattern associated with T. Then, row i and column i are either zero or have exactly one star entry.*

Proof. Denote f the father of node i. There are three possible situations:

- either node i is the out-neighbor of f and vice-versa. In this case, row i and column i have both exactly one star entry,
- or node i has no out-neighbor, but node i is the out-neighbor of its father f. So, row i is zero and column i has only one star,
- or node f is the out-neighbor of node i, whereas node i is not a out-neighbor. In that situation, row i has exactly one star entry and column i is zero.

□

Corollary 3. *Let T be an undamped loop directed tree, node i be a leaf of T and P be the zero-nonzero pattern associated with T. Then, row i and column i in P will be set to zero by the elimination.*

From the previous corollary, the following theorem is proved similarly to Theorem 4.6 in [4].

Theorem 3. *The minimum rank of any undamped loop directed tree can be computed in linear time thanks to the elimination process.*

As shown in [4], computing the minimum rank of a loop directed tree is equivalent to compute a maximum constrained matching in the bipartite graph associated with the loop directed tree. This maximum constrained matching is useful to study the strong structural controllability of the loop directed tree [2]. A maximum constrained matching in a bipartite graph associated to an undamped loop directed tree can thus be computed in linear time. The following results confirm this fact.

Theorem 4. *The bipartite graph associated with an undamped loop directed tree is a forest.*

The proof of this theorem is similar to the proof of Theorem 3.9 in [4].

It is well known [8] that a maximum constrained matching in a tree can be computed in linear time.

Corollary 4. *A maximum constrained matching in a bipartite graph associated with an undamped loop directed tree can be found in linear time.*

6 Conclusion

Currently, there are algorithms computing the minimum rank of any (loop or simple) symmetric directed tree [5]. In [4], it has been shown that the minimum rank of any loop oriented tree can be computed in linear time thanks to an elimination process. In this paper, we have generalized this process so that the minimum rank of any simple oriented tree can also be computed in linear time. Moreover, we have shown that the minimum rank of any undamped loop directed tree is also computable in linear time thanks to the elimination process. However, as stated in [9], an efficient algorithm computing the minimum rank of any (loop or simple) directed tree is still needed.

ACKNOWLEDGMENTS. The authors thank Airlie Chapman for the interesting discussions. They also acknowledge support from the Belgian Programme of Interuniversity Attraction Poles and an Action de Recherche Concertée (ARC) of the Wallonia-Brussels Federation.

References

[1] Liu, Slotine, Barabasi, *Controllability of complex networks*, Nature, vol. 473, no. 7346, pp. 167-173, May 2011.

[2] A. Chapman, M. Mesbahi, *Strong structural controllability of networked dynamics*, to appear in American Control Conference, 2013.

[3] D. Hershkowitz, H. Schneider, *Ranks of zero patterns and sign patters*, Linear Multilinear Algebra, 34(1993), pp. 3-19.

[4] M. Trefois, J.-C. Delvenne, *Zero forcing sets, constrained matchings and minimum rank* , submitted, 2013.

[5] F. Barioli, S. Fallat, D. Hershkowitz, H.T. Hall, L. Hogben, H. van der Holst, B. Shader, *On the minimum rank of not necessarily symmetric matrices: a preliminary study*, Electron. J. Linear Algebra, 18(2009), pp. 126-145.

[6] S. Fallat, L. Hogben, *The Minimum Rank of Symmetric Matrices Described by a Graph : A Survey*, Linear Algebra Appl., 426 (2007), pp. 558-582.

[7] C. Godsil, S. Severini, *Control by quantum dynamics on graphs*, Phys. Rev. A 81, 052316 (2010).

[8] M.C. Golumbic, T. Hirst, M. Lewenstein, *Uniquely Restricted Matchings*, Algorithmica, 31(2001), no. 2, pp. 139-154.

[9] L. Hogben, *Minimum rank problems*, Linear Algebra Appl., 432(8) (2010), pp. 1961-1974.