The Minimum Rank Problem of a Graph

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Outline

Some basic notions

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Definition

A graph G consists of :

- *a set $V(G)$: the vertices*
- *a set $E(G)$: the edges*

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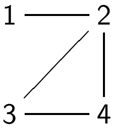
A representation of this graph is :



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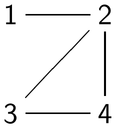

$$\Rightarrow \qquad A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

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The spectrum of a graph is the spectrum of its adjacency matrix.

Inverse eigenvalue problem of a graph :

We associate to a graph G on n vertices the matrix set :

$$\mathcal{Q}(G) = \{B \in \mathbb{R}^{n\times n} | B = B^T, \text{for } i \neq j, b_{ij} \neq 0 \Leftrightarrow \{i,j\} \in E(G)\}$$

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$\Rightarrow$ A first step : let $\mu \in \mathbb{R}$. What is the maximum possible multiplicity of eigenvalue μ for a matrix in $\mathcal{Q}(G)$?

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The *minimum rank* *of a graph G is :*

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Answer :

maximum possible multiplicity of eigenvalue $\mu = |G| - \text{min rank}(G)$,

where $|G|$ denotes the number of vertices in G.

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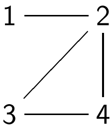
Definition

A hypergraph H consists of :

- *a set $V(H)$: the vertices*
- *a set $E(H)$: the hyperedges*

A hyperedge of size n is a set of n vertices.

Example



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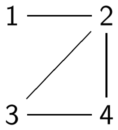

We associate to it the hypergraph H_G :
$V(H_G) = V(G)$ and $E(H_G) = \{\{1,2\},\{1,2,3,4\},\{2,3,4\}\}$.

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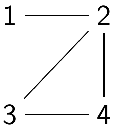

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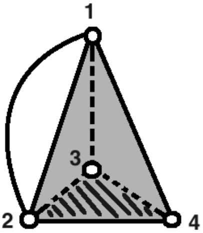

A color change rule on H_G :

- it is applied to each hyperedge of H_G until no more color change is possible.
- if a hyperedge of size n contains exactly $n-1$ black vertices, then its unique white vertex is colored in black.

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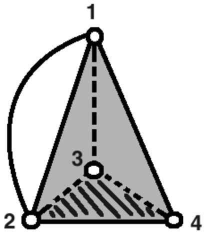

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The generating set number $Z(H_G)$ *is the minimum number of vertices we have to color in black so that after applying the color change rule to* H_G *no more vertex is white.*

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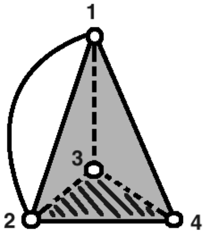

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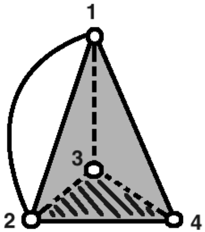

$$\Rightarrow Z(H_G) = 2.$$

$$\Rightarrow |H_G| - Z(H_G) = 4 - 2 = 2 \leq \text{min rank}(G).$$

The rank of the following matrix :

$$\begin{pmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

is 2.
As this matrix is in $\mathcal{Q}(G)$, we conclude :

$$\min \operatorname{rank}(G) = 2.$$

Conjecture : For any graph G,

$$\text{min rank}(G) = |H_G| - Z(H_G).$$

Thank you for your attention !