Minimum Rank and Zero Forcing Number of Directed Graphs

Maguy Trefois   Jean-Charles Delvenne

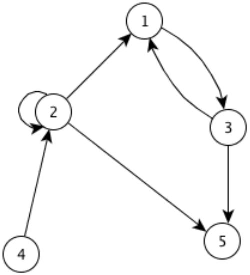
Large Graphs and Networks seminar

October 25, 2013

- The adjacency matrix is one of the most important tools in graph theory
- We are often interested in the rank of the adjacency matrix:
  - Open problem: characterizing the singular graphs
  - The nullity of a bipartite graph is of interest in chemistry
  - ...

In many applications, the rank of any real matrix described by the graph is of interest...

A real matrix described by the graph:



$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ -4 & \sqrt{2} & 0 & 0 & \pi \\ 56 & 0 & 0 & 0 & 2 \\ 0 & 9 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

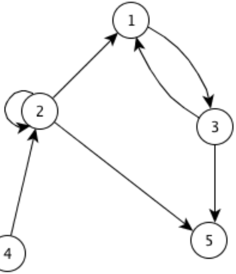
For any directed graph G,

$$\mathcal{Q}(G) := \{A \in \mathbb{R}^{|G|\times|G|} : a_{ij} \neq 0 \Leftrightarrow (i,j) \text{ is an edge in } G\}.$$

For any directed graph G,

$$\mathcal{Q}(G) := \{A \in \mathbb{R}^{|G|\times|G|} : a_{ij} \neq 0 \Leftrightarrow (i,j) \text{ is an edge in } G\}.$$

All the matrices in $\mathcal{Q}(G)$ have the same zero-nonzero pattern P:



$$P = \begin{pmatrix} 0 & 0 & \star & 0 & 0 \\ \star & \star & 0 & 0 & \star \\ \star & 0 & 0 & 0 & \star \\ 0 & \star & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

For any directed graph G,

$$\mathcal{Q}(G) := \{A \in \mathbb{R}^{|G|\times|G|} : a_{ij} \neq 0 \Leftrightarrow (i,j) \text{ is an edge in } G\}.$$

In many applications,

- **we have** a system described by a graph G but with unknown weights, namely the matrix A representing the system is in $\mathcal{Q}(G)$.
- **we need** information about the rank of A ...

Examples: chemical reaction networks

$$\begin{tikzcd} A+B \arrow[r, shift left, "\alpha"] & C \arrow[l, shift left, "\beta"] \arrow[dl, "\delta"] \\ D+E \arrow[u, "\gamma"] & \end{tikzcd}$$

The rate constants α, β, γ, δ are unknown.

For any directed graph G,

$$\mathcal{Q}(G) := \{A \in \mathbb{R}^{|G|\times|G|} : a_{ij} \neq 0 \Leftrightarrow (i,j) \text{ is an edge in } G\}.$$

The minimum rank of G is

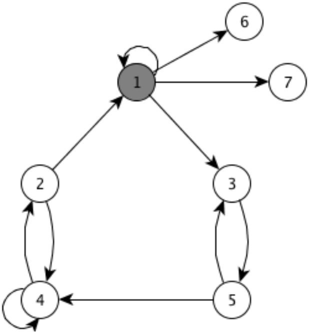
$$mr(G) = \min\{rank(A) : A \in \mathcal{Q}(G)\}.$$

Challenging problem: How compute $mr(G)$?

The zero forcing number of a directed graph G

A color change rule on G: suppose that any node of G is either black or white. If a node j is the only white out-neighbor of node i, then change the color of j to black.

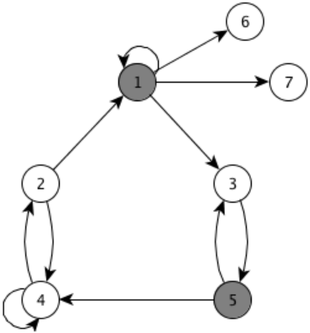
The color change rule is repeatedly applied to each node until no color change is possible.



The zero forcing number of a directed graph G

A color change rule on G: suppose that any node of G is either black or white. If a node j is the only white out-neighbor of node i, then change the color of j to black.

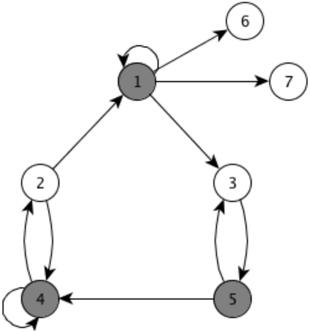
The color change rule is repeatedly applied to each node until no color change is possible.



The zero forcing number of a directed graph G

A color change rule on G: suppose that any node of G is either black or white. If a node j is the only white out-neighbor of node i, then change the color of j to black.

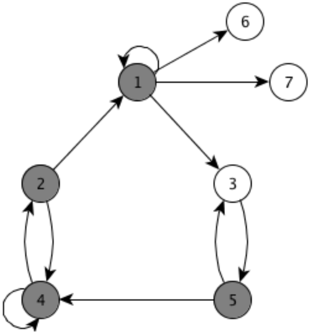
The color change rule is repeatedly applied to each node until no color change is possible.



The zero forcing number of a directed graph G

A color change rule on G: suppose that any node of G is either black or white. If a node j is the only white out-neighbor of node i, then change the color of j to black.

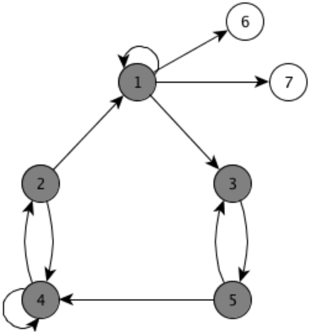
The color change rule is repeatedly applied to each node until no color change is possible.



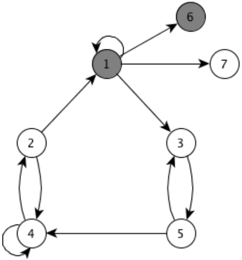
The zero forcing number of a directed graph G

A color change rule on G: suppose that any node of G is either black or white. If a node j is the only white out-neighbor of node i, then change the color of j to black.

The color change rule is repeatedly applied to each node until no color change is possible.



The zero forcing number $Z(G)$ of G is defined as the minimum number of nodes which have to be initially black so that after applying the color change rule all the nodes of G are black.



Its zero forcing number $Z(G)$ equals 2.

$\{1, 6\}$ is called a minimum zero forcing set.

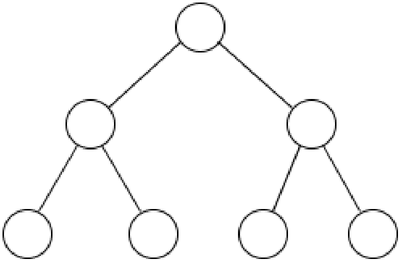
Theorem: For any directed graph G,

$$|G| - Z(G) \leq mr(G).$$

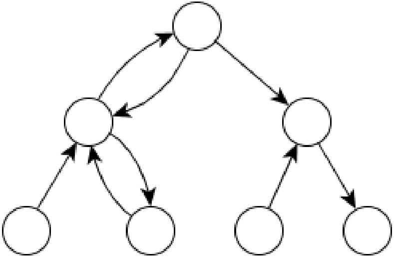
Theorem: The computation of the zero forcing number of any directed graph is NP-hard.

However, the zero forcing number is mostly interesting in the case of **directed trees**.

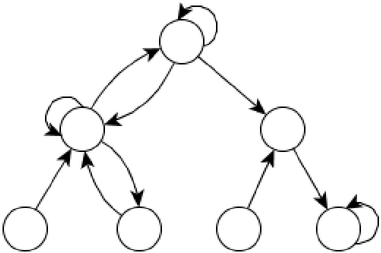
A tree is a connected undirected graph without cycle.



Put (a) direction(s) on the edges, you obtain a directed tree:



Put (a) direction(s) on the edges, you obtain a directed tree.
Loops are allowed.



Theorem: For any directed graph G,

$$|G| - Z(G) \leq mr(G).$$

Theorem: For any directed graph G,

$$|G| - Z(G) \leq mr(G).$$

Theorem: If T is a directed tree,

$$|T| - Z(T) = mr(T).$$

Theorem: For any directed graph G,

$$|G| - Z(G) \leq mr(G).$$

Theorem: If T is a directed tree,

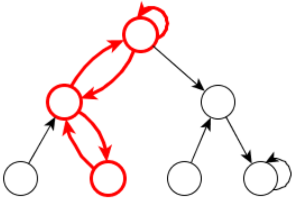
$$|T| - Z(T) = mr(T).$$

$\Rightarrow$ How compute $Z(T)$?

Let T be a directed tree.

A symmetric edge $\{i, j\}$:

Subgraph T_s induced by the symmetric edges of T:

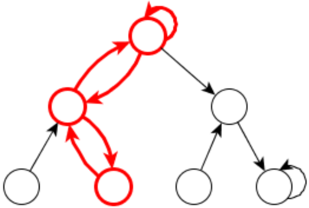


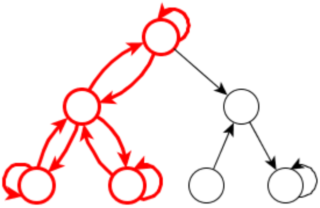
Definition: A directed tree T is said to be of class $C^\star$ if

$$\sharp \text{ symmetric edges } \leq \sharp \text{ loop-less nodes in } T_s.$$

A directed tree T is said to be of class $C^\star$ if

$$\sharp \text{ symmetric edges } \leq \sharp \text{ loop-less nodes in } T_s.$$





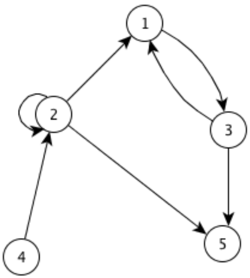
$2 \leq 2.$

$\Rightarrow$ of class $C^\star$

$3 > 1.$

$\Rightarrow$ not of class $C^\star$

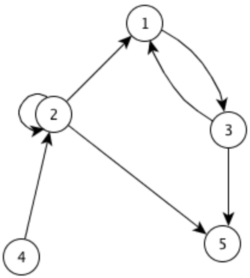
An elimination process:



$$P = \begin{pmatrix} 0 & 0 & \star & 0 & 0 \\ \star & \star & 0 & 0 & \star \\ \star & 0 & 0 & 0 & \star \\ 0 & \star & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$mr(G) = mr \begin{pmatrix} \color{red}{0} & \color{red}{0} & \color{red}{\star} & \color{red}{0} & \color{red}{0} \\ \star & \star & 0 & 0 & \star \\ \star & 0 & 0 & 0 & \star \\ 0 & \star & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

An elimination process:



$$P = \begin{pmatrix} 0 & 0 & \star & 0 & 0 \\ \star & \star & 0 & 0 & \star \\ \star & 0 & 0 & 0 & \star \\ 0 & \star & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$mr(G) = mr \begin{pmatrix} \color{red}{0} & \color{red}{0} & \color{red}{\star} & \color{red}{0} & \color{red}{0} \\ \star & \star & 0 & 0 & \star \\ \star & 0 & 0 & 0 & \star \\ 0 & \star & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 1 + mr \begin{pmatrix} \star & \star & 0 & \star \\ \star & 0 & 0 & \star \\ 0 & \star & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

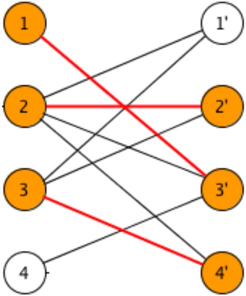
An elimination process:

$$mr(G) = mr\begin{pmatrix} 0 & 0 & \star & 0 & 0 \\ \star & \star & 0 & 0 & \star \\ \star & 0 & 0 & 0 & \star \\ 0 & \star & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 1 + mr\begin{pmatrix} \star & \star & 0 & \star \\ \star & 0 & 0 & \star \\ 0 & \star & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Theorem: The minimum rank of any directed tree of class $C^\star$ can be computed in linear time thanks to the elimination process.

Let $B = (V, V', E)$ be a bipartite graph.

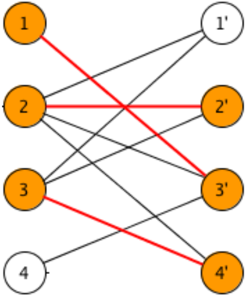
A *t*-matching is a set of t edges such that no two edges have a common node.



$\{1, 3'\}, \{2, 2'\}, \{3, 4'\}$ is a 3-matching.

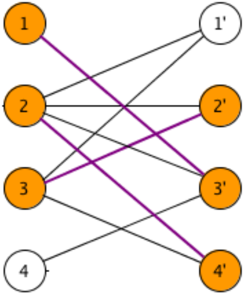
The nodes $1, 2, 3, 2', 3', 4'$ are called matched nodes, whereas $4, 1'$ are unmatched nodes.

A t-matching is a constraint t-matching if it is the only t-matching between the matched nodes.



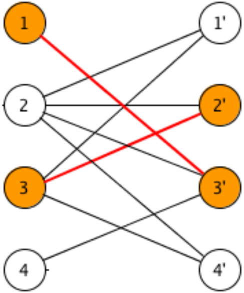
$\{1, 3'\}, \{2, 2'\}, \{3, 4'\}$ is a NOT a constraint matching.

A t-matching is a constraint t-matching if it is the only t-matching between the matched nodes.



$\{1, 3'\}, \{2, 2'\}, \{3, 4'\}$ is a NOT a constraint matching.

A t-matching is a constraint t-matching if it is the only t-matching between the matched nodes.



$\{1, 3'\}, \{3, 2'\}$ is a constraint matching.

A t-matching is a set of t edges such that no two edges have a common node.

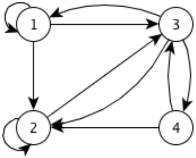
A t-matching is a constraint t-matching if it is the only t-matching between the matched nodes.

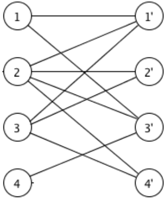
A (constraint) t-matching is maximum if there is no (constraint) s-matching with $s > t$.

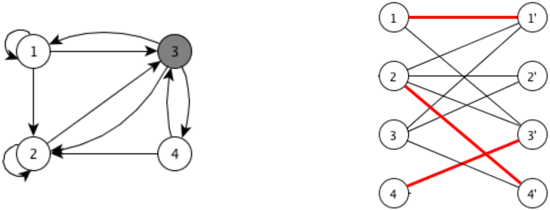
Directed graph and bipartite graph

Let G be a directed graph with nodes $1, \ldots, N$. The bipartite graph associated with G is $B_G = (V, V', E)$ with:

- $V = \{1, \ldots, N\}$ and $V' = \{1', \ldots, N'\}$
- $\{i, j'\} \in E$ if and only if (j, i) is an edge in G.







$\{3\}$ is a minimum zero forcing set of G with chronological list of forces:

$$4 \to 2, 1 \to 1, 3 \to 4$$

iff

$\{2, 4'\}, \{1, 1'\}, \{4, 3'\}$ is a maximum constraint matching in B_G.

We have presented:

- the notion of minimum rank of a directed graph
- a graph invariant: the zero forcing number
  - for any directed graph G, $|G| - Z(G) \leq mr(G)$
  - the computation of $Z(G)$ is NP-hard
  - for any directed tree T, $|T| - Z(T) = mr(T)$
  - the minimum rank of any directed tree of class $C^{\star}$ is computable in linear time
- the equivalence between the minimum zero forcing sets in a directed graph G and the maximum constraint matchings in the associated bipartite graph B_G

We have presented:

- the notion of minimum rank of a directed graph
- a graph invariant: the zero forcing number
  - for any directed graph G, $|G| - Z(G) \leq mr(G)$
  - the computation of $Z(G)$ is NP-hard
  - for any directed tree T, $|T| - Z(T) = mr(T)$
  - the minimum rank of any directed tree of class $C^\star$ is computable in linear time
- the equivalence between the minimum zero forcing sets in a directed graph G and the maximum constraint matchings in the associated bipartite graph B_G

Future work: finding an algorithm for the computation of the zero forcing number/minimum rank of any directed tree.