AN ALGORITHM FOR THE MINIMUM RANK OF A LOOP DIRECTED TREE

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EXTENDED ABSTRACT.

Introduction. The minimum rank of a graph is the minimum possible rank of a real matrix whose zero-nonzero pattern is described by the graph. The minimum rank problem consists in computing the minimum rank of a graph. The Inverse Eigenvalue Problem of a Graph (IEPG) has been an important motivation for this problem. In the IEPG, we consider a simple undirected graph G and we associate with it a set $\mathcal{Q}(G)$ of real symmetric matrices whose zero-nonzero pattern of the off-diagonal entries is described by the graph. Given a sequence $[\mu_1, ..., \mu_n]$ of non increasing real numbers, we would like to know if there exists a matrix in the set $\mathcal{Q}(G)$ whose spectrum is $[\mu_1, ..., \mu_n]$. A first step to study this very difficult problem is to compute the maximum possible multiplicity of an eigenvalue μ for a matrix in $\mathcal{Q}(G)$. This maximum is computed thanks to the minimum rank of the graph G. Indeed, for any real number μ, the maximum possible multiplicity for μ is :

$$|G| - mr(G),$$

where $|G|$ denotes the number of vertices in G (the vertex set of a graph is assumed to be finite) and $mr(G)$ refers to the minimum rank of the graph, that is :

$$mr(G) = \min\{rank(A) : A \in \mathcal{Q}(G)\}.$$

The minimum rank of a graph is also useful in many other problems like the singular graphs or the biclique partition of the edges of a graph (see [1] for a brief description of these problems and interesting references). Moreover, it has natural applications in determining the possible dynamics of a network of interacting systems, whose interaction topology is fixed but interaction strengths are unknown or to be chosen.

The current algorithms can compute efficiently the minimum rank of undirected trees. In this talk, we would like to present an algorithm which computes in polynomial time the minimum rank of a loop directed tree (this last notion is defined in next section).

The minimum rank of a loop directed tree. At first, let us define the notion of loop directed tree. A **directed acyclic graph** is a directed graph with no directed cycle, that is, it is impossible to start at some vertex i and follow a sequence of edges that loops back to i. If G is a directed graph, the associated undirected graph $\hat{G}$ is the graph having the same vertex set, and $\{i, j\}$ is an edge in $\hat{G}$ when at least one of

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(i,j), (j,i) is an edge in G. A **directed tree** is a directed acyclic graph such that its associated undirected graph is a tree. A **loop directed tree** is a directed graph such that if ignoring the loops, the graph is a directed tree.

A graph describes a matrix set whose definition depends on the type of the graph. The matrix set described by a loop directed tree G is :

$$\mathcal{Q}_{ld}(G) = \{A \in \mathbf{R}^{|G|\times|G|} : a_{ij} \neq 0 \Leftrightarrow (i,j) \text{ is an edge in } G\}.$$

The minimum rank of a graph is the minimum possible rank for a matrix in the matrix set described by the graph. So, the minimum rank of a loop directed tree G is :

$$mr(G) = \min\{rank(A) : A \in \mathcal{Q}_{ld}(G)\}.$$

Our algorithm for computing the minimum rank of a loop directed tree is based on the notion of hypergraph : a **hypergraph** H consists of a finite vertex set and a finite hyperedge set. A hyperedge of size s ($s \geq 1$) is a collection of s distinct vertices.

Minimum rank and hypergraph. Our algorithm relies on a color change rule defined on a hypergraph.

Let H be a black-white hypergraph, that is any vertex of H is either black or white. The **color change rule** is applied to a hyperedge e of H in the following way: suppose that the size of e is s. If exactly $s-1$ vertices of e are black, then the last one becomes black too.

Say by abuse of language that when the color change rule is applied to a black-white hypergraph, it is actually applied repeatedly to each hyperedge until no more color change is possible.

Thanks to this rule, we define the generating number of a black-white hypergraph H. This number will allow us to determine the minimum rank of a loop directed tree.

A **generating set** $\mathcal{Y}$ is a white vertex set of H such that if all the vertices of $\mathcal{Y}$ were colored in black in H, then after applying the color change rule to H, no more vertex of H would be white. The **generating number** of H, denoted $Y(H)$, is the minimum number of vertices in a generating set of H.

Let us see the link between the minimum rank of a loop directed tree and the generating number of a black-white hypergraph.

Let P be the zero-nonzero pattern of a matrix of order N. We associate with P the black-white hypergraph H_P defined in the following way : H_P contains N (white) vertices (numbered from 1 to N). Let n be the number of distinct nonzero rows in P. Then, H_P contains exactly n hyperedges : if the nonzero entries in a row lie in columns $i_1, ..., i_l$, then we create hyperedge $\{i_1, ..., i_l\}$.

In particular, we can associate a black-white hypergraph with a loop directed tree G: denote P_G the zero-nonzero pattern of the matrices in $\mathcal{Q}_{ld}(G)$. The hypergraph H_{P_G} is called the hypergraph associated with G.

Following proposition provides the link between the minimum rank of a loop directed tree and the generating number of its associated hypergraph.

PROPOSITION 0.1. *Let G be a loop directed tree. Then,*

$$mr(G) = |G| - Y(H_{P_G}),$$

where $|G|$ is the number of vertices in G.

The generating number of the associated hypergraph H_{P_G} is actually equal to the zero forcing number of the loop directed tree, defined in [2]. This zero forcing number can be computed by brute force computer programs. However, such programs are not efficient for trees of large size. The advantage to work from the hypergraph associated with the loop directed tree instead of the tree itself is the ease with which we can write an efficient algorithm for computing this number.

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