Strong Controllability of Networked Systems and Minimum Rank of a Graph: what can we compute ?

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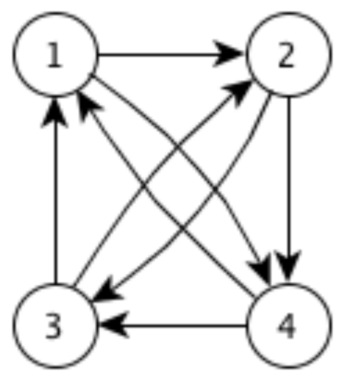
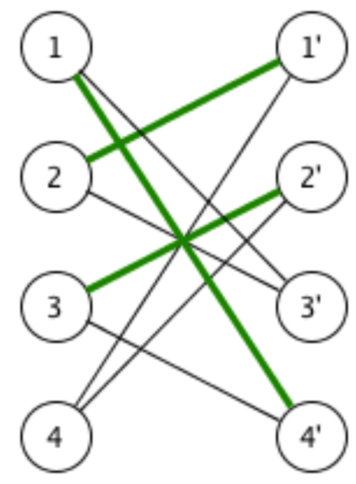
IAP VI/4 - DYSCO

Some equivalent objects

$$A = \begin{pmatrix} 0 & \star & 0 & \star \\ 0 & 0 & \star & \star \\ \star & \star & 0 & 0 \\ \star & 0 & \star & 0 \end{pmatrix}$$

G(A):





Strong Structural Controllability

Consider the system described by the equation:

$$\dot{x}(t) = Ax(t) + B(S)u(t)$$

where the matrix B(S) is such that only the nodes of G(A) which are in S are directly controlled by the outside controller.

Goal: Identify a minimum number n of nodes to control in order to have full control over the system.

Definition [1]: The system (A,B(S)) is strongly s-controllable if ALL realizations (A,B) are controllable.

Interest: Denote $m := \min\{|S| : (A, B(S)) \text{ is strongly s-controllable}\}$.
Then, $n \leq m$.

Question: How to find a set S with minimum size such that (A,B(S)) is strongly s-controllable ?

Tool: A maximum constraint matching in a bipartite graph

Notation: A_X is the matrix A with stars along the diagonal.

Theorem [1]: (A,B(S)) is strongly s-controllable iff the bipartite graphs associated to A(S|.) and A_X(S|.) have both a constraint matching of size 4-|S|.

Remark: This theorem holds only if A has a zero diagonal.

Problem: Computing a maximum constraint matching (and even its size) in a bipartite graph is NP-hard.

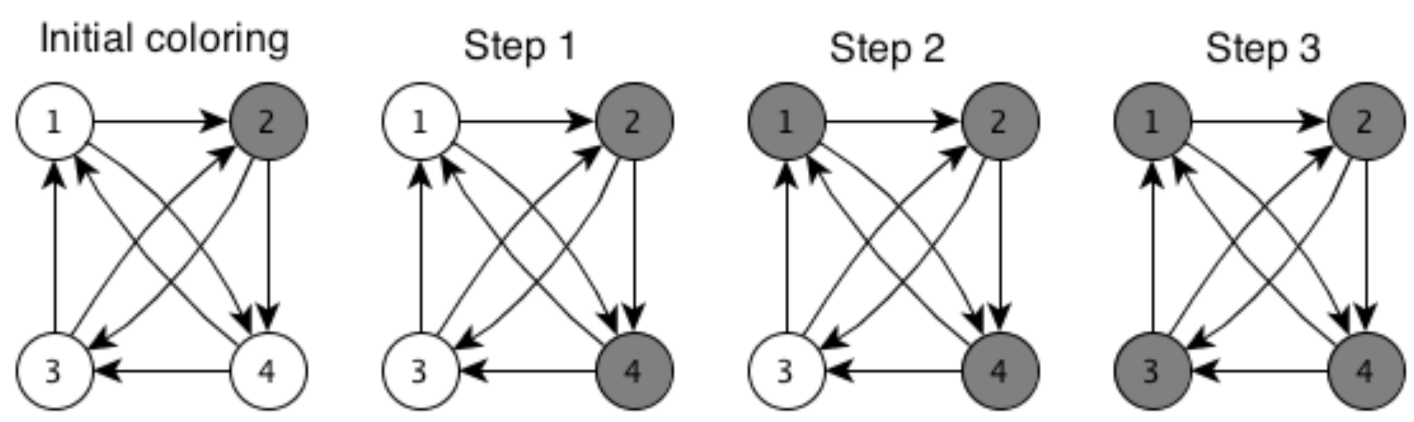
Minimum Rank of a Graph

Definition: The minimum rank of A is the minimum possible rank for a realization of A.

Question: How compute the minimum rank of A ?

Tool: The zero forcing number (ZFN) Z(G) of G(A)

A **color change rule** (CCR):



Definition [2]: The ZFN of a graph is the minimum number of nodes which have to be initially black so that after the CCR the whole graph is black.

Theorem [2]: If G is a loop directed tree, $|G| - Z(G) = mr(G)$

Problem: How compute Z(G) ?

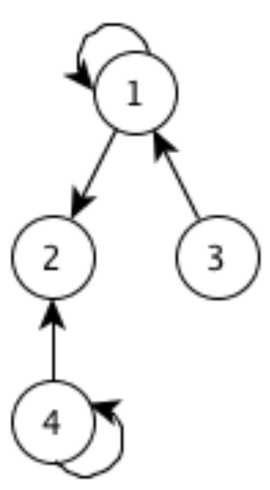
Theorem [3]: Computing Z(G) is equivalent to compute the size of a maximum constraint matching in the bipartite graph associated with G.

Question: Is it easy for trees ?

Minimum Rank of a Loop Oriented Tree

Example

T:



$$A(T) = \begin{pmatrix} \star & \star & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \star & 0 & 0 & 0 \\ 0 & \star & 0 & \star \end{pmatrix}$$

An **elimination process**:

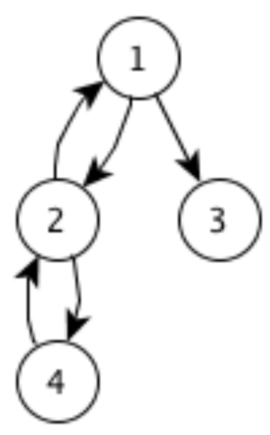
$$mr(T) = mr\begin{pmatrix} \star & \star & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \star & 0 & 0 & 0 \\ 0 & \star & 0 & \star \end{pmatrix} = 1 + mr\begin{pmatrix} \star & 0 & 0 \\ 0 & 0 & 0 \\ \star & 0 & \star \end{pmatrix} = 2 + mr\begin{pmatrix} \star & 0 \\ 0 & 0 \end{pmatrix} = 3 + mr\begin{pmatrix} 0 \end{pmatrix} = 3$$

Theorem [3]: The minimum rank of any loop oriented tree can be computed in linear time thanks to the elimination process.

Minimum Rank of an Undamped Loop Directed Tree

Example

T:



$$A(T) = \begin{pmatrix} 0 & \star & \star & 0 \\ \star & 0 & 0 & \star \\ 0 & 0 & 0 & 0 \\ 0 & \star & 0 & 0 \end{pmatrix}$$

An **elimination process**:

$$mr(T) = mr\begin{pmatrix} 0 & \star & \star & 0 \\ \star & 0 & 0 & \star \\ 0 & 0 & 0 & 0 \\ 0 & \star & 0 & 0 \end{pmatrix} = 1 + mr\begin{pmatrix} 0 & \star & \star \\ 0 & 0 & 0 \\ 0 & \star & 0 \end{pmatrix} = 2 + mr\begin{pmatrix} 0 & \star \\ 0 & 0 \end{pmatrix} = 3 + mr(0) = 3$$

Theorem: The minimum rank of any undamped loop directed tree can be computed in linear time thanks to the elimination process.

Conclusion

If A(S|.) and A_X(S|.) are (a disjoint union of) loop oriented trees, then the strong s-controllability of the system (A,B(S)) can be studied in linear time.

References

[1] A. Chapman, M. Mesbahi, **Strong structural controllability of networked dynamics**, in American Control Conference, 2013.

[2] L. Hogben, **Minimum rank problems**, Linear Algebra Appl., 432(8) (2010), pp. 1961-1974.

[3] M. Trefois, J.-C. Delvenne, **Zero forcing sets, constraint matchings and minimum rank**, submitted, 2013.