IAP VII/19 - DYSCO

Solving Laplacian Systems In Nearly-Linear Time

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Framework

G : - undirected graph
- positive weights along the edges
- n vertices
- m edges

A Laplacian System is a compatible linear system

$$Lv = b$$

where $L \in \mathbb{R}^{n \times n}$ is the Laplacian matrix of G .

Goal

In expected running time $\tilde{\mathcal{O}}(m \log^2 n)$, find $v_K \in \mathbb{R}^n$ such that given $\epsilon > 0$,

$$||v_K - v_{opt}||_L \leq \epsilon \cdot ||v_{opt}||_L$$

where $Lv_{opt} = b$.

Overview of the method

Factorize L as

$$L = B^T R^{-1} B$$

where - $B \in \mathbb{R}^{m \times n}$ is the incidence matrix of G
- $R \in \mathbb{R}^{m \times m}$ is a diagonal matrix with

$$R_{ii} = 1/(\text{weight of edge } e_i)$$

Therefore, the system of interest is

$$B^T R^{-1} B v = b$$

Pose $f = R^{-1} B v$.

Step 1: Find an ϵ - approximation of the unique solution $f_{opt} \in \mathbb{R}^m$ to the system

$$\begin{cases} B^T f = b \\ f^T R q = 0, \text{ for any } q \in \ker B^T, \end{cases}$$

i.e. find $f_K \in \mathbb{R}^m$ such that

$$||f_K - f_{opt}||_R \leq \epsilon \cdot ||f_{opt}||_R$$

Step 2: From $f_K \in \mathbb{R}^m$ and the relation $f_K = R^{-1} B v$, find $v_K \in \mathbb{R}^n$ such that

$$||v_K - v_{opt}||_L \leq \epsilon \cdot ||v_{opt}||_L$$

Details of Step 1

1. Find $q_1, q_2, \ldots, q_{m-n+1} \in \mathbb{R}^m$ a basis of $\ker B^T$

   Basis provided by a well-chosen spanning tree T of G
   (Well-chosen means: the condition number $\tau(T)$ of T is $\tilde{\mathcal{O}}(m \log n)$)

   Define $P(q_i) := \{f \in \mathbb{R}^m | f^T R q_i = 0\}$.

2. Use of an iterative method (the Kaczmarz method)

   (a) Find $f_0 \in \mathbb{R}^m$ nonzero only on the edges of T such that

   $$B^T f_0 = b$$

   (b) Do iteratively:

   - pick randomly a basis vector $q_i \in \mathbb{R}^m$
   - project current $f_k \in \mathbb{R}^m$ onto $P(q_i)$, i.e.

   $$f_{k+1} = f_k - \frac{\langle f_k, q_i \rangle_R}{\langle q_i, q_i \rangle_R} \cdot q_i \qquad (\star)$$

   Number of iterations: $\approx \left\lceil \tau(T) \log\left(\frac{\tau(T)}{\epsilon}\right) \right\rceil$

   Tricky point: perform any projection $(\star)$ in $\mathcal{O}(\log n)$ time.

   $\Rightarrow$ Key idea: Work with two particular bases of $\mathbb{R}^m$ simultaneously.

Details of Step 2

Given the spanning tree T, the incidence matrix $B \in \mathbb{R}^{m \times n}$ of G can be written as

$$B = \begin{bmatrix} U \\ C \end{bmatrix},$$

where $U \in \mathbb{R}^{(n-1) \times n}$ is the incidence matrix of T.

The relation $f_K = R^{-1} B v$ can be written as

$$\begin{bmatrix} f_K(T) \\ f_K(C) \end{bmatrix} = \begin{bmatrix} R_T & 0 \\ 0 & R_C \end{bmatrix}^{-1} \begin{bmatrix} U \\ C \end{bmatrix} v, \qquad (\triangle)$$

where $f_K(T) \in \mathbb{R}^{n-1}$ and $R_T \in \mathbb{R}^{(n-1) \times (n-1)}$ are the restrictions of f_K and R respectively to the edges of T.

We solve the equation $(\triangle)$ on the edges of T, i.e. we solve

$$f_K(T) = R_T^{-1} U v \qquad (\#)$$

We choose $v_K \in \mathbb{R}^n$ to be the solution of minimal euclidean norm to $(\#)$, i.e.

$$v_K = U^\dagger R_T f_K(T)$$

Such a vector $v_K \in \mathbb{R}^n$ can be easily computed in $\mathcal{O}(n)$ time.

References

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[2] T. Strohmer, R. Vershynin, *A randomized Kaczmarz algorithm with exponential convergence*, Journal of Fourier Analysis and Applications, Vol. 15, pp. 262-278, 2009.