Unicity of optimal topology of communication in the average consensus problem

M. Trefois J-C. Delvenne J-P. Tignol

Department of Applied Mathematics
Université catholique de Louvain (UCL)
Bâtiment Euler-4, Avenue G. Lemaître B-1348 Louvain-La-Neuve
Belgium
Email: maguy.trefois@uclouvain.be

1 Introduction

Let N communicating agents compute the average of their initial positions. Each agent communicates with a given small number of other agents (Average consensus problem). It has been proved that the de Bruijn's graph provides the optimal topology of communication. With this topology, the consensus can be reached in finitely many steps. We would like to prove that the de Bruijn's graph is the essentially unique communication graph resolving the consensus problem in finitely many steps.

2 Problem formulation

Let $\vec{x}(0) = (x_1(0), ..., x_N(0))^T \in \mathbb{R}^{N\times 1}$ be the vector of the initial positions (The agents positions are assumed to be scalar. The problem can easily be generalized for higher dimensions.)

The dynamics of the problem in discrete time is described in the following way :

$$\vec{x}(t+1) = A.\vec{x}(t),$$

where A is a squared matrix of size N. So, the problem consists in finding a matrix A such that

$$\lim_{k\to\infty} A^k.\vec{x}(0) = \vec{x_A},$$

where $\vec{x_A} = (\frac{1}{N}.\sum_{1\le i\le N} x_i(0), ..., \frac{1}{N}.\sum_{1\le i\le N} x_i(0))$ is the vector whose all entries are equal to the average of the initial positions.

Let $A \in \mathbb{R}^{N\times N}$. The communication graph $G(V,E)$ associated to A is a directed graph with N vertices and for all $1 \le i, j \le N$, $(i, j) \in E \Leftrightarrow a_{ij} \ne 0$.

We assume that each agent is allowed to communicate with at most v agents.

3 The de Bruijn's graph

Definition. *A $n-$dimensional de Bruijn's graph of m symbols is a directed graph $G(V,E)$ where the vertices set is $V = \{0, ..., m^n - 1\}$ and the edges set is $E = \{(i, ni), (i, ni+1), ..., (i, ni+(m-1))($ all modulo $m^n)|0 \le i \le m^n - 1\}$.*

Assume that $N = 8$. Let's consider the following matrix :

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

This matrix resolves the consensus problem. Moreover, the communication graph of A is the 3-dimensional de Bruijn's graph of 2 symbols.

4 Suggested approach

The only possible communication graph is essentially the de Bruijn's graph. We will show this problem may be related to the minimum rank problem, defined as follows : let $G(V,E)$ be a simple digraph, we associate to G the following set of matrices :

$$Q(G) = \{A \in \mathbb{R}^{|G|\times|G|} : \text{ for all } i, j, a_{ij} \ne 0 \leftrightarrow (i, j) \in E\}.$$

The minimum rank of G, denoted $mr(G)$, is defined in this way :

$$mr(G) = \min\{rank(A) : A \in Q(G)\}.$$

References

[1] J-C. Delvenne, C. Ruggero, S. Zampieri, "Optimal strategies in the average consensus problem", *Systems & Control Letters*, 58 (10-11), p.759, Oct.2009.

[2] S. Fallat, L. Hogben, "The minimum rank of symmetric matrices described by a graph : a survey", *Linear Algebra and Applications*, 426 : 558-582, 2007.

[3] L. Hogben, "Minimum rank problem", *Linear Algebra and its Applications*, 432(8), p.1961, Apr. 2010.