Structural controllability, minimum rank and constrained matchings

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1 Introduction

The structural controllability of networked systems and the minimum rank of a graph seem to be unrelated problems. In this presentation, we will show that these problems are actually strongly connected through the notion of constrained matching in a bipartite graph.

2 Structural controllability

Consider the dynamical system described by the equation:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$

where the $N \times N$ matrix A describes the interaction strengths between the components of the system.

It is well known that such a system is controllable if and only if its controllability matrix is full rank. However, in many real networks the interaction strengths between the components of the system (that is, the matrix A) are unknown. In such cases, the rank of the controllability matrix cannot be used to study the controllability of the system. That is why the structural controllability has been introduced. What we are really interested in is identifying a minimum node set to control (the driver nodes) in order to have full control over the system. In this presentation, we will present the strong structural controllability, briefly presented below.

The given networked system defines a pattern $\mathbf{A}$, that is a matrix whose each entry is either a zero or a star. A realization of $\mathbf{A}$ is a real matrix A such that by replacing each nonzero entry of A by a star we obtain the pattern $\mathbf{A}$.

Given an input node set S, the pattern $\mathbf{B}(S)$ is such that only the nodes in S are controlled by an outside controller.

A pair $(\mathbf{A}, \mathbf{B}(S))$ is **strongly structurally controllable** if all realizations (A,B) of $(\mathbf{A}, \mathbf{B}(S))$ are controllable.

An input node set S of minimum size such that the pair $(\mathbf{A}, \mathbf{B}(S))$ is strongly structurally controllable provides an upper bound on the number of driver nodes. We will show how to study the strong structural controllability through the notion of constrained matching.

3 Minimum rank of a graph

The rank of real matrices whose zero-nonzero pattern is described by a given graph is useful in many applications. A directed graph G allowing loops defines a matrix set as follows:

$$\mathscr{Q}(G) = \{A \in \mathbb{R}^{N\times N} : a_{ij} \neq 0 \text{ iff } (i,j) \text{ is an edge in } G\}.$$

The minimum rank of G is defined to be the minimum possible rank for a matrix in $\mathscr{Q}(G)$. In order to study the minimum rank of a graph many graph parameters have been introduced. In this talk, we will present one of them called the zero forcing number and we will show how this parameter is related to the constrained matchings.

References

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