# Community detection in complex networks

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# Outline

Network : definition

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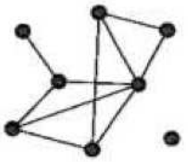
Detecting groups of vertices with the same behavior ...

A network (or graph) is a set of points joined by lines.

Example



A point is called a vertex.
A line is called an edge.

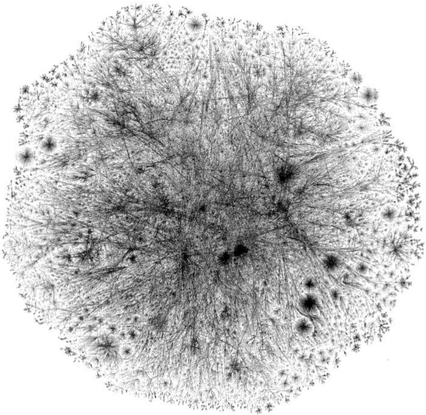
**Why networks ?**

Networks allow to model systems with interacting agents.

The structure of the network is fundamental for the understanding of the underlying system.

# Examples

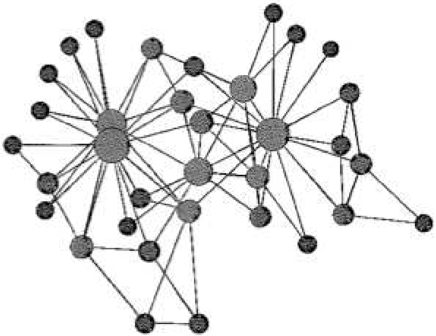
- *Technological networks*
  - *Internet*



*M.E.J. Newman, Networks : an introduction, Oxford University Press, Oxford UK, 2010, page 5.*

  - *Telephone networks*
  - *Transportation networks*

- Social networks
    - Friendship network between members of a club



M.E.J. Newman, *Networks : an introduction*, Oxford University Press, Oxford UK, 2010, page 6.

And so many others ...

**Community detection :**

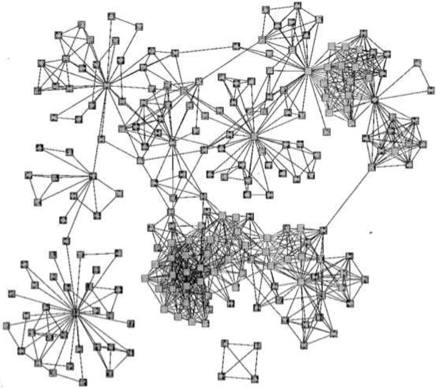
Partitioning the vertices of the network into groups, called communities, with many edges within the communities et few links between them.

**Utility :**

Revealing the structure and the organisation of the network.

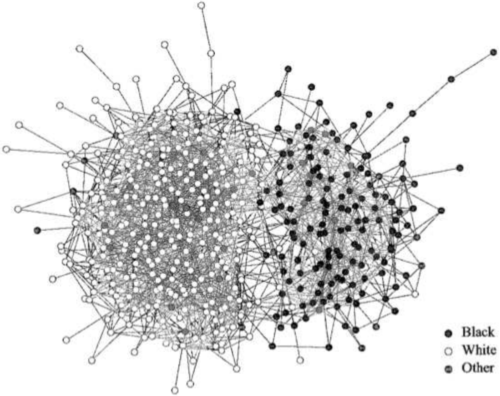
## Examples

- *Network of coauthorship in a university department*



*M.E.J. Newman, Networks : an introduction, Oxford University Press, Oxford UK, 2010, page 355.*

- Friendship network at a US high school




M.E.J. Newman, *Networks : an introduction*, Oxford University Press, Oxford UK, 2010, page 221.

Suppose that the network has $N$ vertices numbered from 1 to $N$.

The network can be mathematically represented thanks to its adjacency matrix $A \in \mathbb{R}^{N \times N}$ :

$$A_{ij} = \begin{cases} 1 & \text{if vertices } i \text{ and } j \text{ are connected} \\ 0 & \text{otherwise} \end{cases}$$

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The degree $k_i$ of vertex $i$ is the number of edges connected to it, that is :

$$k_i = \sum_{j=1}^{N} A_{ij}$$

The structure of the network can also be mathematically represented thanks to the Laplacian, strongly related to the adjacency matrix.

Denote $D$ the diagonal matrix with the degrees of the vertices on its diagonal, that is :

$$D = \begin{pmatrix} k_1 & 0 & 0 & \dots & 0 \\ 0 & k_2 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & k_N \end{pmatrix}$$

The Laplacian of the network is then the matrix : $L := D - A$, where $A$ is the adjacency matrix of the network.

**Goal** : partitioning the network into two communities $C_1$ and $C_2$.

The cut function counts the number of edges between communities $C_1$ and $C_2$, that is :

$$R = \sum_{i \in C_1, j \in C_2} A_{ij}.$$

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$\Rightarrow$ We have to find a partition which minimizes $R$.

**Spectral partitioning method of Fiedler :**

Define the following vector $s \in \mathbb{R}^{N \times 1}$ :

$$s_i = \begin{cases} +1 & \text{if vertex } i \in C_1 \\ -1 & \text{if vertex } i \in C_2 \end{cases}$$

Then,

$$R = \frac{1}{4} s^T L s,$$

where $L$ is the Laplacian of the network to partition.

Denote $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_N$ the spectrum of $L$.

Let $v_2$ be the eigenvector related to eigenvalue $\lambda_2$.

The vector $s$ given by Fiedler's method is :

$$s_i = \begin{cases} +1 & \text{if } [v_2]_i > 0 \\ -1 & \text{if } [v_2]_i < 0 \end{cases}$$

If one entry $i$ of $v_2$ is zero, then $s_i$ can take equivalently value $+1$ or -1.

## Configuration model :

In this model of random graph :

- the number $N$ of vertices is fixed
- the degree sequence $[k_1, ..., k_N]$ is given.

As a consequence, the number $m$ of edges is fixed. Indeed,

$$m = \frac{1}{2} \sum_{i=1}^{N} k_i.$$

Let us place randomly the edges in the graph.

The expected number of edges between vertices $i$ and $j$ is :

$$\frac{k_i k_j}{2m}.$$

The modularity function compares the fraction of edges between two vertices in a same communtity and the expected fraction of edges (given the degree sequence), that is :

$$Q = \frac{1}{2m} \sum_{i,j} \left( A_{ij} - \frac{k_i k_j}{2m} \right) \delta(c_i, c_j),$$

where :

- $m$ is the number of edges in the network
- $c_i$ is the community index of vertex $i$
- $\delta$ is the Kronecker symbol.

$\Rightarrow$ We have to find a partition which maximizes $Q$.

There exist several methods. The fastest one is the Louvain method.

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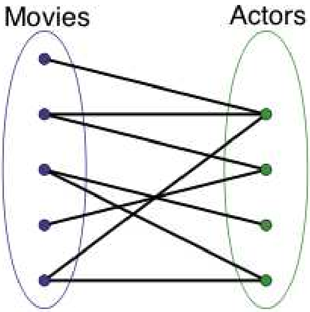
Modularity maximization

Detecting groups of vertices with the same behavior ...

We would like to detect groups of vertices with the same behavior or of the same type.

## Example

*A Movie-Actor network.*



**Actual work :**

1. How to define the behavior of a vertex ?
2. Developing algorithms to detect such groups of vertices.

Thank you for your attention !