Binary Factorizations of the Matrix of All Ones

Maguy Trefois Paul Van Dooren Jean-Charles Delvenne

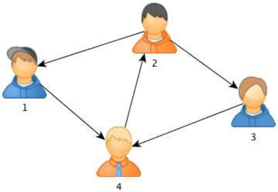
Université catholique de Louvain, Belgium

ILAS, June 2013

Motivation: the finite-time average consensus problem

We have:

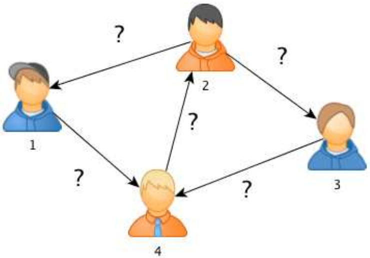
- n communicating agents with an initial position
- a communication topology



At each time step:

- each agent sends its current position to some other agents according to the communication pattern
- with the received information, each agent changes its position

The goal: after a finite time, all the agents meet at the average of their initial positions



Vector of initial positions:

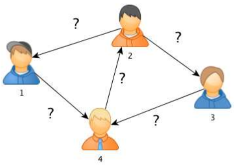
$$\vec{x}(0)$$

Dynamics:

$$\vec{x}(t+1) = A^{t+1}.\vec{x}(0)$$

The matrix A respects the communication topology:

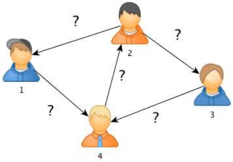
$$A = \begin{pmatrix} 0 & ? & 0 & 0 \\ 0 & 0 & 0 & ? \\ 0 & ? & 0 & 0 \\ ? & 0 & ? & 0 \end{pmatrix}$$



$$\vec{x}(t+1) = A^{t+1}.\vec{x}(0)$$

The matrix A is a solution to the consensus if:

- A is of the form $\begin{pmatrix} 0 & ? & 0 & 0 \\ 0 & 0 & 0 & ? \\ 0 & ? & 0 & 0 \\ ? & 0 & ? & 0 \end{pmatrix}$
- After a finite time m, $A^m = \frac{1}{4}.\begin{pmatrix} 1 & \dots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$



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Question: for which communication patterns is it possible to reach the consensus ?

We should study the solutions to the equation:

$$A^m = \frac{1}{n}.\begin{pmatrix} 1 & \dots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \dots & 1 \end{pmatrix},$$

where $A \in \mathbb{R}^{n \times n}$.

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$\Rightarrow$ difficult to tackle directly

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$\Rightarrow$ difficult to tackle directly

Simpler problem: study the solutions to:

$$A^m = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \dots & 1 \end{pmatrix},$$

where $A \in \{0, 1\}^{n \times n}$.

Outline

Factorization problem

The De Bruijn matrices

Factorizations into commuting factors

General form of a root of $\mathbb{I}_n$ with minimum rank

A root class of $\mathbb{I}_n$

Conclusion

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We are looking for the solutions to

$$\prod_{i=1}^{m} A_i = A_1 A_2 ... A_m = \mathbb{I}_n,$$

where

- $\mathbb{I}_n$ is the $n \times n$ matrix with all ones
- each factor A_i is an $n \times n$ binary matrix.

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where

- $\mathbb{I}_n$ is the $n \times n$ matrix with all ones
- each factor A_i is an $n \times n$ binary matrix.

In particular, we are investigating the solutions to:

$$A^m = \mathbb{I}_n,$$

where A is a binary matrix.

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Lemma

If $A \in \{0,1\}^{n\times n}$ is such that $A^m = \mathbb{I}_n$, then

- *A is p-regular, i.e $A.\mathbf{1} = p.\mathbf{1}$ and $\mathbf{1}^T.A = p.\mathbf{1}^T$*
- *$n = p^m$*

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- *A is p-regular, i.e $A.\boldsymbol{1} = p.\boldsymbol{1}$ and $\boldsymbol{1}^T.A = p.\boldsymbol{1}^T$*
- *$n = p^m$*

Definition

The De Bruijn matrix of order p and dimension n is a matrix of the form:

$$D(p,n) := \boldsymbol{1}_p \otimes I_{n/p} \otimes \boldsymbol{1}_p^T,$$

where

- *$I_{n/p}$ is the identity matrix of dimension n/p*
- *$\boldsymbol{1}_p$ is the $p \times 1$ vector with all ones*
- *$\otimes$ denotes the Kronecker product*

Moreover, it is imposed that $n = p^m$, for some integer m.

$$D(2,8) = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

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Proposition

The De Bruijn matrix $D(p,n)$ with $n = p^m$ is such that

$$D(p,n)^m = \mathbb{I}_n.$$

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Proposition

The De Bruijn matrix $D(p,n)$ with $n = p^m$ is such that

$$D(p,n)^m = \mathbb{I}_n.$$

Question: Can we characterize all the roots from the De Bruijn matrices ?

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Factorization into commuting factors:

Looking for the solutions to:

$$AB = BA = \mathbb{I}_n,$$

where

- A and B are binary matrices
- A is p-regular
- B is l-regular

Factorization problem:

$$AB = BA = \mathbb{I}_n,$$

where A is p-regular and B is l-regular.

Theorem

If A and B are commuting factors, then

- $p.l = n$
- $rank(A) \geq n/p$ *and* $rank(B) \geq n/l$
- *if* $rank(A) = n/p$ *(resp.* $rank(B) = n/l$*), then there exist permutation matrices* P_1, P_2 *such that*

$$P_1 A P_2^T = D(p, n) \qquad (resp.\ P_2 B P_1^T = D(l, n)).$$

Question: Is it possible that $rank(A) > n/p$?

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

- A and B are 2-regular
- $AB = BA = \mathbb{I}_4$
- BUT, $rank(A) = 3 > 4/2$

Question: Can we choose $P_1 = P_2$?

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

- A is 3-regular, B is 2-regular and $AB = BA = \mathbb{I}_6$
- $rank(A) = 6/3$, $rank(B) = 6/2$
- BUT, A is not isomorphic to $D(3,6)$ since

$$A^2 = \begin{pmatrix} 3 & 0 & 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 & 0 & 3 \\ 3 & 0 & 3 & 0 & 3 & 0 \\ 0 & 3 & 0 & 3 & 0 & 3 \end{pmatrix}, D(3,6)^2 = \begin{pmatrix} 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 2 & 2 & 2 & 1 & 1 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 \end{pmatrix}$$

Corollary

Let A be a binary matrix satisfying $A^m = \mathbb{I}_n$. Then,

- *A is p-regular*
- *if $rank(A) = n/p$, then there are permutation matrices P_1, P_2 such that*

$$P_1 A P_2^T = D(p,n).$$

As previously,

- A may have a rank greater than n/p
- A may not be isomorphic to $D(p,n)$ even though $rank(A) = n/p$

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Theorem

Let $A \in \{0,1\}^{n \times n}$ such that $A^m = \mathbb{I}_n$, A is p-regular and $p^m = n$. If $rank(A) = n/p$, then A is isomorphic to a matrix

$$P_1 D(p, n),$$

where $P_1 = diag(Q_1, ..., Q_p)$ with each $Q_i \in \{0,1\}^{n/p \times n/p}$ is a permutation matrix.

Theorem

Let $A \in \{0,1\}^{n\times n}$ *such that* $A^m = \mathbb{I}_n$*, A is p-regular and* $p^m = n$*.*
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where $P_1 = diag(Q_1, ..., Q_p)$ *with each* $Q_i \in \{0,1\}^{n/p\times n/p}$ *is a permutation matrix.*

Not all the matrices of that form are solutions. Indeed, consider

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

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A 2-circulant matrix:

$$\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$

Theorem (Wu, 2002)

Let $A \in \{0,1\}^{n\times n}$ be g-circulant and such that $A^m = \mathbb{I}_n$. If

- $g^m \equiv 0 \mod n$
- *A is p-regular,*

then A is isomorphic to $D(p,n)$.

Definition

A *nice permutation matrix* *is built as follows: start with a $p \times p$ permutation matrix. Then, replace all the zeros by a zero $p \times p$ matrix and each one by a $p \times p$ permutation matrix. Repeat this m times. You obtain a permutation matrix of dimension p^m.*

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Theorem

Any matrix of the form $P_1 D(p, n)$ $(n = p^m)$ with P_1 a nice permutation matrix is a m-th root of $\mathbb{I}_n$.

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Theorem

Any matrix of the form $P_1 D(p, n)$ $(n = p^m)$ with P_1 a nice permutation matrix is a m-th root of $\mathbb{I}_n$.

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Any nice permutation of the De Bruijn matrix $D(p, n)$ is isomorphic to $D(p, n)$.

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  - $rank(A) = n/p$ implies that A is essentially a De Bruijn matrix

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- Any nice permutation of the De Bruijn matrix is a root of $\mathbb{I}_n$ isomorphic to the De Bruijn matrix
- **Future work**: characterize all the roots of $\mathbb{I}_n$ with minimum rank.

Thank you for your attention !