Unicity of optimal topology of communication in the average consensus problem

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Plan

The average consensus problem

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The minimum rank problem

Suggested approach

Idea :

We have :

- N communicating agents
- each agent communicates with a given small number ν of agents

We would like :

Each agent moves to the average of their initial positions

Problem formulation :

- Vector of the initial positions :

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where $A \in \mathbb{R}^{N \times N}$.

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where $A \in \mathbb{R}^{N\times N}$.

$\Rightarrow$ Problem : To find a matrix $A \in \mathbb{R}^{N\times N}$ such that

$$A^k = \frac{1}{N}.\begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix},$$

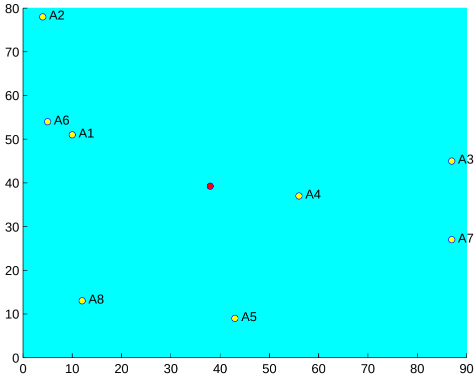
for a certain $k \in \mathbb{N}$.

Example : For $N = 8$ and $\nu = 2$, the matrix

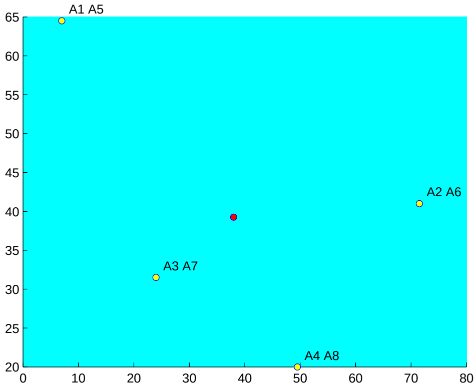
$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

is a solution.

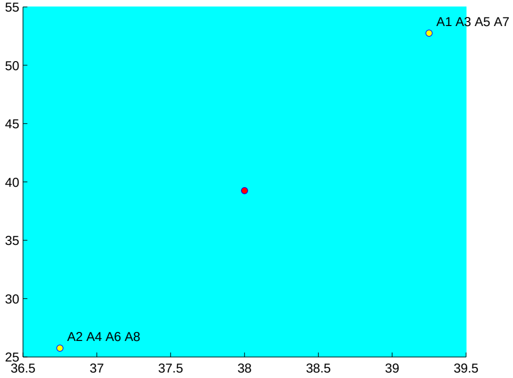
Initial positions



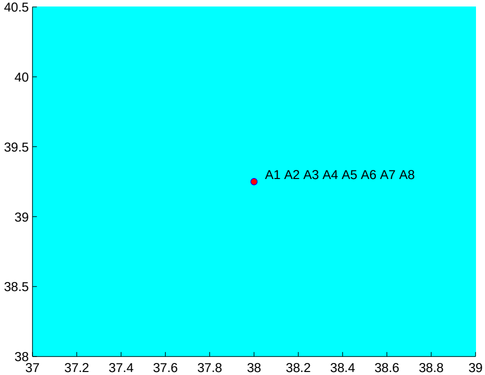
Positions at time 1



Positions at time 2



Positions at time 3



Definition

*Let $A \in \mathbb{R}^{N \times N}$. The graph $G(V, E)$ is the **communication graph** of A if :*

- $|V| = N$
- *(i, j) is an edge of G*
  iff
  the entry (i, j) of A is different from zero

Remark : The communication graph of the matrix A, in the previous example :

$$A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix},$$

is a de Bruijn's graph.

Conjecture : The only possible communication graph for the consensus is essentially a de Bruijn's graph.

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The minimum rank problem

Suggested approach

The minimum rank of a loop digraph :

Let G be a loop digraph. We define the matrices set $\mathcal{Q}(G)$ in this way :

A matrix $A \in \mathcal{Q}(G)$ is a real matrix such that the communication graph of A is G.

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A matrix $A \in \mathcal{Q}(G)$ is a real matrix such that the communication graph of A is G.

The **minimum rank** of G, denoted $\mathrm{mr}(G)$, is the minimum possible rank of a matrix in $\mathcal{Q}(G)$.

Example : consider the following loop digraph :

$$\begin{tikzcd} 1 \arrow[r] \arrow[d] & 2 \arrow[d] \arrow[dl] \\ 3 \arrow[r] & 4 \end{tikzcd}$$

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The matrices in $\mathcal{Q}(G)$ have the following pattern :

$$P = \begin{pmatrix} 0 & \star & \star & 0 \\ 0 & 0 & \star & \star \\ 0 & 0 & 0 & \star \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

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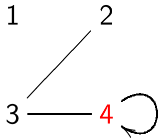
We define a hypergraph H in this way :

$$\begin{tikzcd} 1 & 2 \arrow[dl, dash] \\ 3 \arrow[r, dash] & 4 \arrow[loop right] \end{tikzcd}$$

As vertex 4 has a loop, we color it :



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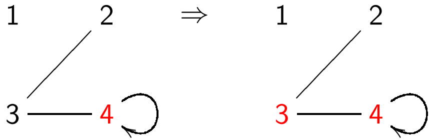

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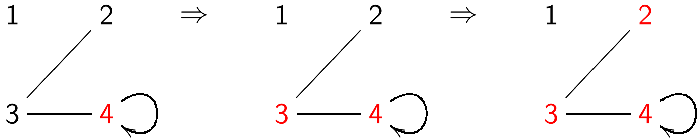

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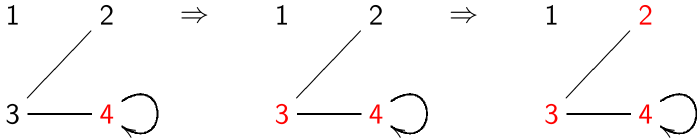

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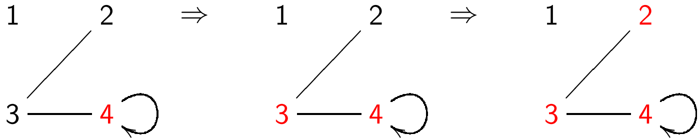

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$$Z(H_G) = 1.$$

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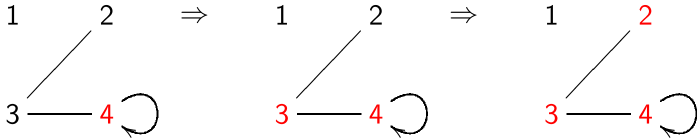

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And finally,

$$\mathrm{mr}(G) = 4 - 1 = 3.$$

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4. we compute the generating set number $Z(H_G)$: the minimum number of black vertices we have to color ourselves in order to deduce, thanks to the CCR, all the other black vertices of H_G
5. $\mathrm{mr}(G) = N - Z(H_G)$, where N is the order of G.

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If A resolves the consensus in k steps, then

$$A^k = \frac{1}{N} \cdot \begin{pmatrix} 1 & 1 & \dots & 1 \\ \vdots & \vdots & \dots & \vdots \\ 1 & 1 & \dots & 1 \end{pmatrix},$$

and trivially,

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$\Rightarrow$ We hope to use the minimum rank to deduce the solutions for the consensus.

Thank you for your attention !