Structural controllability, minimum rank and constrained matchings

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Goal of the presentation

- Strong Structural Controllability of Networked Systems
- Minimum Rank of a Loop Directed Graph

Same scope

Computing a maximum constrained matching in a bipartite graph
Outline

Constrained matchings in a bipartite graph

Structural controllability of networked systems

Minimum rank of a loop directed graph

Conclusion
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Conclusion
Let $B = (V, V', E)$ be a bipartite graph.

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\{1, 3'\}, \{2, 2'\}, \{3, 4'\} is a 3-matching.

The nodes 1, 2, 3, 2', 3', 4' are called matched nodes, whereas 4, 1' are unmatched nodes.
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$\{1, 3'\}, \{2, 2'\}, \{3, 4'\}$ is **not** a constrained matching.
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    \item \{1, 3\}', \{2, 2\}', \{3, 4\}' is \textbf{NOT} a constrained matching.
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$\{1, 3\}', \{3, 2\}'$ is a constrained matching.
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A *t*-matching is a **constrained *t*-matching** if it is the only *t*-matching between the matched nodes.

A (constrained) *t*-matching is **maximum** if there is no (constrained) *s*-matching with *s* > *t*. 
Directed graph and bipartite graph

Let $G$ be a directed graph with nodes $1, \ldots, N$. The bipartite graph associated with $G$ is $B_G = (V, V', E)$ with:

- $V = \{1, \ldots, N\}$ and $V' = \{1', \ldots, N'\}$
- $\{i, j'\} \in E$ if and only if $(j, i)$ is an edge in $G$. 
Directed graph and bipartite graph

Let \( G \) be a directed graph with nodes \( 1, ..., N \). The **bipartite graph associated with** \( G \) is \( B_G = (V, V', E) \) with:

- \( V = \{1, ..., N\} \) and \( V' = \{1', ..., N'\} \)
- \( \{i, j'\} \in E \) if and only if \((j, i)\) is an edge in \( G \).
More generally,

\[ A = \begin{pmatrix} 1 & 0 & 3 \\ 5 & 1 & 0 \\ 0 & 0 & 7 \\ 4 & 8 & 0 \end{pmatrix} \]

By abuse of language, a (constrained) \( t \)-matching of \( B_A \) will be called a (constrained) \( t \)-matching of \( A \).
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Conclusion
Consider the networked system described by the equation:

\[ \dot{x}(t) = Ax(t) + Bu(t), \]

where

- the vector \( x^T(t) = (x_1(t), \ldots, x_N(t)) \) captures the state of the system with \( N \) nodes at time \( t \)
- the \( N \times N \) matrix \( A \) describes the interaction strengths between the components of the system
- the \( N \times M \) matrix \( B \) identifies the nodes controlled by an outside controller
- the system is controlled by the vector \( u(t) \) imposed by the controller.
**Goal:** Identify the minimum number of nodes (the driver nodes) whose control is sufficient to control the whole system.
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**Theorem**

The system

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is **controllable** if and only if the controllability matrix

\[ C = (B, AB, A^2B, ..., A^{N-1}B) \]

is full rank.
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**Problem:** for most real networks, the interaction strengths (the matrix $A$) are unknown or approximately known.
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⇒ structural controllability
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- A real matrix $A$ is a realization of the pattern matrix $A$ if $A$ can be obtained by replacing all nonzero entries of $A$ by stars. In that case, we denote $A \in A$.

\[
A = \begin{pmatrix}
\star & 0 & \star \\
\star & \star & 0 \\
0 & 0 & \star \\
\star & \star & 0
\end{pmatrix}
\quad A = \begin{pmatrix}
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5 & 1 & 0 \\
0 & 0 & 7 \\
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\end{pmatrix}
\]
Given an input node set $S$, pattern $B(S)$ is such that only the nodes in $S$ are controlled by an outside controller.

**Example:** $N = 4, S = \{2, 4\}$,

$$B(S) = \begin{pmatrix} 0 & 0 \\ \star & 0 \\ 0 & 0 \\ 0 & \star \end{pmatrix}.$$
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Goal: given the pattern \(A\), finding a node subset \(S_1\) (resp. \(S_2\)) of minimum size such that the pair \((A, B(S_1))\) (resp. \((A, B(S_2))\)) is weakly (resp. strongly) structurally controllable.
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Given a networked system described by a matrix \(A \in \mathbb{A}\), \(S_1\) and \(S_2\) provide respectively a lower and an upper bound on the minimum number of driver nodes.
Notation
Let $A$ be a $N \times N$ pattern matrix and $S \subset \{1, \ldots, N\}$. Denote $A(S\mid\cdot)$ the pattern matrix obtained from $A$ by deleting the rows indexed by $S$.

Notation
Given a square pattern matrix $A$, denote $A_\times$ the matrix pattern obtained from $A$ by setting stars on its diagonal.

Definition
An undamped pattern is a pattern with only zeros along the diagonal.
Theorem (Chapman et al, 2012)

In the case of an undamped pattern $\mathbf{A}$, the pair $(\mathbf{A}, \mathcal{B}(S))$ is strongly $s$-controllable from a $m$-input set $S$ iff $\mathbf{A}(S|.)$ and $\mathbf{A}_\times(S|.)$ have both a constrained $(N - m)$-matching.
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**Problem:** computing a maximum constrained matching in a bipartite graph is NP-hard.
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Problem: computing a maximum constrained matching in a bipartite graph is NP-hard.

Challenge: approximate the size of $S_2$ providing the strong structural controllability.
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Conclusion
Let $G$ be a directed graph allowing loops with $N$ nodes.

The graph $G$ defines a family of real matrices:

$$Q(G) = \{ A \in \mathbb{R}^{N \times N} : a_{ij} \neq 0 \text{ iff } (i, j) \text{ is an edge in } G \}.$$  

The minimum rank of $G$ is the minimum possible rank for a matrix in $Q(G)$, that is:

$$mr(G) = \min \{ \text{rank}(A) : A \in Q(G) \}.$$
The zero forcing number of a loop directed graph

A color change rule on $G$: suppose that any node of $G$ is either black or white. If a node $j$ is the only white out-neighbor of node $i$, then change the color of $j$ to black.

The color change rule is repeatedly applied to each node until no color change is possible.
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Its zero forcing number $Z(G)$ equals 2.

Theorem
For any loop directed graph $G$,

$$|G| - Z(G) \leq mr(G).$$

In particular cases of loop directed graphs, equality holds.
Theorem

Computing the zero forcing number of any loop directed graph is equivalent to computing a maximum constrained matching in the associated bipartite graph.
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Wanted: a good approximation of the zero forcing number for any loop directed graph.
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- We have seen two applications of the constrained matchings in a bipartite graph:
  - the strong structural controllability of networked systems
  - the minimum rank of a loop directed graph
- computing a maximum constrained matching in a bipartite graph as well as its size is NP-hard
- a good approximation of the size of a maximum constrained matching?
Thank you for your attention!