Structural controllability, minimum rank and constrained matchings

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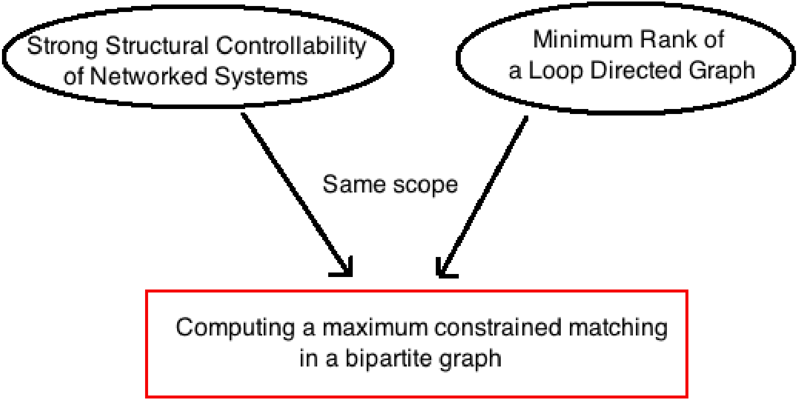
Goal of the presentation



Outline

Constrained matchings in a bipartite graph

Structural controllability of networked systems

Minimum rank of a loop directed graph

Conclusion

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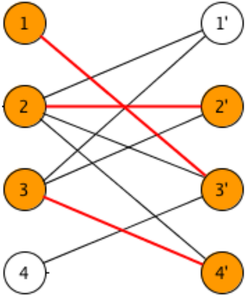

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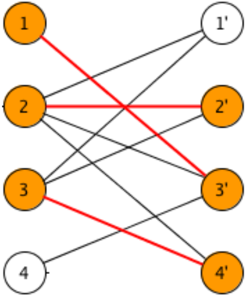
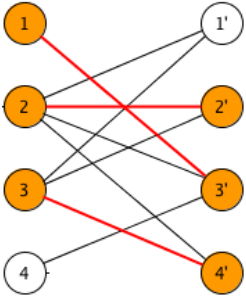


$\{1, 3'\}, \{2, 2'\}, \{3, 4'\}$ is a 3-matching.

The nodes $1, 2, 3, 2', 3', 4'$ are called matched nodes, whereas $4, 1'$ are unmatched nodes.

A t-matching is a constrained t-matching if it is the only t-matching between the matched nodes.



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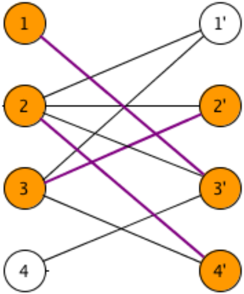

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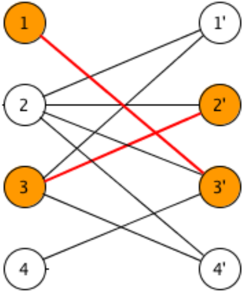

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A (constrained) t-matching is maximum if there is no (constrained) s-matching with $s > t$.

Directed graph and bipartite graph

Let G be a directed graph with nodes $1, ..., N$. The bipartite graph associated with G is $B_G = (V, V', E)$ with:

- $V = \{1, ..., N\}$ and $V' = \{1', ..., N'\}$
- $\{i, j'\} \in E$ if and only if (j, i) is an edge in G.

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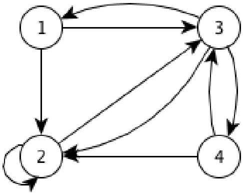
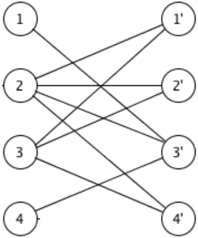
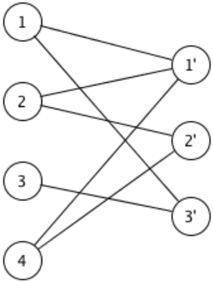




More generally,

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 5 & 1 & 0 \\ 0 & 0 & 7 \\ 4 & 8 & 0 \end{pmatrix}$$

B_A



By abuse of language, a (constrained) t-matching of B_A will be called a (constrained) t-matching of A.

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Consider the networked system described by the equation:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$

where

- the vector $\mathbf{x}^T(t) = (x_1(t), ..., x_N(t))$ captures the state of the system with N nodes at time t
- the $N \times N$ matrix A describes the interaction strengths between the components of the system
- the $N \times M$ matrix B identifies the nodes controlled by an outside controller
- the system is controlled by the vector $\mathbf{u}(t)$ imposed by the controller.

Goal: Identify the minimum number of nodes (the driver nodes) whose control is sufficient to control the whole system.

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is controllable if and only if the controllability matrix

$$C = (B, AB, A^2B, ..., A^{N-1}B)$$

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$\Rightarrow$ structural controllability

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- *A real matrix* A *is a realization of the pattern matrix* $\boldsymbol{A}$ *if* $\boldsymbol{A}$ *can be obtained by replacing all nonzero entries of* A *by stars. In that case, we denote* $A \in \boldsymbol{A}$.

$$\mathbf{A} = \begin{pmatrix} \star & 0 & \star \\ \star & \star & 0 \\ 0 & 0 & \star \\ \star & \star & 0 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 & 3 \\ 5 & 1 & 0 \\ 0 & 0 & 7 \\ 4 & 8 & 0 \end{pmatrix}$$

Given an input node set S, pattern $\mathbf{B}(S)$ is such that only the nodes in S are controlled by an outside controller.

Example: $N = 4, S = \{2, 4\}$,

$$\mathbf{B}(S) = \begin{pmatrix} 0 & 0 \\ \star & 0 \\ 0 & 0 \\ 0 & \star \end{pmatrix}.$$

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Goal: given the pattern $\mathbf{A}$, finding a node subset S_1 (resp. S_2) of minimum size such that the pair $(\mathbf{A}, \mathbf{B}(S_1))$ (resp. $(\mathbf{A}, \mathbf{B}(S_2))$) is weakly (resp. strongly) structurally controllable.

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Given a networked system described by a matrix $A \in \mathbf{A}$, S_1 and S_2 provide respectively a lower and an upper bound on the minimum number of driver nodes.

Notation

Let $\boldsymbol{A}$ be a $N \times N$ pattern matrix and $S \subset \{1, ..., N\}$.
Denote $\boldsymbol{A}(S|.)$ the pattern matrix obtained from $\boldsymbol{A}$ by deleting the rows indexed by S.

Notation

Given a square pattern matrix $\boldsymbol{A}$, denote $\boldsymbol{A}_\times$ the matrix pattern obtained from $\boldsymbol{A}$ by setting stars on its diagonal.

Definition

An undamped pattern is a pattern with only zeros along the diagonal.

Theorem (Chapman *et al*, 2012)

In the case of an undamped pattern $\boldsymbol{A}$*, the pair* $(\boldsymbol{A}, \boldsymbol{B}(S))$ *is strongly s-controllable from a m-input set* S *iff* $\boldsymbol{A}(S|.)$ *and* $\boldsymbol{A}_{\times}(S|.)$ *have both a constrained* $(N-m)$*-matching.*

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Challenge: approximate the size of S_2 providing the strong structural controllability.

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Conclusion

Let G be a directed graph allowing loops with N nodes.

The graph G defines a family of real matrices:

$$\mathcal{Q}(G) = \{A \in \mathbb{R}^{N \times N} : a_{ij} \neq 0 \text{ iff } (i, j) \text{ is an edge in } G\}.$$

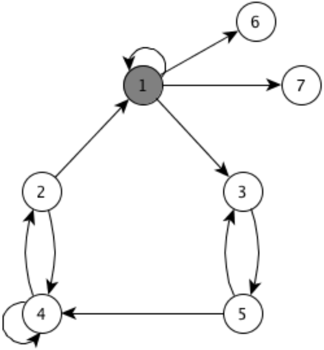
The minimum rank of G is the minimum possible rank for a matrix in $\mathcal{Q}(G)$, that is:

$$mr(G) = \min\{\text{rank}(A) : A \in \mathcal{Q}(G)\}.$$

The zero forcing number of a loop directed graph

A color change rule on G: suppose that any node of G is either black or white. If a node j is the only white out-neighbor of node i, then change the color of j to black.

The color change rule is repeatedly applied to each node until no color change is possible.



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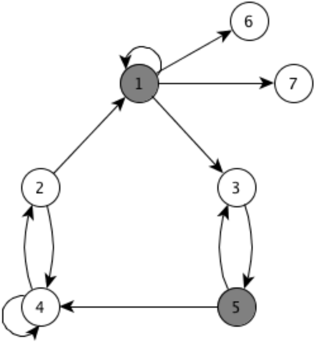

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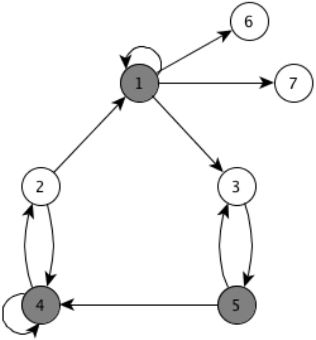

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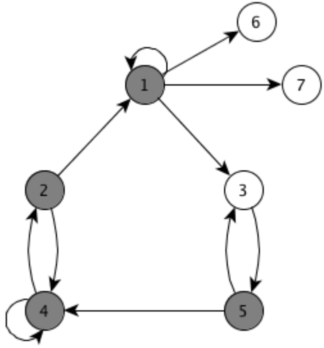

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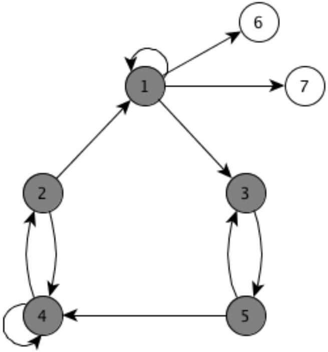

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The zero forcing number $Z(G)$ of G is defined to be the minimum number of nodes which have to be initially black so that after applying the color change rule all the nodes of G are black.

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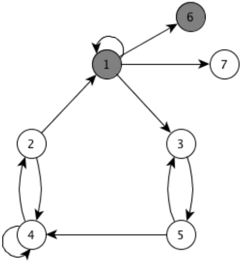

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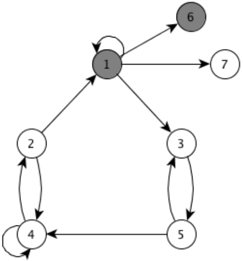

Its zero forcing number $Z(G)$ equals 2.

Theorem

For any loop directed graph G,

$$|G| - Z(G) \leq mr(G).$$

In particular cases of loop directed graphs, equality holds.

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Computing the zero forcing number of any loop directed graph is equivalent to computing a maximum constrained matching in the associated bipartite graph.

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Wanted: a good approximation of the zero forcing number for any loop directed graph

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- We have seen two applications of the constrained matchings in a bipartite graph:
  - the strong structural controllability of networked systems
  - the minimum rank of a loop directed graph
- computing a maximum constrained matching in a bipartite graph as well as its size is NP-hard
- a good approximation of the size of a maximum constrained matching ?

Thank you for your attention!