Structural Controllability of Networked Systems, Minimum Rank Problem of a Graph and Maximum Constrained Matchings

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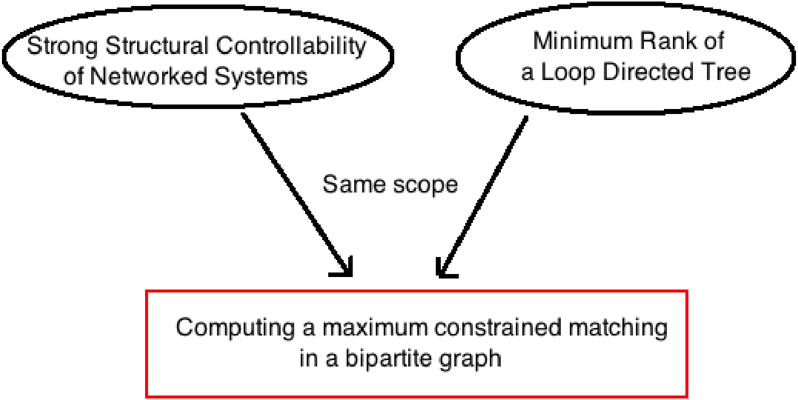
Goal of the presentation



Outline

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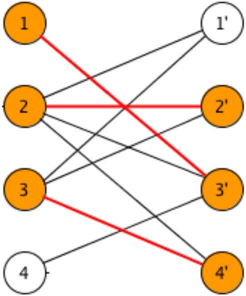

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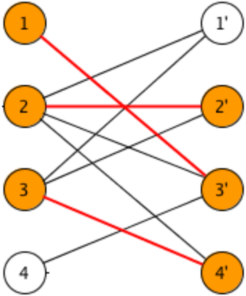

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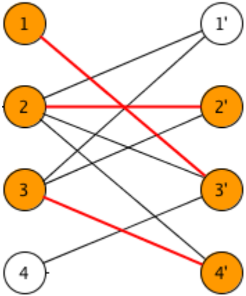
The nodes $1, 2, 3, 2', 3', 4'$ are called matched nodes, whereas $4, 1'$ are unmatched nodes.

A t-matching is a constrained t-matching if it is the only t-matching between the matched nodes.



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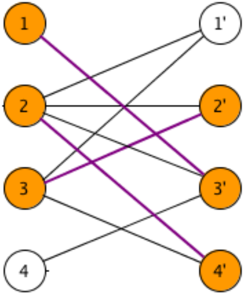

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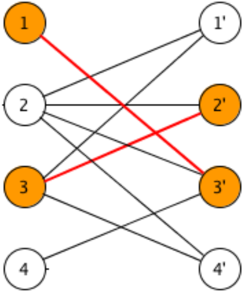

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A (constrained) t-matching is maximum if there is no (constrained) s-matching with $s > t$.

Directed graph and bipartite graph

Let G be a directed graph with nodes $1, ..., N$. The bipartite graph associated with G is $B_G = (V, V', E)$ with:

- $V = \{1, ..., N\}$ and $V' = \{1', ..., N'\}$
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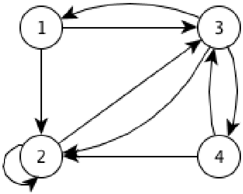
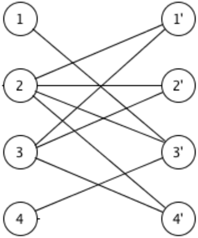
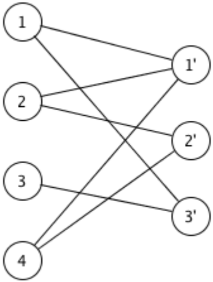




More generally,

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 5 & 1 & 0 \\ 0 & 0 & 7 \\ 4 & 8 & 0 \end{pmatrix}$$

B_A



By abuse of language, a (constrained) t-matching of B_A will be called a (constrained) t-matching of A.

Consider the networked system described by the equation:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t),$$

where

- the vector $\mathbf{x}^T(t) = (x_1(t), ..., x_N(t))$ captures the state of the system with N nodes at time t
- the $N \times N$ matrix A describes the interaction strengths between the components of the system
- the $N \times M$ matrix B identifies the nodes controlled by an outside controller
- the system is controlled by the vector $\mathbf{u}(t)$ imposed by the controller.

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$\Rightarrow$ structural controllability

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- *A real matrix A is a realization of the pattern matrix $\boldsymbol{A}$ if $\boldsymbol{A}$ can be obtained by replacing all nonzero entries of A by stars. In that case, we denote $A \in \boldsymbol{A}$.*

$$\mathbf{A} = \begin{pmatrix} \star & 0 & \star \\ \star & \star & 0 \\ 0 & 0 & \star \\ \star & \star & 0 \end{pmatrix} \qquad A = \begin{pmatrix} 1 & 0 & 3 \\ 5 & 1 & 0 \\ 0 & 0 & 7 \\ 4 & 8 & 0 \end{pmatrix}$$

Given an input node set $S = \{i_1, ..., i_m\}$ ($m \leq N$), pattern $\mathsf{B}(S)$ is such that only the nodes in S are controlled by an outside controller.

Example: $N = 4, S = \{2, 4\}$,

$$\mathsf{B}(S) = \begin{pmatrix} 0 & 0 \\ \star & 0 \\ 0 & 0 \\ 0 & \star \end{pmatrix}.$$

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Goal: given the pattern $\mathbf{A}$, finding a node subset S_1 (resp. S_2) of minimum size such that the pair $(\mathbf{A}, \mathbf{B}(S_1))$ (resp. $(\mathbf{A}, \mathbf{B}(S_2))$) is weakly (resp. strongly) structurally controllable.

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Given a networked system described by a matrix $A \in \mathbf{A}$, S_1 and S_2 provide respectively a lower and an upper bound on the minimum number of driver nodes.

Notation

Let $\boldsymbol{A}$ *be a* $N \times N$ *pattern matrix and* $S \subset \{1, ..., N\}$*.*
Denote $\boldsymbol{A}(S|.)$ *the pattern matrix obtained from* $\boldsymbol{A}$ *by deleting the rows indexed by* S*.*

Theorem (Liu *et al*, 2011)

The pair $(\boldsymbol{A}, \boldsymbol{B}(S))$ *is weakly s-controllable from a m-input set* S *iff* $\boldsymbol{A}(S|.)$ *has a* $(N - m)$*-matching.*

A maximum matching in a bipartite graph can be found in $\mathcal{O}(\sqrt{N}|E|)$ time.

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In the case of an undamped pattern $\boldsymbol{A}$, the pair $(\boldsymbol{A}, \boldsymbol{B}(S))$ is strongly s-controllable from a m-input set S iff $\boldsymbol{A}(S|.)$ and $\boldsymbol{A}_{\times}(S|.)$ have both a constrained $(N-m)$-matching.

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Challenge: approximate the size of S_2 providing the strong structural controllability.

Motivation

- The adjacency matrix is the most important tool in graph theory
- We are often interested in the rank of the adjacency matrix:
    - Open problem: to characterize the graphs whose adjacency matrix is singular
    - The nullity of a bipartite graph is of interest in chemistry
    - Progress in characterizing the nullity of a general graph is still needed

The Inverse Eigenvalue Problem of a Graph

Consider a simple undirected graph G and we define the matrix set:

$$\mathcal{Q}(G) = \{A \in \mathbb{R}^{N \times N} : A = A^T, \text{for any } i \neq j, a_{ij} \neq 0 \text{ iff } \{i, j\} \in E\}.$$

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First step: The maximum possible multiplicity of a number μ as an eigenvalue of a matrix in $\mathcal{Q}(G)$ is:

$$|G| - mr(G),$$

where $mr(G)$ denotes the minimum rank of G.

Let G be a directed graph allowing loops with N nodes.

The graph G defines a family of real matrices:

$$\mathcal{Q}_d(G) = \{A \in \mathbb{R}^{N \times N} : a_{ij} \neq 0 \text{ iff } (i,j) \text{ is an edge in } G\}.$$

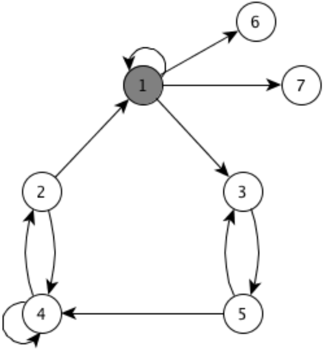
The minimum rank of G is the minimum possible rank for a matrix in $\mathcal{Q}_d(G)$, that is:

$$mr(G) = \min\{\text{rank}(A) : A \in \mathcal{Q}_d(G)\}.$$

The zero forcing number of a loop directed graph

A color change rule on G: suppose that any node of G is either black or white. If a node j is the only white out-neighbor of node i, then change the color of j to black.

The color change rule is repeatedly applied to each node until no color change is possible.



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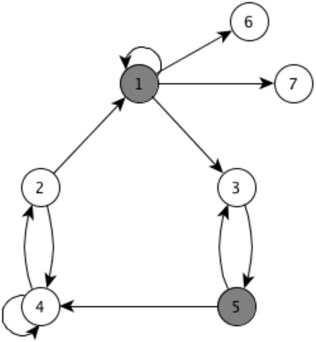

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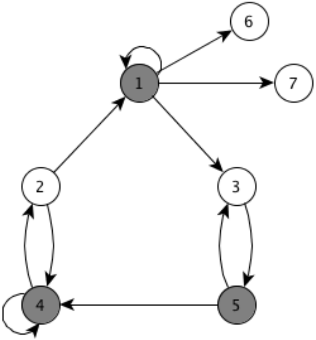

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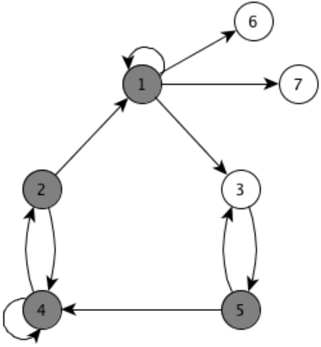

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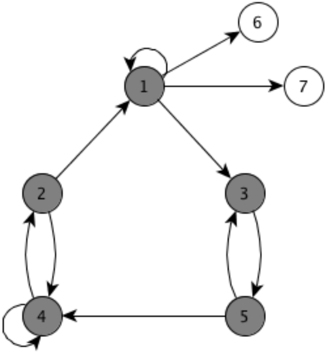

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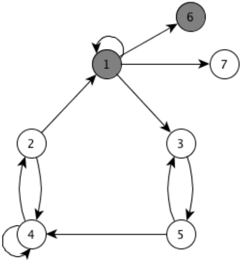

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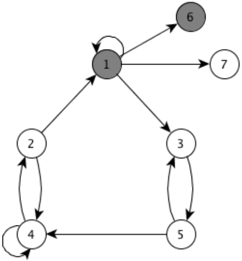

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Theorem

For any loop directed graph G,

$$|G| - Z(G) \leq mr(G).$$

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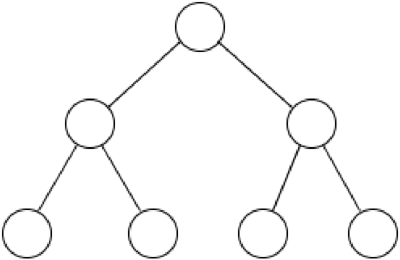

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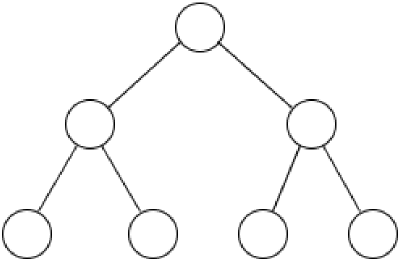
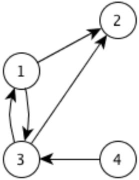


What is a directed tree ?


1
2
3
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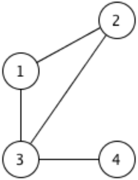
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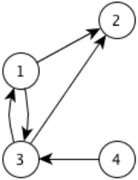

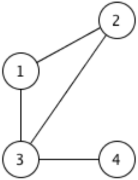

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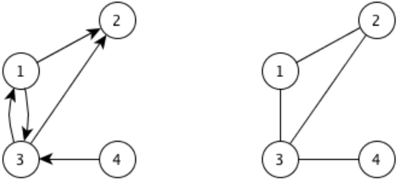

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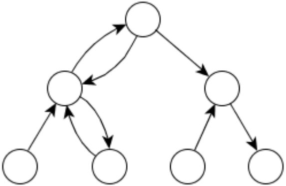

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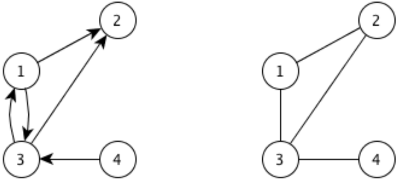

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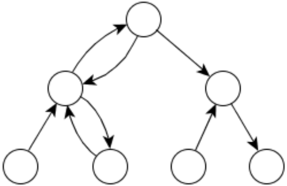

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A loop directed tree is a directed tree allowing loops.

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Theorem

If T is a loop directed tree,

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$\Rightarrow$ How compute $Z(T), Z(G)$?

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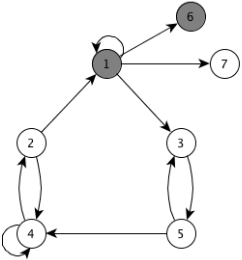

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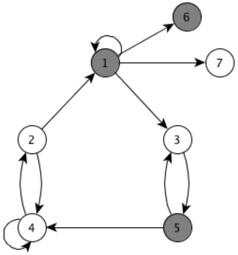

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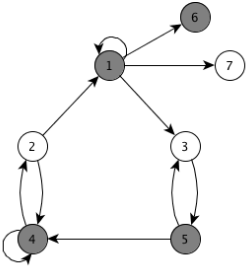

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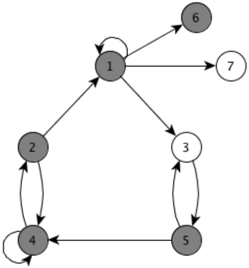
- *Given a minimum zero forcing set of G, we can list the forces in order in which they were performed to color the vertices of G in black. This list is called a chronological list of forces.*



A min zero forcing set: $\{1, 6\}$.

A chronological list:

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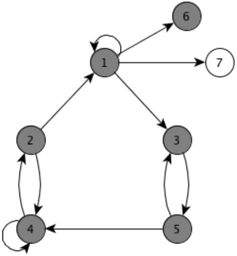

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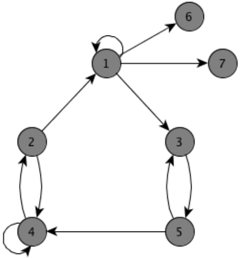

A min zero forcing set: $\{1, 6\}$.

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$$5 \to 3, 1 \to 7$$

Theorem

Let G be a loop directed graph and B_G the bipartite graph associated with G. Then, a node subset of G is a minimum zero forcing set of G with a chronological list of forces

$$j_1 \to i_1, j_2 \to i_2, ..., j_t \to i_t$$

if and only if

$$\{i_1, j_1'\}, \{i_2, j_2'\}, ..., \{i_t, j_t'\}$$

is a maximum constrained matching in B_G.

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Corollary

Computing the zero forcing number of any loop directed graph is NP-hard.

Definition

A loop oriented tree is a loop directed tree with no antiparallel edges:

$$\textit{for any } i \neq j, \ \textit{if } (i,j) \in E, \ \textit{then } (j,i) \notin E$$

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Let $\boldsymbol{A}$ be a pattern matrix having row s (or column t) that has exactly one star entry a_{st}. Then,

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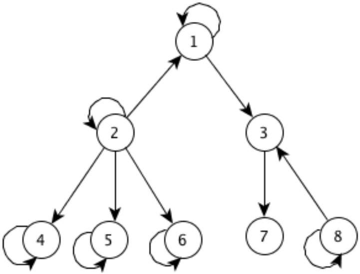
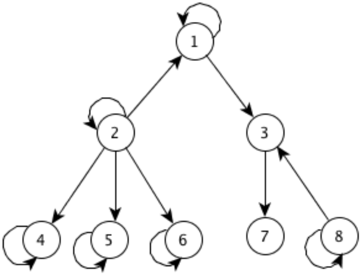
Theorem

The minimum rank of any loop oriented tree can be computed in linear time thanks to the elimination process.



$$mr(T) = mr\begin{pmatrix} \star & 0 & \star & 0 & 0 & 0 & 0 & 0 \\ \star & \star & 0 & \star & \star & \star & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \star & 0 \\ 0 & 0 & 0 & \star & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \star & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \star & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \star & 0 & 0 & 0 & 0 & \star \end{pmatrix}$$



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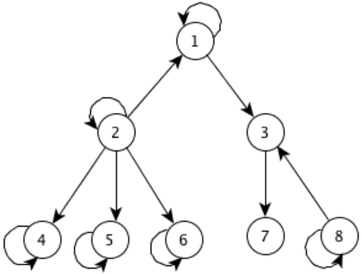
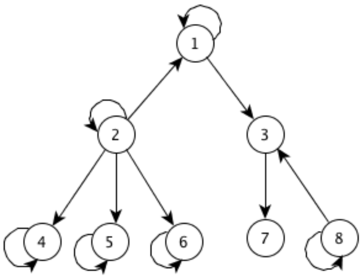


$$mr(T) = 1 + mr\begin{pmatrix} \star & 0 & \star & 0 & 0 & 0 & 0 \\ \star & \star & 0 & \star & \star & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \star & 0 \\ 0 & 0 & 0 & \star & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \star & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \star & 0 & 0 & 0 & \star \end{pmatrix}$$



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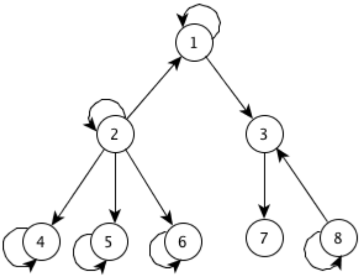
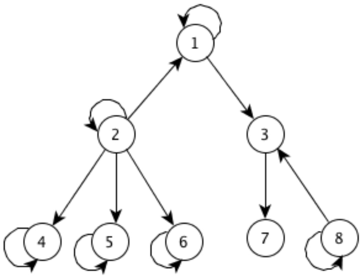


$$mr(T) = 2 + mr \begin{pmatrix} \star & 0 & \star & 0 & 0 & 0 \\ \star & \star & 0 & \star & \star & 0 \\ 0 & 0 & 0 & \star & 0 & 0 \\ 0 & 0 & 0 & 0 & \star & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \star & 0 & 0 & \star \end{pmatrix}$$



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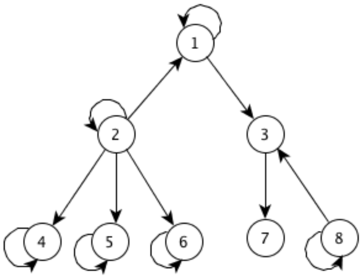

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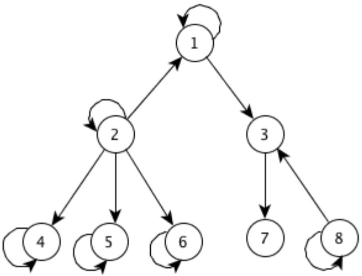

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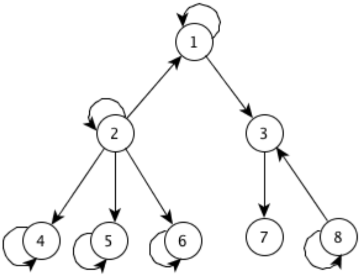
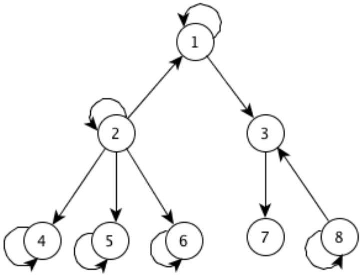
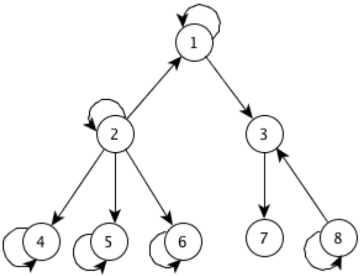


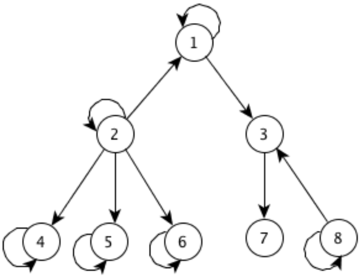
$$mr(T) = 4 + mr\begin{pmatrix} \star & 0 & \star & 0 \\ \star & \star & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \star & \star \end{pmatrix}$$



$$mr(T) = 4 + mr\begin{pmatrix} \star & 0 & \star & 0 \\ \star & \star & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \star & \star \end{pmatrix}$$



$$mr(T) = 5 + mr\begin{pmatrix} \star & \star & 0 \\ 0 & 0 & 0 \\ 0 & \star & \star \end{pmatrix}$$



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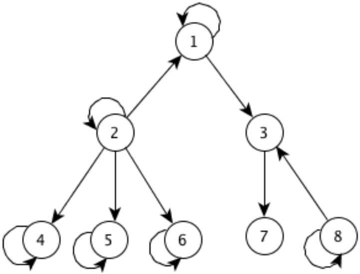
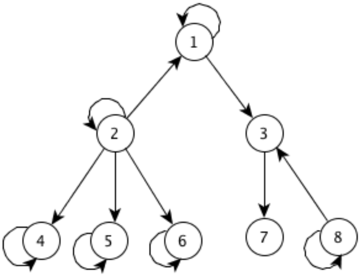


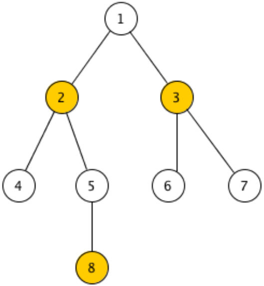
$$mr(T) = 6 + mr\begin{pmatrix} 0 & 0 \\ \star & \star \end{pmatrix}$$

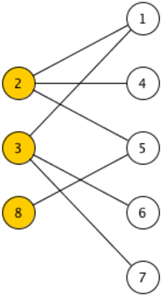


$$mr(T) = 7$$

A rooted (undirected) tree T_u is a bipartite graph (V_e, V_o, E) where:

- V_e is the node subset of T_u with an even height
- V_o is the node subset of T_u with an odd height
- E is the edge set of T_u.





Theorem

Let T_u be a rooted tree. Then, there is a loop oriented tree T such that the bipartite graph B_T associated with T is T_u with eventual additional isolated nodes in B_T.

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A maximum matching in an (undirected) tree can be computed in linear time.

Constrained matchings in a bipartite graph

Structural controllability of networked systems

Minimum rank of a loop directed graph

Conclusion and references

Conclusion

- We have seen two applications of the constrained matchings in a bipartite graph:
    - the strong structural controllability of networked systems
    - the minimum rank of a loop directed tree
- computing a maximum constrained matching in a bipartite graph as well as its size is NP-hard
- good approximation of the size of a maximum constrained matching ?
- what about the case where the bipartite graph is defined from a loop directed tree ?

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