An Algorithm for the Minimum Rank of a Loop Directed Tree

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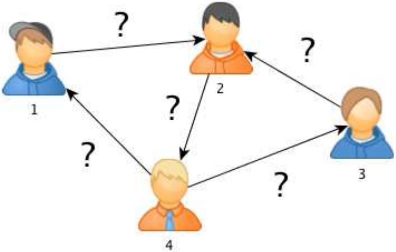
Motivation

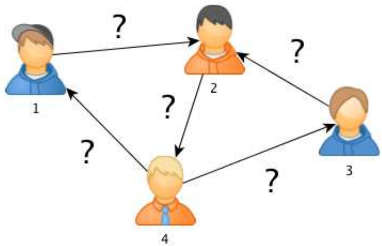
- The adjacency matrix is the most important tool in graph theory
- We are often interested in the rank of the adjacency matrix:
    - Open problem: to characterize the graphs whose adjacency matrix is singular
    - The nullity of a bipartite graph is of interest in chemistry
    - Progress in characterizing the nullity of a general graph is still needed

In some cases, we are interested in the rank of matrices whose pattern is defined by the graph.

Example: **The finite-time average consensus problem**

- Suppose we know the communication topology of the interacting agents
- Is it possible to choose the interaction strengths in order to get the consensus in finite time ?





Vector of initial positions:

$$\vec{x}(0)$$

Dynamics:

$$\vec{x}(t+1) = A^{t+1}.\vec{x}(0)$$

A matrix A is a solution if:

- A is of the form $\begin{pmatrix} 0 & ? & 0 & 0 \\ 0 & 0 & 0 & ? \\ 0 & ? & 0 & 0 \\ ? & 0 & ? & 0 \end{pmatrix}$
- After a finite time T, $A^T = \frac{1}{4}.\begin{pmatrix} 1 & \dots & 1 \\ \vdots & \vdots & \vdots \\ 1 & \dots & 1 \end{pmatrix}$

Outline

The Minimum Rank Problem

Hypergraph, generating number and minimum rank

An algorithm for the minimum rank of a loop directed tree

Conclusion

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Let G be a directed graph allowing loops with N vertices.

The graph G defines a family of real matrices:

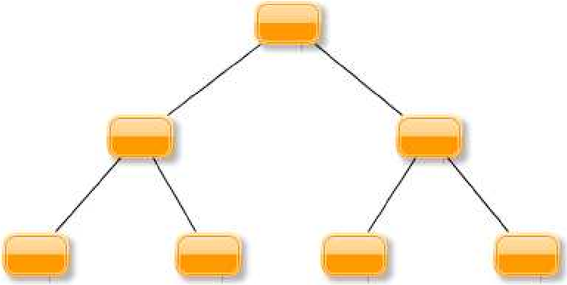
$$\mathcal{Q}(G) = \{A \in \mathbb{R}^{N\times N} | a_{ij} \neq 0 \text{ iff } (i,j) \text{ is an edge in } G\}.$$

The minimum rank of G is the minimum possible rank for a matrix in $\mathcal{Q}(G)$, that is:

$$mr(G) = \min\{\text{rank}(A) | A \in \mathcal{Q}(G)\}.$$

We will focus on a particular type of directed graphs: the loop directed trees.

A tree is a connected undirected graph without cycle.



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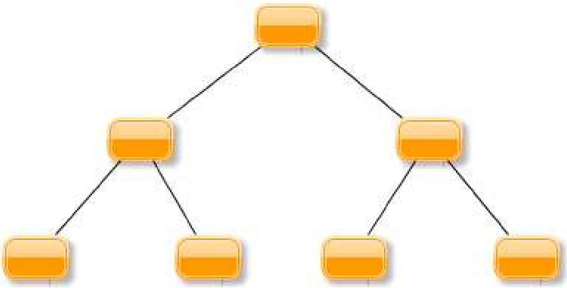
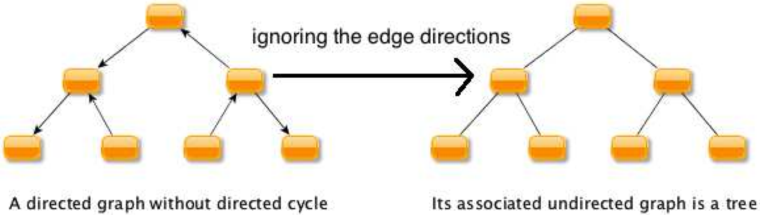
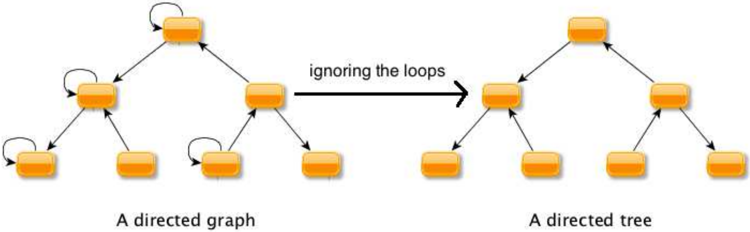


What is a directed tree ?

A directed tree:



A loop directed tree:



Problem:

How to compute the minimum rank of a loop directed tree ?

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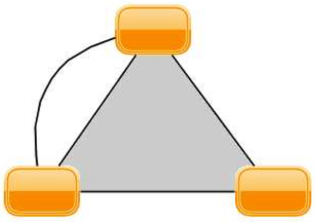
A hypergraph consists of:

- a vertex set
- a hyperedge set

A hyperedge of size n ($n \geq 1$) is a set of n distinct vertices.

Example

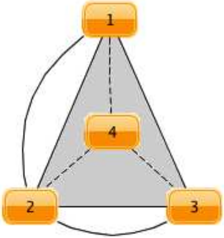


How can we define a hypergraph from a graph ?

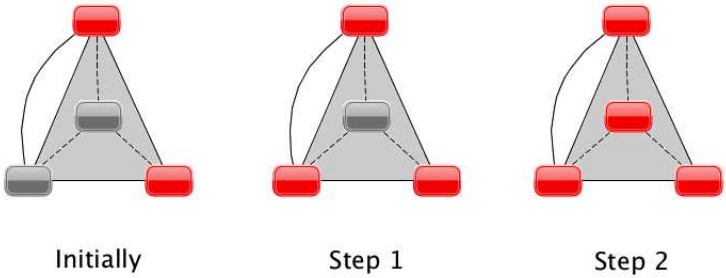
Suppose the adjacency matrix of the graph is:

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

Its associated hypergraph is:



A color change rule on a hypergraph:



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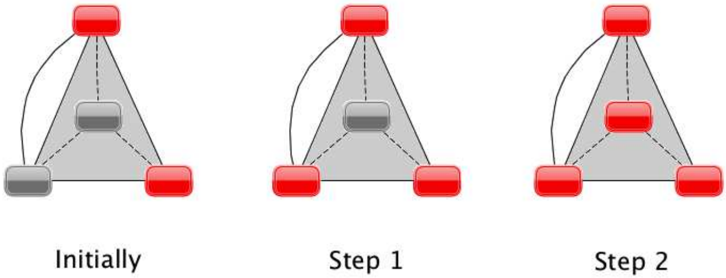

The generating number of a hypergraph is the minimum number of vertices which have to be initially red so that after applying repeatedly the color change rule, all the vertices are red.

Proposition

Let G be a loop directed tree. Then,

$$mr(G) = |G| - gen\ num(H_G),$$

where

- $|G|$ *is the number of vertices in* G
- H_G *is the hypergraph associated with* G

Problem: how to compute the generating number of a hypergraph?

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Goal: to compute the generating number of a hypergraph associated with a loop directed tree

To do so, we distinguish three kinds of vertices in the hypergraph:

- the isolated vertices which do not belong to a hyperedge
- the pendent vertices which belong to exactly one hyperedge
- the vertices which belong to several hyperedges

Trivially, we have the following lemma:

Lemma

Let H be a hypergraph,

$$gen\ num(H) = \sharp \text{ of isolated vertices}(H) + gen\ num(H'),$$

where H' is the hypergraph obtained from H by deleting its isolated vertices.

So, without loss of generality, we can consider that H does not contain isolated vertices.

To deal with the pendent vertices, we have the following result:

Proposition

Let H be a hypergraph and e be a hyperedge of H with $p(e)$ (≥ 1) pendent vertices. Then,

$$gen\ num(H) = (p(e) - 1) + gen\ num(H'),$$

where H' is the hypergraph obtained from H by deleting hyperedge e and the pendent vertices of e.

To deal with the pendent vertices, we have the following result:

Proposition

Let H be a hypergraph and e be a hyperedge of H with $p(e)$ (≥ 1) pendent vertices. Then,

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where H' is the hypergraph obtained from H by deleting hyperedge e and the pendent vertices of e.

Denote pendent(H) the number obtained by iterating this process until the resulting hypergraph does not contain pendent vertices anymore and H_f the resulting hypergraph at the end of the process.

Theorem

If H is the hypergraph of a loop directed tree, then H_f is empty.

To resume ...

If H is the hypergraph of a loop directed tree, then

$$\text{gen num}(H) = \sharp \text{ of isolated vertices}(H) + \text{pendent}(H).$$

Complexity of this algorithm : $\mathcal{O}(n^2 m)$,
where:

- n is the number of hyperedges in H
- m is the number of vertices in H

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- In many problems, we are interested in the rank of matrices whose pattern is described by a graph.
- In particular, we could be interested in computing the minimum possible rank of a real matrix whose pattern is given by a graph (minimum rank problem).
- Hypergraphs seem to be a natural structure to formulate the minimum rank problem of a loop directed tree.
- The presented algorithm can also be applied to loop undirected trees.

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- In many problems, we are interested in the rank of matrices whose pattern is described by a graph.
- In particular, we could be interested in computing the minimum possible rank of a real matrix whose pattern is given by a graph (minimum rank problem).
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Future work: Computing the minimum rank of a general graph seems to be much harder.

- Is it a NP-hard problem ?
- Could this hypergraph formalism be extended in order to compute the minimum rank of a more general graph ?

Thank you for your attention !