Strong structural controllability and zero forcing sets

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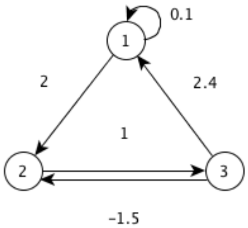
Framework (1)

Networked system:



Dynamics:

$$\dot{x}(t) = Ax(t) + B(S)u(t),$$

where

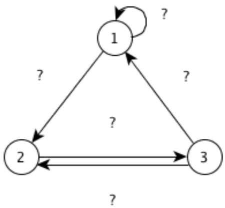
$$A = \begin{bmatrix} 0.1 & 0 & 2.4 \\ 2 & 0 & -1.5 \\ 0 & 1 & 0 \end{bmatrix}$$

For example, take $S = \{3\}$ and $B(S) = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$.

Since the controllability matrix $C = \begin{bmatrix} B(S) & AB(S) & A^2B(S) \end{bmatrix}$ is full rank, this system $(A, B(S))$ is controllable.

Framework (2)

Networked system:



Dynamics:

$$\dot{x}(t) = \underline{A}x(t) + B(S)u(t),$$

where

$$\underline{A} = \begin{bmatrix} \star & 0 & \star \\ \star & 0 & \star \\ 0 & \star & 0 \end{bmatrix}$$

A star $\star$ can be any nonzero real value.

For any realization A of $\underline{A}$, is the system $(A, B(S))$ controllable ?

For example, take $S = \{3\}$.

No, if all the weights equal 1, the controllability matrix is not full rank.

Framework (3)

Given:

- A directed graph $G = (V, E)$ with no idea of the weights
- A node subset $S \subset V$

The given graph G defines the matrix set:

$$\mathcal{Q}(G) := \{A \in \mathbb{R}^{n \times n} | \text{ for any } i, j, a_{ij} \neq 0 \Leftrightarrow (j, i) \in E\}.$$

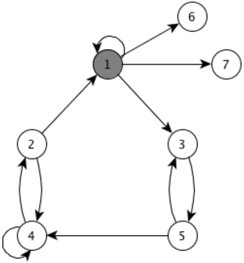
Question (strong structural controllability):

$$\text{For any } A \in \mathcal{Q}(G), \text{ is the system } (A, B(S)) \text{ controllable ?}$$

Goal of the talk: presenting a combinatorial tool allowing to answer this question.

Color change rule on a loop graph

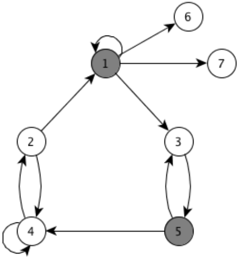
A loop graph = graph allowing loops



- Initial coloring: black nodes and white nodes
- Color change rule: if node i has exactly one white out-neighbor j, j becomes black.
- Apply the rule repeatedly on each node of G until no more color change is possible.

Color change rule on a loop graph

A loop graph = graph allowing loops



Chronological list of forces:
$3 \rightarrow 5$

- Initial coloring: black nodes and white nodes
- Color change rule: if node i has exactly one white out-neighbor j, j becomes black.
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Color change rule on a loop graph

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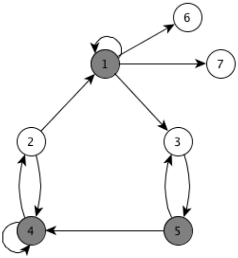
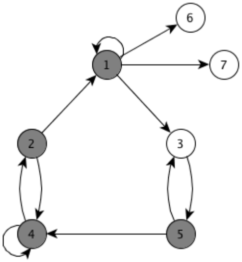


Chronological list of forces:
$3 \to 5$
$2 \to 4$

- Initial coloring: black nodes and white nodes
- Color change rule: if node i has exactly one white out-neighbor j, j becomes black.
- Apply the rule repeatedly on each node of G until no more color change is possible.

Color change rule on a loop graph

A loop graph = graph allowing loops



Chronological list of forces:

$3 \rightarrow 5$

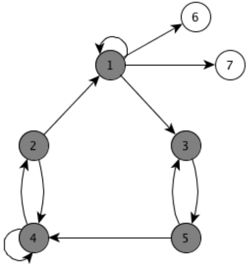
$2 \rightarrow 4$

$4 \rightarrow 2$

- Initial coloring: black nodes and white nodes
- Color change rule: if node i has exactly one white out-neighbor j, j becomes black.
- Apply the rule repeatedly on each node of G until no more color change is possible.

Color change rule on a loop graph

A loop graph = graph allowing loops



Chronological list of forces:

$3 \to 5$

$2 \to 4$

$4 \to 2$

$5 \to 3$

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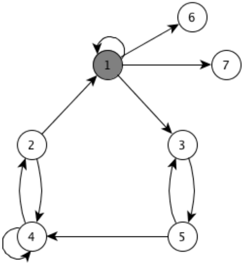
- Apply the rule repeatedly on each node of G until no more color change is possible.



Definition (Barioli, 2009)

- *A loop directed graph G with black and white nodes.*
- *S the set of initially black nodes in G.*

If after the color change rule, all the nodes of G are black, then S is a zero forcing set of G.

In the example, $S = \{1\}$ is not a zero forcing set, unlike $S = \{1, 6\}$.

Given:

- A loop directed graph $G = (V, E)$ with no idea of the weights
- A node subset $S \subset V$

The given graph G defines the matrix set:

$$\mathcal{Q}(G) := \{A \in \mathbb{R}^{n \times n} | \text{ for any } i, j, a_{ij} \neq 0 \Leftrightarrow (j, i) \in E\}.$$

Definition

System (G, S) is said strongly (structurally) S-controllable if for any $A \in \mathcal{Q}(G)$, the system (A,B(S)) is controllable.

In the literature, use of constraint matchings in a bipartite graph defined from G.

$G_\times$ = G with a loop on each node

Theorem (Trefois, Delvenne, 2014)

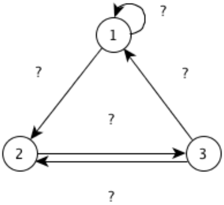
- *a loop directed graph G*
- *a node subset S*

System (G, S) is strongly S-controllable if and only if

- *S is a zero forcing set in G and*
- *S is a zero forcing set in $G_\times$ with a chronological list of forces with no force of the form $i \to i$, $(i, i) \in E(G)$.*

System (G, S) is strongly S-controllable if and only if

- S is a zero forcing set in G and
- S is a zero forcing set in $G_\times$ with a chronological list of forces with no force of the form $i \to i$, $(i, i) \in E$.



$$\dot{x}(t) = \underline{A}x(t) + B(S)u(t),$$

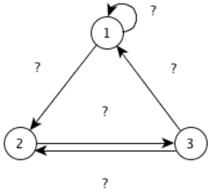
$$\underline{A} = \begin{bmatrix} \star & 0 & \star \\ \star & 0 & \star \\ 0 & \star & 0 \end{bmatrix}$$

For any realization A of $\underline{A}$, is the system $(A, B(S))$ controllable with $S = \{3\}$?

No, $S = \{3\}$ is not a zero forcing set in G.

System (G, S) is strongly S-controllable if and only if

- S is a zero forcing set in G and
- S is a zero forcing set in $G_\times$ with a chronological list of forces with no force of the form $i \to i$, $(i, i) \in E(G)$.



And with $S = \{1\}$? Yes,

- $S = \{1\}$ is a zero forcing set in G
- S is also a zero forcing set in $G_\times$ with chronological list of forces:

$$1 \to 2, 3 \to 3$$

Intermediate conclusion: given G on n nodes and S,

- is the system (G, S) strongly S-controllable ? answer in time $\mathcal{O}(n^2)$
- tool: the definition of zero forcing set in a loop graph

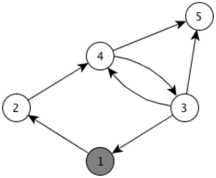
New question: given G, find S of minimum size such that (G, S) is strongly S-controllable...

In the next:

- answer for particular systems
- tool: zero forcing sets in a simple graph

Color change rule on a simple graph

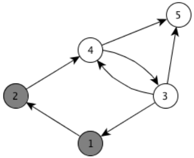
A simple graph = graph that prohibits loops



- Initial coloring: black nodes and white nodes
- Color change rule: if node i is black and has exactly one white out-neighbor j, j becomes black.
- Apply the rule repeatedly on each node of G until no more color change is possible.

Color change rule on a simple graph

A simple graph = graph that prohibits loops

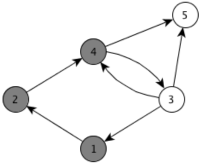


Chronological list of forces:
$1 \rightarrow 2$

- Initial coloring: black nodes and white nodes
- Color change rule: if node i is black and has exactly one white out-neighbor j, j becomes black.
- Apply the rule repeatedly on each node of G until no more color change is possible.

Color change rule on a simple graph

A simple graph = graph that prohibits loops



Chronological list of forces:
$1 \to 2$
$2 \to 4$

- Initial coloring: black nodes and white nodes
- Color change rule: if node i is black and has exactly one white out-neighbor j, j becomes black.
- Apply the rule repeatedly on each node of G until no more color change is possible.

Definition (AIM minimum rank, 2008)

- *A simple graph G with black and white nodes.*
- *S the set of initially black nodes in G.*

If after the color change rule, all the nodes of G are black, then S is a zero forcing set of G.

When graph without loops, always specify if it is considered as a simple or as a loop graph.

Self-damped system: loop directed graph with a loop on each node.

System (G, S) is strongly S-controllable if and only if

- S is a zero forcing set in G and
- S is a zero forcing set in $G_\times$ with a chronological list of forces with no force of the form $i \to i$, $(i, i) \in E(G)$.

Corollary (Trefois, Delvenne, 2014)

- *a self-damped system G*
- *a node subset S*

System (G, S) is strongly S-controllable if and only if S is a zero forcing set in the simple graph G_s.

- a self-damped system G
- a node subset S

System (G, S) is strongly S-controllable if and only if S is a zero forcing set in the simple graph G_s.

finding S of minimum size for strong controllability of G

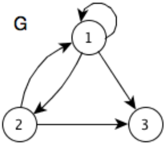
$$=$$

finding a minimum-size zero forcing set S in G_s (NP-hard)

Theorem (Trefois, Delvenne, 2014)

G a self-damped system on n nodes with a tree structure. Then, one can compute, in time $\mathcal{O}(n^2)$, a set S of minimum size such that (G, S) is strongly S-controllable.

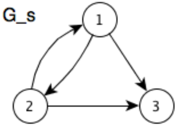
Our framework:



$$\begin{pmatrix} \star & \star & 0 \\ \star & 0 & 0 \\ \star & \star & 0 \end{pmatrix}$$

$$\mathcal{Q}(G) = \{A \in \mathbb{R}^{3\times 3} : \text{ for any } i, j, a_{ij} \neq 0 \Leftrightarrow (j,i) \in E(G)\}$$

Another framework (Monshizadeh, Zhang, Camlibel, 2013):



$$\begin{pmatrix} ? & \star & 0 \\ \star & ? & 0 \\ \star & \star & ? \end{pmatrix}$$

$$\mathcal{Q}(G_s) = \{A \in \mathbb{R}^{3\times 3} : \text{ for any } {\color{red} i \neq j}, a_{ij} \neq 0 \Leftrightarrow (j,i) \in E(G)\}$$

$$\Rightarrow \mathcal{Q}(G) \subseteq \mathcal{Q}(G_s)$$

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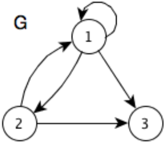
Theorem (Trefois, Delvenne, 2014)

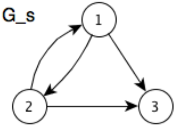
System (G, S) is strongly S-controllable if and only if

- *S is a zero forcing set in G and*
- *S is a zero forcing set in $G_\times$ with a chronological list of forces with no force of the form $i \to i$, $(i,i) \in E(G)$.*

Theorem (Monshizadeh, Zhang, Camlibel, 2013)

System (G_s, S) is strongly S-controllable if and only if S is a zero forcing set in G_s.





$S = \{1\}$ is

- a zero forcing set in G and
- a zero forcing set in $G_\times$ with list of forces $3 \to 3, 1 \to 2$.

$$\Rightarrow (G, S) \text{ is strongly } S \text{ - controllable.}$$

$S = \{1\}$ is NOT a zero forcing set in G_s.

$$\Rightarrow (G_s, S) \text{ is not strongly } S \text{ - controllable.}$$

Zero forcing set (loop graph)

Strong structural controllability

Zero forcing set (simple graph)

Self-damped systems

Remark on existing results

Conclusion

We have presented:

- the strong structural controllability of a networked system
- the notion of zero forcing set in a loop/simple graph
- the role of the zero forcing sets in the study of the strong structural controllability of a system

References

- M. Trefois, J.-C. Delvenne, *Zero forcing number, constraint matchings and strong structural controllability*, submitted 2014. arXiv:1405.6222
- N. Monshizadeh, S. Zhang, M.K. Camlibel, *Zero forcing sets and controllability of dynamical systems defined on graphs*, accepted for publication in IEEE Transaction on Automatic Control.
- D. Burgarth, D. D'Alessandro, L. Hogben, S. Severini, M. Young, *Zero forcing, linear and quantum controllability for systems evolving on networks*, IEEE Trans. on Automatic Control 58, 2349 (2013).

Take home one message:

The zero forcing sets seem to play an important role in the study of the dynamics of networked systems.