

THE AUTOMORPHISM GROUP OF ACCESSIBLE GROUPS

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ABSTRACT. In this article, we study the outer automorphism group of a group G decomposed as a finite graph of group with finite edge groups and finitely generated vertex groups with at most one end. We show that $\text{Out}(G)$ is essentially obtained by taking extensions of relative automorphism groups of vertex groups, groups of Dehn twists and groups of automorphisms of free products. We apply this description and obtain a criterion for $\text{Out}(G)$ to be finitely presented, as well as a necessary and sufficient condition for $\text{Out}(G)$ to be finite. Consequences for hyperbolic groups are discussed.

1. INTRODUCTION

In this article, we study the structure of the automorphism group of accessible groups. A group is called *accessible* if it admits a decomposition as a finite graph of group with finite edge groups and finitely generated vertex groups with at most one end. Recall that finitely presented groups are accessible [4], as well as finitely generated groups with a uniform bound on the order of finite subgroups [11]. We reduce the study of the automorphism group of an accessible group to that of certain relative automorphism groups of vertex groups, groups of Dehn twists and automorphism groups of free products.

We recall the notion of relative automorphism groups. Let G be a group, and let \mathcal{H} be a family of subgroups of G , closed under taking conjugates. The relative automorphism group $\text{Out}_{\mathcal{H}}(G)$ is the subgroup of $\text{Out}(G)$ of those outer automorphisms that preserve the conjugacy class of each element of \mathcal{H} . This is an algebraic analogue of the following geometric situation: S is an orientable surface with boundary, and \mathcal{H} is the family of cyclic subgroups of $G = \pi_1(S)$ corresponding to boundary curves, then $\text{Out}_{\mathcal{H}}(G)$ is exactly the mapping class group $\text{Mod}^{\partial}(S)$ preserving the boundary componentwise.

Suppose G is a free product of finitely many freely indecomposable groups. Fousse-Rabinovitch [5, 6] gave a presentation of $\text{Aut}(G)$ in terms of the freely indecomposable free factors of G and their automorphism group. In particular he shows that if each freely indecomposable free factor of G is finitely presented and has a finitely presented automorphism group, then $\text{Aut}(G)$ is finitely presented.

Our study of the structure of the automorphism group of an accessible group G combined with a suitable adaptation of Fousse-Rabinovitch's result yield the following consequence for the finite presentability of $\text{Out}(G)$.

Theorem 1.1. *Let \mathbb{A} be a finite reduced graph of groups with finite edge groups and finitely generated vertex groups with at most one end. Let $G = \pi_1(\mathbb{A})$ act on the Bass-Serre tree $T_{\mathbb{A}}$. Suppose the following two conditions are satisfied:*

- (1) *For each edge e of $T_{\mathbb{A}}$ and each vertex v stabilized by G_e , the normalizer N_e of G_e in G_v is finitely presented, and its center $Z(N_e)$ is finitely generated.*

The author is a FNRS Research Fellow (Belgium). Part of this research was supported by a Marie Curie fellowship.

- (2) For each vertex v of T_A , the group $\text{Out}_{\mathcal{H}_v}(G_v)$ of automorphisms relative to the family \mathcal{H}_v of edge stabilizers contained in G_v is finitely presented.

Then $\text{Out}(G)$ is finitely presented.

Our study also leads to a simple characterization of accessible groups with finite outer automorphism group in terms of splittings over finite groups and relative automorphism groups of maximal elliptic subgroups.

Theorem 1.2. *Let \mathbb{A} be a finite reduced graph of groups with finite edge groups and finitely generated vertex groups with at most one end. Let $G = \pi_1(\mathbb{A})$ act on the Bass-Serre tree T_A . Then $\text{Out}(G)$ is infinite if and only if one of the following holds:*

- (1) *There is a vertex stabilizer G_v of T_A such that $\text{Out}_{\mathcal{H}_v}(G_v)$ is infinite, where \mathcal{H}_v is the family of edge stabilizers contained in G_v ;*
- (2) *There is a splitting of G as an amalgam $A *_C B$ over a finite group with $B \neq C$ such that the center of A has infinite index in the centralizer of C in A ;*
- (3) *There is a splitting of G as an HNN extension $A *_C$ over a finite group such that the centralizer of \tilde{C} in A is infinite, where \tilde{C} is one of the two isomorphic copies of C in A given by the HNN extension.*

The paper is organized as follows. We introduce notation and make some preliminary observations in section 2. The structure of the automorphism group of an accessible group is described in section 3, summarized by proposition 3.12 which is the main technical result of this paper. We apply our structural results inductively in section 4 to prove theorem 1.1. Section 5 is devoted to the proof of theorem 1.2. In section 6, theorems 1.1 and 1.2 are applied to show that the automorphism group of any hyperbolic group is finitely presented and to characterize hyperbolic groups with finite outer automorphism group.

2. PRELIMINARIES

We denote the center of a group G by $Z(G)$. If H is a subgroup of G , we write $Z_G H$ for the centralizer of H in G and $N_G H$ for the normalizer of H in G . For an element $g \in G$, we define the associated inner automorphism as $i_g : G \rightarrow G : x \mapsto gxg^{-1}$.

2.1. Graphs of groups. In order to fix notation, we recall some definitions and results of Bass-Serre theory. The unfamiliar reader is referred to [14] for more details.

A *graph* A is given by the following data: a set of vertices VA , a set of edges EA , a boundary map $\alpha : EA \rightarrow VA$ and an involution $^{-1} : EA \rightarrow EA$ such that $e^{-1} \neq e$ for each edge $e \in EA$. The vertex $\alpha(e)$ is called the *initial vertex* of e . The *terminal vertex* of e is defined as $\omega(e) := \alpha(e^{-1})$.

A *graph of groups* \mathbb{A} is given by the following data: a connected graph A , a vertex group \mathbb{A}_v for each vertex $v \in VA$, an edge group \mathbb{A}_e for each edge $e \in EA$ such that $\mathbb{A}_e = \mathbb{A}_{e^{-1}}$ and injections $\alpha_e : \mathbb{A}_e \rightarrow \mathbb{A}_{\alpha(e)}$ of each edge group in the initial vertex group. Given such data, we also define the map $\omega_e : \mathbb{A}_e \rightarrow \mathbb{A}_{\omega(e)}$ by $\omega_e := \alpha(e^{-1})$. An \mathbb{A} -path is a sequence $a_0, e_1, a_1, \dots, e_n, a_n$ where a_i is an element of a vertex group \mathbb{A}_{v_i} and e_i is an edge of \mathbb{C} such that $\omega(e_i) = v_i = \alpha(e_{i+1})$. Two \mathbb{A} -paths γ_1 and γ_2 are *elementarily equivalent* if either

$$\gamma_1 = \gamma, a_i, e, 1, e^{-1}, a_{i+2}, \gamma' \text{ and } \gamma_2 = \gamma, a_i a_{i+2}, \gamma'$$

or if

$$\gamma_1 = \gamma, a_i, e, a_{i+1}, \gamma' \text{ and } \gamma_2 = \gamma, a_i \alpha_e(c), e, \omega_e(c^{-1}) a_{i+1}, \gamma' \text{ where } c \in \mathbb{A}_e$$

Let \sim denote the equivalence relation on the set of \mathbb{A} -paths generated by this elementary equivalence. If u_0 is a vertex of A , then the *fundamental group* $\pi_1(\mathbb{A}, u_0) = \{\text{closed } \mathbb{A}\text{-paths based at } u_0\} / \sim$ is a group with the operation of concatenation. The equivalence class of an \mathbb{A} -path γ is denoted by $[\gamma]$. If \mathbb{A} is a graph of groups, we use the same letter A to denote the underlying graph and we let T_A be the Bass-Serre tree on which the fundamental group $\pi_1(\mathbb{A}, u_0)$ acts. The isomorphism class of $\pi_1(\mathbb{A}, u_0)$ does not depend on the basepoint u_0 , so we will often write $\pi_1(\mathbb{A})$.

All actions on trees are assumed not to invert edges. We say a group G *splits* over a subgroup C if either $G = A *_C B$ with $A \neq C \neq B$ or if $G = A *_C$.

Let \mathbb{A} be a finite graph of group. A subgroup H of $\pi_1(\mathbb{A})$ is *elliptic* if it fixes a vertex in T_A . For a vertex or edge x of T_A we let G_x be the stabilizer of x in G . \mathbb{A} is *minimal* if \mathbb{A} does not have any vertex v of valence 1 such that the boundary monomorphism of the adjacent edge is surjective. \mathbb{A} is *reduced* if for any edge $e \in EA$, either $\alpha(e) = \omega(e)$ or at least one of the monomorphisms α_e or ω_e is not surjective. Note that a reduced graph of group is automatically minimal.

2.2. Normalizers of elliptic subgroups.

Lemma 2.1. *Let \mathbb{A} be a graph of group, and G_v be a vertex stabilizer in $G = \pi_1(\mathbb{A})$ that does not stabilize an edge of T_A . Then G_v is its own normalizer in G .*

Proof. The normalizer $N_G G_v$ acts on the set of fixed points of G_v in T_A , which only consists of the vertex v , thus showing that $N_G G_v \subset G_v$. \square

The following lemma computes the normalizer of an elliptic subgroup of $G = \pi_1(\mathbb{A})$ from normalizers in vertex groups.

Lemma 2.2. *Let \mathbb{A} be a finite graph of groups with finite edge groups. Let H be an elliptic subgroup of $G = \pi_1(\mathbb{A})$.*

- (1) *If $N_{G_v} H$ is finitely generated (resp. presented) for every vertex $v \in T_A$ fixed by H then $N_G H$ is finitely generated (resp. presented).*
- (2) *If the center of $N_{G_v} H$ is finitely generated for each $v \in T_A$ fixed by H , then the center of $N_G H$ is finitely generated.*

Proof. Let T_H be the maximal subtree of T_A fixed by H . Since H is elliptic, T_H is nonempty. The normalizer $N_G H$ acts on T_H , and let \mathbb{B} be the corresponding graph of group. If v is a vertex of T_H , then the stabilizer of v in $N_G H$ is $N_G H \cap G_v = N_{G_v} H$. Let us show that \mathbb{B} has finitely many edges. The inclusions $T_H \hookrightarrow T_A$ and $N_G H \hookrightarrow G$ induce a graph map: $\mathbb{B} \rightarrow A$. We claim that for each edge $e \in EA$, there are only finitely many preimages in EB . Let f_1, f_2 be two edges of T_H in the same G -orbit, and let X be the set of elements of G sending f_2 to f_1 . Then f_1 and f_2 are in the same $N_G H$ orbit if and only if $H^X = H^{G_{f_1}}$. Therefore there cannot be more preimages of e in EB than there are conjugacy classes of groups isomorphic to H in the finite group G_e . Hence \mathbb{B} is a finite graph of groups such that each vertex group is isomorphic to some $N_{G_v} H$ and each edge group is finite. Thus $\pi_1(\mathbb{B}) = N_G H$ is finitely generated (resp. presented) provided that each $N_{G_v} H$ is.

We now consider the center of $N_G H = \pi_1(\mathbb{B})$. Note that if \mathbb{B}' is obtained from \mathbb{B} by collapsing some non-loop edges with at least one surjective boundary monomorphism, then the set of vertex stabilizer for \mathbb{B}' is a subset of the set of vertex stabilizer for \mathbb{B} . Thus without loss of generality, we can suppose that \mathbb{B} is reduced. If \mathbb{B} consists of a single vertex, the center of $N_G H$ is the same as the center of $N_{G_v} H$ for some vertex $v \in VT_A$. If \mathbb{B} is a mapping torus, then the center of $N_G H$ is virtually infinite cyclic. In any other case, the center of $N_G H$ is contained in all edge stabilizers of T_B , and so must be finite. \square

2.3. Relative automorphism groups. The following lemma is well-known, but we include a proof for the sake of completeness.

Lemma 2.3. *Let \mathbb{A} and \mathbb{A}' be finite reduced graphs of groups with finite edge groups and finitely generated vertex groups with at most one end. Let \mathcal{H} and \mathcal{H}' be the families of edge stabilizers of $G = \pi_1(\mathbb{A})$ and $G' = \pi_1(\mathbb{A}')$ respectively. If $\varphi : G \rightarrow G'$ is an isomorphism, then φ maps \mathcal{H} to \mathcal{H}' . In particular $\text{Out}_{\mathcal{H}}(G)$ has finite index in $\text{Out}(G)$.*

Proof. We prove the first assertion by induction on the number of edges of \mathbb{A} . If \mathbb{A} has no edge, then \mathcal{H} is empty and G has at most one end. Hence G' also has at most one end, so \mathbb{A}' has no edge and \mathcal{H}' is empty.

Suppose now that $|EA| > 0$. Let \mathcal{F} be the family of finite subgroups of G over which G splits, and let \mathcal{F}_{\min} be the minimal elements of this family with respect to inclusion. Defining \mathcal{F}' and \mathcal{F}'_{\min} similarly, it is clear that φ maps \mathcal{F} to \mathcal{F}' and \mathcal{F}_{\min} to \mathcal{F}'_{\min} .

Let \mathbb{B} be the graph of groups obtained by collapsing all edges of \mathbb{A} whose edge stabilizer are not in \mathcal{F}_{\min} . As edge stabilizers of T_A are finite, any finite group over which G splits contains a minimal subgroup over which G splits, so that \mathbb{B} is not a single vertex and \mathcal{F}_{\min} is nonempty. Let \mathcal{S} be the family of vertex stabilizers of T_B .

Note that if a subgroup of G does not split over a subgroup of an element of \mathcal{F}_{\min} then it acts elliptically on T_B . Therefore, \mathcal{S} is characterized as the family of maximal subgroups of G which do not split over a subgroup of an element of \mathcal{F}_{\min} . Again, defining \mathbb{B}' and \mathcal{S}' similarly, it is clear that φ maps \mathcal{S} to \mathcal{S}' . Therefore, φ maps vertex stabilizers of T_B to vertex stabilizers of $T_{B'}$. Since \mathbb{A} and \mathbb{A}' are reduced, so are \mathbb{B} and \mathbb{B}' . Hence vertex groups of \mathbb{B} (resp. \mathbb{B}') correspond bijectively to conjugacy classes of elements of \mathcal{S} (resp. \mathcal{S}'). Therefore φ induces bijection $\varphi : VT_B \rightarrow VT_{B'}$ which projects to a map on the orbits $\varphi : VB \rightarrow VB'$.

For each $v \in VB$, let $\mathbb{A}(v)$ be the subgraph of group of \mathbb{A} collapsed to v in \mathbb{B} . Note that $G_v := \pi_1(\mathbb{A}(v)) \cong \mathbb{B}_v$ and that $\mathbb{A}(v)$ has strictly fewer edges than \mathbb{A} . Let \mathcal{H}_v be the set of edge stabilizers of $T_{\mathbb{A}(v)}$. It is exactly the set of elements of $\mathcal{H} \setminus \mathcal{F}_{\min}$ contained in G_v . Since \mathbb{A} is reduced, so is $\mathbb{A}(v)$. We apply induction to $(\mathbb{A}(v), G_v, \mathcal{H}_v)$ and conclude that $\varphi|_{G_v}$ maps G_v to $G'_{\varphi(v)}$ and \mathcal{H}_v to $\mathcal{H}'_{\varphi(v)}$.

Finally we observe that \mathcal{H} is the union of \mathcal{F}_{\min} with the set of conjugates in G of elements of $\cup_{v \in VB} \mathcal{H}_v$. The same statement holds for \mathcal{H}' . Since φ maps \mathcal{F}_{\min} to \mathcal{F}'_{\min} and $\cup_{v \in VB} \mathcal{H}_v$ to $\cup_{v' \in VB'} \mathcal{H}'_{v'}$, we conclude that φ maps \mathcal{H} to \mathcal{H}' .

Since \mathbb{A} is finite, there are finitely many conjugacy classes of edge stabilizers in G . The last assertion follows since $\text{Out}_{\mathcal{H}}(G)$ is the subgroup of $\text{Out}(G)$ which induces a trivial permutation of the conjugacy classes of elements of \mathcal{H} . \square

3. THE STRUCTURE OF $\text{Out}_{\mathcal{H}}(G)$

Throughout this section, we fix a triple $(\mathbb{A}, G, \mathcal{H})$ as follows: \mathbb{A} is a finite reduced graph of groups with finite edge groups and finitely generated vertex groups with at most one end. \mathcal{H} is a family of elliptic subgroups of $G = \pi_1(\mathbb{A})$ containing each edge stabilizer. Suppose moreover that \mathbb{A} does not consist of a single vertex (i.e. G is not one-ended). The following two conditions will be used in the sequel:

- (C1) For any $v \in VT_A$ the set of G_v -conjugacy classes of elements in \mathcal{H}_v is finite.
- (C2) The normalizer of any edge group in any vertex group G_v containing it is finitely generated.

3.1. The graph of group \mathbb{B} . The set of edge stabilizers of T_A is nonempty as G is assumed to have more than one end. As edge stabilizers are finite, an edge stabilizer can never be conjugate to a proper subgroup of itself, so the set of edge

stabilizers partially ordered by inclusion admits a minimal element. Let G_0 be such a minimal edge stabilizer, and let \mathcal{E} be the set of edges of \mathbb{A} whose edge stabilizer is conjugate to G_0 . We set $\bar{\mathbb{B}}$ to be the graph of group obtained by collapsing all edges of \mathbb{A} not in \mathcal{E} .

Let $V = V\bar{\mathbb{B}}$. For each $v \in V$, let $\mathbb{A}(v)$ be the connected subgraph of groups of \mathbb{A} collapsed to v in $\bar{\mathbb{B}}$. As not all edges of \mathbb{A} are collapsed in $\bar{\mathbb{B}}$, each $\mathbb{A}(v)$ has fewer edges than \mathbb{A} . These graphs of groups will not be referred to in this section, but will be used in sections 4 and 5 to prove theorems 1.1 and 1.2 respectively.

Observe that $\bar{\mathbb{B}}$ is a finite reduced graph of group with fundamental group G and Bass-Serre tree $T_{\bar{\mathbb{B}}}$ such that:

- All edge stabilizers of G are conjugate;
- No edge stabilizer is conjugate to a proper subgroup of itself.

Remark. The remainder of subsection 3.1 can be applied to any graph of group $\bar{\mathbb{B}}$ satisfying the two conditions just stated. In fact, all the statements of section 3 except corollary 3.6 remain true if the edge groups of $\bar{\mathbb{B}}$ are assumed to satisfy the two above conditions and to have finite outer automorphism group.

In order to fix notation we modify $\bar{\mathbb{B}}$ to a more symmetric graph of group \mathbb{B} without changing the set of elliptic subgroups nor the set of edge groups. Subdivide an edge of $\bar{\mathbb{B}}$ and call b_0 the new vertex. Now slide the beginning of each edge to b_0 whenever possible. This makes a graph of group \mathbb{B} having vertex set $\{b_0\} \cup V$ with the following properties: b_0 has a vertex group isomorphic to some edge group in $\bar{\mathbb{B}}$; for each vertex v of V the vertex group \mathbb{B}_v properly contains each incoming edge group; all loop edges begin at b_0 ; all non-loop edges that begin at b_0 end in V and vice versa; for any two edges $e \neq f$ having a common terminal vertex $v \in V$, the subgroups $\omega_e(\mathbb{B}_e)$ and $\omega_f(\mathbb{B}_f)$ are not conjugate in \mathbb{B}_v .

Write the set of loop edges as $\{e_s \mid s \in S^{\pm 1}\}$ in such a way that $e_s^{-1} = e_{s^{-1}}$. Write also the set of non-loop edges as $\{e_k \mid k \in K^{\pm 1}\}$ in such a way that $e_k^{-1} = e_{k^{-1}}$ and that for each $k \in K$ the edge e_k originates at b_0 and ends in V . We identify the vertex group of b_0 with G_0 . We introduce the following notation until the end of the section: $\omega(e_k)$ is denoted by v_k for each $k \in K$. For each $k \in K$, we write G_k for the subgroup $[1, e_k, \mathbb{B}_{v_k}, e_k^{-1}, 1] \subset \pi_1(\mathbb{B}, b_0)$. Choosing a maximal tree in B , we identify \mathbb{B}_v with the group G_k for some k such that $v_k = v$. The normalizer of G_0 in G_k is written as N_k and the normalizer of G_0 in G is denoted by N . We do not distinguish between $s \in S$ and the corresponding element $[1, e_s, 1] \in \pi_1(\mathbb{B}, b_0)$.

Observe that the normalizer of G_0 in G has a very simple decomposition $\bar{\mathbb{N}}$ as an amalgam of the groups N_k for $k \in K$ and some mapping tori along G_0 . Similarly, the group N/G_0 inherits a decomposition $\bar{\mathbb{F}}$ obtained from $\bar{\mathbb{N}}$ by taking the quotient of every edge and vertex group by G_0 . Thus edge groups of $\bar{\mathbb{F}}$ are trivial and $\bar{\mathbb{F}}$ is decomposition of N/G_0 as a free product. The decompositions \mathbb{B} and $\bar{\mathbb{N}}$ are described in figure 1.

It could happen that $\bar{\mathbb{N}}$ is not minimal, as there can be some k such that $N_k = G_0$. This is represented by a dashed edge in figure 1. We define the graph of group \mathbb{N} and \mathbb{F} by removing these vertices and edges from the decomposition $\bar{\mathbb{N}}$ as well as the corresponding ones from $\bar{\mathbb{F}}$. Let I be the set of $i \in K$ such that $N_i = G_0$, and let $J = K \setminus I$ be the other indices.

Lemma 3.1. *The family of vertex stabilizers of T_B is preserved by $\text{Out}_{\mathcal{H}}(G)$.*

Proof. By construction of \mathbb{B} , all edge stabilizers of G acting on T_B are conjugate to G_0 . Let $\hat{\varphi} \in \text{Out}_{\mathcal{H}}(G)$ be a relative automorphism and φ a representative of $\hat{\varphi}$. Since G_0 lies in \mathcal{H} the automorphism φ preserves the conjugacy class of G_0 , and also preserves the family of subgroups of conjugates of G_0 .

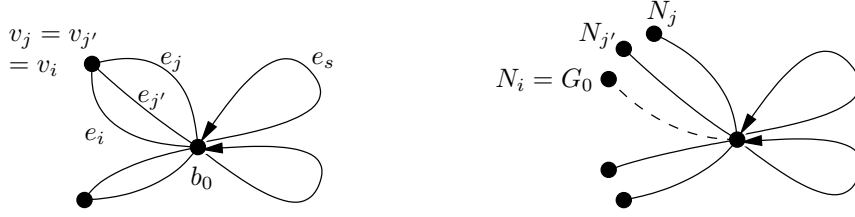


FIGURE 1. The decomposition \mathbb{B} of G and $\bar{\mathbb{N}}$ of $N = N_G(G_0)$

Recall that G_0 is a minimal element among the family of edge stabilizers of G acting on T_A and that vertex groups of \mathbb{A} have at most one end. Therefore if a subgroup of G does not split over a subgroup of a conjugate of G_0 then it is contained in a vertex stabilizer of T_B . So vertex stabilizers of T_B are either conjugates of G_0 or maximal subgroups of G which do not split over a subgroup of a conjugate of G_0 . Since φ preserves the conjugacy class of G_0 it must preserve the family of vertex stabilizers. \square

3.2. The maps ρ . For every subgroup H of a group Γ , let $\text{Out}_H(\Gamma)$ be the subgroup of the outer automorphism group preserving the conjugacy class of H . Then there is a map $\rho_H : \text{Out}_H(\Gamma) \rightarrow \text{Out}(N_\Gamma(H))$ defined as follows. Fix an element $\hat{\alpha}$ in $\text{Out}_H(\Gamma)$. Choose a representative α of $\hat{\alpha}$ in $\text{Aut}(\Gamma)$ such that $\alpha(H) = H$. This representative is unique up to right multiplication by an inner automorphism normalizing H . Moreover, $\alpha(N_\Gamma(H)) = N_\Gamma(H)$. Define $\rho_H(\hat{\alpha})$ to be the element of $\text{Out}(N_\Gamma(H))$ represented by the restriction of α to $N_\Gamma(H)$. By the discussion above, this map is well defined. ρ_H is easily checked to be a homomorphism.

Having made this observation, we can now define $\text{Out}^0(G)$. By lemma 3.1 the group $\text{Out}_{\mathcal{H}}(G)$ permutes the (finite) set of conjugacy classes of vertex stabilizers of \mathbb{B} . Let $\text{Out}'_{\mathcal{H}}(G)$ be the subgroup of $\text{Out}_{\mathcal{H}}(G)$ which induce the trivial permutation of this set. Moreover lemma 2.1 implies that the vertex stabilizer G_v is self-normalized in G for each $v \in V$, so there are maps $\rho_v : \text{Out}'_{\mathcal{H}}(G) \rightarrow \text{Out}(G_v)$. The image of ρ_v preserves the family \mathcal{H}_v , but does not necessarily preserve the G_v -conjugacy class of each element. Let $\text{Out}^0(G)$ be the largest subgroup of $\text{Out}'_{\mathcal{H}}(G)$ which maps to $\text{Out}_{\mathcal{H}_v}(G_v)$ for each $v \in V$, i.e. the subgroup of $\text{Out}'_{\mathcal{H}}(G)$ which preserves the G_v -conjugacy class of each element of \mathcal{H}_v . If condition (C1) is fulfilled, i.e. if there are finitely many such conjugacy classes for each $v \in V$, $\text{Out}^0(G)$ has finite index in $\text{Out}_{\mathcal{H}}(G)$.

Putting all the maps ρ_v together, we define the map

$$\rho_{\mathbb{B}} : \text{Out}^0(G) \xrightarrow{\prod \rho_v} \prod_{v \in V} \text{Out}_{\mathcal{H}_v}(G_v)$$

Let \mathcal{N} be the set of vertex stabilizers of N acting on T_N , i.e. the family of subgroups conjugate to either G_0 or to one of the N_j . Let \mathcal{F} be set of vertex stabilizers of F acting on T_F . Remark that \mathcal{F} is precisely the image of \mathcal{N} in F under the projection $\pi : N \rightarrow F$. Similarly to above, we have maps $\rho'_j : \text{Out}_{\mathcal{N}}(N) \rightarrow \text{Out}_{G_0}(N_j)$ and $\rho''_j : \text{Out}_{\mathcal{F}}(F) \rightarrow \text{Out}(F_j)$ and we again combine these maps to form

$$\rho_{\mathbb{N}} : \text{Out}_{\mathcal{N}}(N) \xrightarrow{\prod \rho'_j} \prod_{j \in J} \text{Out}_{G_0}(N_j) \quad ; \quad \rho_{\mathbb{F}} : \text{Out}_{\mathcal{F}}(F) \xrightarrow{\prod \rho''_j} \prod_{j \in J} \text{Out}(F_j)$$

Since $\text{Out}^0(G)$ preserves in the conjugacy class of G_0 and the conjugacy class of each G_j , we define the map $\sigma : \text{Out}^0(G) \rightarrow \text{Out}_{\mathcal{N}}(N)$ as above. For each $k \in I \cup J$ there is a map $\sigma_k : \text{Out}_{\mathcal{H}_{v_k}}(G_{v_k}) \rightarrow \text{Out}_{G_0}(N_k)$. Define $\sigma_I = \prod_{i \in I} \sigma_i$ and $\sigma_J = \prod_{j \in J} \sigma_j$.

Since any automorphism representing an element of $\text{Out}_{\mathcal{N}}(N)$ must fix G_0 , the projection $\pi : N \rightarrow N/G_0$ induces a map $\tau : \text{Out}_{\mathcal{N}}(N) \rightarrow \text{Out}_{\mathcal{F}}(F)$, and maps $\tau_j : \text{Out}_{G_0}(N_j) \rightarrow \text{Out}(F_j)$. Define $\tau_J = \prod_{j \in J} \tau_j$.

Let $\mathcal{K}_{\mathbb{B}}$, $\mathcal{K}_{\mathbb{N}}$ and $\mathcal{K}_{\mathbb{F}}$ denote the kernels of the maps $\rho_{\mathbb{B}}$, $\rho_{\mathbb{N}}$ and $\rho_{\mathbb{F}}$ respectively. These groups are exactly those outer automorphisms $\hat{\varphi}$ such that any representative φ of $\hat{\varphi}$ acts on each vertex stabilizer as a conjugation by some element in the corresponding group. Since the squares at the right of the diagram in figure 2 are commutative, the maps σ and τ induce maps $\sigma^* : \mathcal{K}_{\mathbb{B}} \rightarrow \mathcal{K}_{\mathbb{N}}$ and $\tau^* : \mathcal{K}_{\mathbb{N}} \rightarrow \mathcal{K}_{\mathbb{F}}$ on the kernels.

$$\begin{array}{ccccc}
\mathcal{K}_{\mathbb{B}} \hookrightarrow & \text{Out}^0(G) & \xrightarrow{\rho_{\mathbb{B}}} & \prod_{v \in V} \text{Out}_{\mathcal{H}_v}(G_v) & \\
\sigma^* \downarrow & \sigma \downarrow & & \downarrow \sigma_J & \searrow \sigma_I \\
\mathcal{K}_{\mathbb{N}} \hookrightarrow & \text{Out}_{\mathcal{N}}(N) & \xrightarrow{\rho_{\mathbb{N}}} & \prod_{j \in J} \text{Out}_{G_0}(N_j) & \prod_{i \in I} \text{Out}(G_0) \\
\tau^* \downarrow & \tau \downarrow & & \downarrow \tau_J & \\
\mathcal{K}_{\mathbb{F}} \hookrightarrow & \text{Out}_{\mathcal{F}}(F) & \xrightarrow{\rho_{\mathbb{F}}} & \prod_{j \in J} \text{Out}(F_j) &
\end{array}$$

FIGURE 2. The diagram defining the maps σ^* and τ^* .

The following lemma extends, under certain conditions, an automorphism of a vertex stabilizer to a group acting on a tree, and will be our main tool for constructing automorphisms.

Lemma 3.2. *Γ be a group acting on a tree T and let v be a vertex of T . Let φ be an automorphism of Γ_v that acts by conjugation (in Γ_v) on each edge stabilizer contained in Γ_v . Then φ extends to an automorphism $\tilde{\varphi}$ of Γ . Moreover, one can choose $\tilde{\varphi}$ so that it acts by conjugation on each vertex stabilizer Γ_w for any vertex w not in the Γ -orbit of v .*

Proof. Let \mathbb{C} be the quotient graph of group and u be the projection of v in \mathbb{C} . For each edge $e \in EC$ such that $\alpha(e) = u$, choose $\gamma_e \in \mathbb{C}_u$ such that $\varphi|_{\mathbb{C}_e} = i_{\gamma_e}|_{\mathbb{C}_e}$. For all other edges $f \in EC$, let $\gamma_f = 1$. Letting X be the set of \mathbb{C} -paths, we define

$$\tilde{\varphi} : X \rightarrow X : a_0, e_1, a_1, \dots, e_n, a_n \mapsto \bar{a}_0, e_1, \bar{a}_1, \dots, e_n, \bar{a}_n$$

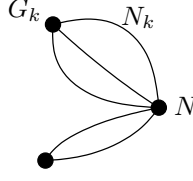
where

$$\bar{a}_i = \begin{cases} a_i & \text{if } a_i \notin \mathbb{C}_u \\ \gamma_{e_i}^{-1} \varphi(a_i) \gamma_{e_{i+1}} & \text{if } a_i \in \mathbb{C}_u \end{cases}$$

($\gamma_{e_0} = \gamma_{e_{n+1}} := 1$). Routine computations show that $\tilde{\varphi}$ induces a well defined endomorphism of $\Gamma = \pi_1(\mathbb{C}, u)$. Moreover, if we define $\tilde{\varphi}'$ similarly by extending $\varphi' = \varphi^{-1}$ and $\gamma'_e := \sigma^{-1}(\gamma_e^{-1})$ it can be checked that $\tilde{\varphi}\tilde{\varphi}' = \tilde{\varphi}'\tilde{\varphi} = \text{Id}_{\Gamma}$ so that $\tilde{\varphi}$ is an automorphism of Γ .

If w is a vertex of T with projection $x \neq u$ in \mathbb{C} , the vertex stabilizer Γ_w can be written as $[p\mathbb{C}_x p^{-1}]$ for some \mathbb{C} -path p starting at u and ending at x . It is clear that the restriction of $\tilde{\varphi}$ to Γ_w is conjugation by $[\tilde{\varphi}(p)p^{-1}]$. \square

Remark. If we are given the trivial automorphism of Γ_v and choose $\gamma_e = 1$ for all edges $e \in EC$ except for one edge f starting at u for which we take $\gamma_f \in \mathbb{Z}_{\mathbb{C}_u} \alpha_f(\mathbb{C}_f)$,

FIGURE 3. The decomposition \mathbb{B}' of G

then the automorphism constructed in the last lemma is called a *Dehn twist* along the edge f .

Corollary 3.3. *The map σ^* is surjective. If $\text{Out}(G_0)$ is finite, the image of $\rho_{\mathbb{B}}$ has finite index in $\prod_{v \in V} \text{Out}_{\mathcal{H}_v}(G_v)$.*

Proof. We introduce an auxiliary graph of group \mathbb{B}' obtained from \mathbb{B} as follows (see figure 3): collapse all loop edges of \mathbb{B} and slide N_k along each edge e_k , thus setting N to be the vertex group of b_0 . Let us show that σ^* is surjective. If $\hat{\varphi}$ is an outer automorphism in $\mathcal{K}_{\mathbb{N}}$, any representative φ of $\hat{\varphi}$ acts as a conjugation on each N_k so one can apply lemma 3.2 and extend φ to an automorphism ψ of G such that ψ acts as a conjugation on each G_v , so that ψ is in $\mathcal{K}_{\mathbb{B}}$ and $\sigma^*(\hat{\psi}) = \hat{\varphi}$.

If $\text{Out}(G_0)$ is finite the subgroup $\text{Out}^*(G_v)$ of $\text{Out}_{\mathcal{H}_v}(G_v)$ which acts as a conjugation on each edge stabilizer contained in G_v is of finite index in $\text{Out}_{\mathcal{H}_v}(G_v)$. Applying lemma 3.2 for each $v \in V$ yields a lift of any element in $\prod_{v \in V} \text{Out}^*(G_v)$, which finishes the proof. \square

3.3. The kernels of σ^* and τ^* . Let \mathbb{B}' be the graph of group defined in the proof of corollary 3.3. We describe the kernel of σ^* in terms of Dehn twists in that splitting of G . For $k \in K$ and $g_k \in Z_G(N_k)$, define the automorphism $D_{g_k}^k$ of G as the Dehn twist of \mathbb{B}' along the edge e_k and with element g_k . Clearly, $\widehat{D_{g_k}^k}$ lies in the kernel of σ^* , and $D_{g_k}^k$ commutes with $D_{g_{k'}}^{k'}$ if k is different from k' . This defines a homomorphism

$$q : \prod_{k \in K} Z_G(N_k) \rightarrow \ker \sigma^* : (g_k)_{k \in K} \mapsto \prod_{k \in K} \widehat{D_{g_k}^k}$$

Lemma 3.4. *The kernel of σ^* fits into the exact sequence*

$$1 \rightarrow Z(G) \rightarrow Z(N) \times \prod_{v \in V} Z(G_v) \xrightarrow{p} \prod_{k \in K} Z_G(N_k) \xrightarrow{q} \ker \sigma^* \rightarrow 1$$

where p embeds $Z(N)$ diagonally into $\prod_{k \in K} Z_G(N_k)$ and embeds $Z(G_v)$ diagonally into $\prod_{v_k=v} Z_G(N_k)$.

Proof. Let $\hat{\varphi}$ be an outer automorphism in the kernel of σ^* . By definition of σ^* we can choose a representative φ of $\hat{\varphi}$ that restricts to the identity on N . For each k, k' such that $v_k = v_{k'}$ we let $t_{kk'} = [1, e_k, 1, e_{k'}^{-1}, 1] \in G = \pi_1(\mathbb{B}', b_0)$. Note that if $k, k' \in K$ are such that $v_k = v_{k'}$, we have $G_{k'} = t_{k'k} G_k t_{k'k}^{-1}$. Therefore, if $g_k \in G$ is such that $\varphi|_{G_k} = i_{g_k}|_{G_k}$ then $\varphi|_{G_{k'}} = i_{\varphi(t_{k'k})g_k}|_{G_{k'}}$. Moreover, since φ restricts to the identity on N the element g_k must belong to $Z_G(N_k)$. Fix a tuple $(g_k)_{k \in K}$ such that

- (1) $\varphi|_{G_k} = i_{g_k}|_{G_k}$
- (2) $g_{k'} = \varphi(t_{k'k})g_k$ for each k, k' with $v_k = v_{k'}$

Notice that the set

$$\bigcup_{k \in K} G_k \bigcup N \bigcup \{t_{kk'} \mid v_k = v_{k'}\}$$

generates $G = \pi_1(\mathbb{B}', b_0)$. Therefore the tuple $(g_k)_{k \in K}$ defines φ from the fact that φ restricts to the identity on N and from properties (1) and (2). In conclusion, $\varphi = \prod_{k \in K} D_{g_k}^k$, and the kernel of σ^* can be viewed as a quotient of $\prod_{k \in K} Z_G(N_k)$.

Let now $(g_k) \in \prod_{k \in K} Z_G(N_k)$ be a tuple in the kernel of q . In other words $\varphi = \prod_{k \in K} D_{g_k}^k$ is an inner automorphism $i_n \in \text{Inn}(G)$. Since φ acts as the identity on N , the element n must lie in the centralizer of N , which equals the center of N as $G_0 \subset N$. Hence $\tilde{\varphi} := \prod_{k \in K} D_{n^{-1}g_k}^k \varphi$ is the identity on G . Therefore $\tilde{g}_k := n^{-1}g_k$ lies in the center of G_k for each k . It remains to show that if $v_k = v_{k'}$ then $\tilde{g}_k = \tilde{g}_{k'}$. Since φ is the identity on G , we have $\varphi(t_{kk'}) = [1, e_k, \tilde{g}_k \tilde{g}_{k'}^{-1}, e_{k'}^{-1}, 1] = [1, e_k, 1, e_{k'}^{-1}, 1] = t_{kk'}$ which implies that $\tilde{g}_k \tilde{g}_{k'}^{-1} = 1$.

The kernel of p is exactly those elements of $Z(N)$ which commute with all elements of the generating set of G described above. Therefore, the kernel of p is isomorphic to the center of G . \square

Remark. Note that for each $k \in K$, either $N_k = G_0$ in which case $Z_G(N_k) = Z_G(G_0)$, or $N_k \neq G_0$ and $Z_G(N_k) = Z(N_k)$ by lemma 2.1. Hence the exact sequence in lemma 3.4 can be rewritten as

$$1 \rightarrow Z(G) \rightarrow Z(N) \times \prod_{v \in V} Z(G_v) \xrightarrow{p} \prod_{i \in I} Z_G(G_0) \times \prod_{j \in J} Z(N_j) \xrightarrow{q} \ker \sigma^* \rightarrow 1$$

Let \mathcal{T} be the subgroup of the kernel of τ^* consisting of those outer automorphisms $\hat{\psi}$ that have a representative ψ that restricts to the identity on G_0 and that induce the identity on the quotient N/G_0 . It is clear that if $\text{Out}(G_0)$ is finite, \mathcal{T} has finite index in the kernel of τ^* . Let K_j denote the subgroup of N_j of those elements that act trivially by conjugation on N_j/G_0 .

If n_j is an element of $Z_{K_j}(G_0)$, let $D_{n_j}^j$ be the automorphism acting on N_j as conjugation by n_j and as the identity on N_j' for j' different from j and on $s \in S$. If n_s is an element of the center of G_0 and $s \in S$, let $D_{n_s}^s$ be the automorphism of N acting as the identity on N_j for each $j \in J$, fixing s' for each s' different from s and mapping s to $n_s s$. Clearly, $\widehat{D_{n_l}^l}$ lies in \mathcal{T} for each $l \in J \cup S$, and $D_{n_l}^l$ commutes with $D_{n_{l'}}^{l'}$ if l is different from l' . Therefore there is a map

$$g : \prod_{s \in S} Z(G_0) \times \prod_{j \in J} Z_{K_j}(G_0) \rightarrow \mathcal{T} : (n_l)_{l \in S \cup J} \mapsto \prod_{l \in S \cup J} \widehat{D_{n_l}^l}$$

Lemma 3.5. *g is surjective and $(1)_{s \in S} \times \prod_{j \in J} Z(N_j)$ lies in the kernel of g .*

Proof. Let $\hat{\psi}$ be an outer automorphism of N in \mathcal{T} , and choose a representative ψ of $\hat{\psi}$ that restricts to the identity on G_0 and such that $\tau(\psi)$ that acts as the identity on F . Choose elements n_j such that $\psi|_{N_j} = i_{n_j}|_{N_j}$. Since $\tau^*(\psi)$ acts as the identity on F , this means that $n_j \in Z_{N_j}(G_0) \cap K_j = Z_{K_j}(G_0)$. It also implies that there exist elements n_s in the center of G_0 such that $\psi(s) = n_s s$. Hence ψ is the product of the automorphisms $D_{n_l}^l$ where l ranges through $J \cup S$, and g is surjective.

If $n_j \in Z(N_j)$ then it is clear that $D_{n_j}^j = \text{Id}_N$. Hence $(1)_{s \in S} \times \prod_{j \in J} Z(N_j)$ lies in the kernel of g . \square

Corollary 3.6. *If G_0 is finite and N_j is finitely generated for each $j \in J$, then the kernel of τ^* is finite.*

Proof. In light of lemma 3.5, it suffices to show that if G_0 is finite and N_j is finitely generated, then $Z(N_j)$ has finite index in $Z_{K_j}(G_0)$. Let n_1, \dots, n_p be a finite generating set for N_j . Observe that $Z_{K_j}(G_0) = Z_{N_j}(G_0) \cap K_j$. Moreover, if $n \in N_j$

and $k \in K_j$, then the commutator $[n, k] := nkn^{-1}k^{-1}$ lies in G_0 by definition of K_j . For each $1 \leq i \leq p$, define a map $\beta_i : Z_{K_j}(G_0) \rightarrow G_0$ sending k to $[n_i, k]$. This is a homomorphism since

$$\beta_i(k)\beta_i(k') = n_i k n_i^{-1} k^{-1} (n_i k' n_i^{-1} k'^{-1}) = n_i k n_i^{-1} (n_i k' n_i^{-1} k'^{-1}) k^{-1} = \beta_i(kk')$$

The center of N_j is a subgroup of $Z_{K_j}(G_0)$ and is exactly the intersection of the kernels of all β_i , which are all finite index subgroups of $Z_{K_j}(G_0)$ since G_0 is finite. Hence $Z(N_j)$ has finite index in $Z_{K_j}(G_0)$ for each $j \in J$. \square

If G_0 is not finite or there is an N_j which is not finitely generated, there is still some control over \mathcal{T} , hence over $\ker \tau^*$ if $\text{Out}(G_0)$ is finite. Indeed, the following proposition holds:

Proposition 3.7 ([10, Proposition 3.1]). *If \mathbb{N} is minimal and not a mapping torus (i.e. $|J| + |S| \geq 2$) there is an exact sequence*

$$1 \rightarrow Z(N) \rightarrow Z(G_0) \times \prod_{s \in S} Z(G_0) \times \prod_{j \in J} Z(N_j) \xrightarrow{f} \prod_{s \in S^{\pm 1}} Z(G_0) \times \prod_{j \in J} Z_{K_j}(G_0) \xrightarrow{g} \mathcal{T} \rightarrow 1.$$

3.4. Labellings and the image of τ^* . Let $\chi : F \rightarrow \text{Out}(G_0)$ be the homomorphism defined by $f \mapsto \widehat{i_{\tilde{f}}|_{G_0}}$ for some lift $\tilde{f} \in N$ of f . In order to describe the image of τ^* we introduce the labelling of an outer automorphism $\hat{\psi} \in \mathcal{K}_{\mathbb{F}}$ as follows. First, choose a representative $\psi \in \hat{\psi}$. For each $j \in J$, choose an element f_j^ψ of F such that $\psi|_{F_j} = i_{f_j^\psi}|_{F_j}$ and let $f_s^\psi = \psi(s)$ for each $s \in S$. Such a choice of f_l^ψ for $l \in J \cup S$ is called a *defining tuple* for ψ . The *labelling* $\mathcal{L}(\hat{\psi})$ of $\hat{\psi}$ is the subset of $\prod_{l \in J \cup S} \text{Out}(G_0)$ consisting of all tuples $(\chi(f_l^\psi))_{l \in J \cup S}$ obtained from all defining tuples f_l^ψ for all representatives ψ of $\hat{\psi}$.

Lemma 3.8. *For any outer automorphisms $\hat{\psi}, \hat{\psi}'$ in $\mathcal{K}_{\mathbb{F}}$, either the labellings $\mathcal{L}(\hat{\psi})$ and $\mathcal{L}(\hat{\psi}')$ do not intersect or they are equal.*

Proof. Suppose that ψ is a representative of $\hat{\psi}$, and let $(f_l)_{l \in J \cup S}$ be a defining tuple for ψ . Then \tilde{f}_l is another defining tuple for ψ if and only if $f_j^{-1} \tilde{f}_j$ lies in the center of F_j for each $j \in J$ and $f_s = \tilde{f}_s$. Moreover, all other representatives of $\hat{\psi}$ are of the form $\bar{\psi} = i_g \psi$ for some $g \in F$, and in that case $\bar{f}_j := g f_j$ and $\bar{f}_s := g f_s g^{-1}$ is a defining tuple for $\bar{\psi}$.

Therefore, if $(\chi_l)_{l \in J \cup S}$ is a tuple in a labelling $\mathcal{L}(\hat{\psi})$ then all other tuples in $\mathcal{L}(\hat{\psi})$ are obtained by performing all finite sequences of the following operations:

- Replace $(\chi_l)_{l \in J \cup S}$ by $((\chi(g)\chi_j)_{j \in J}, (\chi(g)\chi_s\chi(g^{-1}))_{s \in S})$ For some $g \in F$.
- Replace the j th entry of $(\chi_l)_{l \in J \cup S}$ by $\chi_j\chi(g_j)$ for some element g_j in the center of F_j .

This finishes the proof as this way to obtain all tuples in a labelling from one given tuple does not depend on $\hat{\psi}$. \square

Lemma 3.9. *Any element of $\mathcal{K}_{\mathbb{F}}$ having the same labelling as the identity lies in the image of τ^* .*

Proof. Suppose that $\hat{\psi}$ has the same labelling as the identity. Thus there is a representative ψ of $\hat{\psi}$ and elements $f_l \in F$ such that $\chi(f_j) = \widehat{\text{Id}}|_{G_0}$ and such that $\chi(f_s) = \widehat{i_s}|_{G_0}$. Thus if n_l are lifts in N of f_l , then there are elements g_l in G_0 such that $i_{n_j g_j^{-1}}|_{G_0} = \text{Id}|_{G_0}$ and such that $i_{n_s g_s^{-1}}|_{G_0} = i_s|_{G_0}$. Define a homomorphism $\varphi : N \rightarrow N$ on a generating set of N as follows:

$$\varphi|_{N_j} = i_{n_j g_j^{-1}}|_{N_j} \text{ and } \varphi(s) = n_s g_s^{-1}$$

This choice ensures that the image of the generating set satisfy the relations presenting N so that this map extends to an endomorphism of N . Denoting by π the projection of N on $N/G_0 = F$, it follows from the construction of ϕ that $\pi \circ \varphi = \psi \circ \pi$. Therefore, $\pi(\ker \varphi) \subset \ker \psi = \{\text{Id}\}$ and $\pi(\text{im } \varphi) \supset \text{im } \psi = F$. Since φ is injective on G_0 , it follows that φ is an automorphism of N , and $\tau^*(\hat{\varphi}) = \hat{\psi}$ by construction. \square

Proposition 3.10. $\mathcal{K}_{\mathbb{F}}$ acts from the right on the set of labellings by the formula

$$\mathcal{L}(\hat{\psi}_1) \cdot \hat{\psi}_2 := \mathcal{L}(\hat{\psi}_1 \hat{\psi}_2)$$

Proof. Assume for a moment that this is well-defined. Then the formula indeed defines an action since

$$(\mathcal{L}(\hat{\psi}_1) \cdot \hat{\psi}_2) \cdot \hat{\psi}_3 = \mathcal{L}(\hat{\psi}_1 \hat{\psi}_2 \hat{\psi}_3) = \mathcal{L}(\hat{\psi}_1) \cdot (\hat{\psi}_2 \hat{\psi}_3)$$

Let us now show that the formula is well defined. In other words, we have to show that if $\hat{\psi}, \hat{\psi}'$ and $\hat{\psi}^*$ are outer automorphisms in $\mathcal{K}_{\mathbb{F}}$ such that $\hat{\psi}$ and $\hat{\psi}'$ have the same labelling, then $\hat{\psi} \hat{\psi}^*$ and $\hat{\psi}' \hat{\psi}^*$ have the same labelling. Let ψ, ψ', ψ^* and f_l, f'_l, f_l^* be defining tuples for $\hat{\psi}, \hat{\psi}', \hat{\psi}^*$ respectively, and suppose that $\chi(f_l) = \chi(f'_l)$ for each l . The automorphisms $\bar{\psi} := \psi \psi^*$ and $\bar{\psi}' := \psi' \psi^*$ are representatives of $\hat{\psi} \hat{\psi}^*$ and $\hat{\psi}' \hat{\psi}^*$ respectively. Computation shows that the elements $\bar{f}_j := \psi(f_j^*) f_j$ and $\bar{f}_s := \psi(f_s^*)$ are such that $\bar{\psi}|_{F_j} = i_{\bar{f}_j}|_{F_j}$ and such that $\bar{\psi}(s) = \bar{f}_s$. Similarly the elements $\bar{f}'_j := \psi'(f_j^*) f'_j$ and $\bar{f}'_s := \psi'(f_s^*)$ are a defining tuple for $\bar{\psi}'$.

Let us show that $\chi(\psi(g)) = \chi(\psi'(g))$ for any $g \in F$. It suffices to show it for a generating set of F . If g lies in F_j for some $j \in J$, then

$$\chi(\psi(g)) = \chi(f_j) \chi(g) \chi(f_j^{-1}) = \chi(f'_j) \chi(g) \chi(f_j^{-1}) = \chi(\psi'(g)).$$

If $g = s$ for some $s \in S$, then

$$\chi(\psi(g)) = \chi(f_s) = \chi(f'_s) = \chi(\psi'(g)).$$

It follows that $\chi(\bar{f}_l) = \chi(\bar{f}'_l)$. Thus the labellings of $\hat{\psi} \hat{\psi}^*$ and $\hat{\psi}' \hat{\psi}^*$ intersect, hence they must coincide by lemma 3.8. \square

Corollary 3.11. *The image of τ^* has finite index in $\mathcal{K}_{\mathbb{F}}$.*

Proof. Since $\text{Out}(G_0)$ is finite, then so is the set of labellings. An automorphism $\hat{\varphi}$ has the same labelling as the identity if and only if it is in the stabilizer of $\mathcal{L}(\text{Id})$ for the action defined in proposition 3.10. Hence, it follows from lemma 3.9 that the image of τ^* has finite index in $\mathcal{K}_{\mathbb{F}}$. \square

3.5. Conclusion. A short sequence $A \xrightarrow{\lambda} C \xrightarrow{\mu} B$ is *virtually exact* if $\lambda(A)$ is a finite index subgroup of $\ker \mu$ and if $\mu(C)$ has finite index in B . The results of this section can be summarized by the following

Proposition 3.12. *If we let $D = \ker \sigma^*$ and suppose condition (C2) holds, then the following short sequences*

$$\mathcal{K}_{\mathbb{B}} \hookrightarrow \text{Out}^0(G) \xrightarrow{\rho_{\mathbb{B}}} \prod_{v \in VB} \text{Out}_{\mathcal{H}_v}(G_v) \text{ and } D \hookrightarrow \mathcal{K}_{\mathbb{B}} \xrightarrow{\tau^* \circ \sigma^*} \mathcal{K}_{\mathbb{F}}$$

are virtually exact. If condition (C1) holds, $\text{Out}^0(G)$ has finite index in $\text{Out}_{\mathcal{H}}(G)$.

Proof. $\mathcal{K}_{\mathbb{B}} = \ker \rho_{\mathbb{B}}$ by definition. $\rho_{\mathbb{B}}$ is virtually surjective by the second part of corollary 3.3. Condition (C2) ensures that the normalizer of any edge group in a vertex group is finitely generated. Thus corollary 3.6 applies and $\ker \tau^*$ is finite. Therefore $D = \ker \sigma^*$ has finite index in $\ker(\tau^* \circ \sigma^*)$. Finally $\tau^* \circ \sigma^*$ is virtually surjective by combining the first part of corollary 3.3 with corollary 3.11.

The last assertion was mentioned when defining $\text{Out}^0(G)$. \square

Remark that lemma 3.4 describes the structure of the group of Dehn twists D . A presentation of $\mathcal{K}_{\mathbb{F}}$ is discussed in theorem 4.1.

4. FINITE PRESENTATION FOR $\text{Out}(G)$

Throughout this section, one should always keep in mind the following basic facts about finite presentations. If H is a finite index subgroup of G , then H is finitely presented if and only if G is. If $Q = G/N$ and both N and Q are finitely presented, then so is G . If G is a finitely presented group and N is a finitely generated normal subgroup, then G/N is finitely presented.

Using the notations from section 3 we have the following analogue of Fouxé-Rabinovitch's theorem:

Theorem 4.1. *$\mathcal{K}_{\mathbb{F}}$ is finitely presented provided that F_j is finitely presented for each $j \in J$.*

Proof. This is a slight variation of Gilbert's presentation [8] of the automorphism group of a free product. Let A be the preimage of $\mathcal{K}_{\mathbb{F}}$ in $\text{Aut}(F)$, that is the group of those automorphisms that act by conjugation on each F_j . By using the relations between factor automorphisms and the other generators, one shows that a generating set of A is obtained by dropping the automorphisms acting on one factor as a non-inner automorphism and the automorphisms permuting the factors. The peak reduction method goes through unchanged using only the relations not involving the dropped generators, showing that they are sufficient to present A . Since F is assumed to be finitely generated, then so is $\text{Inn}(F)$ and $\mathcal{K}_{\mathbb{F}} \cong A/\text{Inn}(F)$ is finitely presented. \square

To prove theorem 1.1, we need to introduce a relative version of it.

Theorem 4.2 (Relative version of theorem 1.1). *Let $(\mathbb{A}, G, \mathcal{H})$ be a triple as in section 3 satisfying condition (C1). Suppose the following holds:*

- (1) *For each vertex v of $T_{\mathbb{A}}$ and each group H in \mathcal{H} contained in G_v , the normalizer $N_{G_v} H$ of H in G_v is finitely presented, and its center $Z(N_{G_v} H)$ is finitely generated.*
- (2) *For each vertex v of $T_{\mathbb{A}}$, the group $\text{Out}_{\mathcal{H}_v}(G_v)$ of automorphisms relative to the family \mathcal{H}_v of elements of \mathcal{H} contained in G_v is finitely presented.*

Then $\text{Out}_{\mathcal{H}}(G)$ is finitely presented.

Proof. We proceed by induction on the number of edges of \mathbb{A} . If $|EA| = 0$ then \mathbb{A} is a single vertex, and the statement is trivially true by condition (2). Suppose $|EA| > 0$. Let \mathbb{B} be as in section 3. Recall that for every vertex $v \in VB$ we have that $\mathbb{A}(v)$ has fewer edges than \mathbb{A} . Moreover $(\mathbb{A}(v), G_v, \mathcal{H}_v)$ is a triple as in section 3 satisfying condition (C1) and conditions (1) and (2) hold for $\mathcal{A}(v)$ since they hold for \mathcal{A} . Hence induction applies and $\text{Out}_{\mathcal{H}_v}(G_v)$ is finitely presented for each $v \in V$.

Lemma 2.2 and condition (1) imply that N is finitely presented and that $Z(N_j)$ is finitely generated for each $j \in J$. The centralizer $Z_G(G_0)$ is finitely presented since it is a finite index subgroup of $N = N_G(G_0)$. Observe that subgroups of finitely generated abelian groups are still finitely generated abelian, and that finitely generated abelian groups are finitely presented. Therefore $Z(N_j)$ is finitely presented. Moreover $Z(G_v) \subset Z(N_k)$ whenever $N_k \subset G_v$, so that $Z(G_v)$ is finitely generated for each $v \in V$. If \mathbb{B} is a mapping torus (e.g. $|S| = 1$ and $K = \emptyset$) then $Z(N)$ is virtually infinite cyclic, and $Z(N)$ is finite otherwise. In particular, $Z(N)$ is finitely generated. Putting all of this together with the exact sequence in the remark following lemma 3.4, we get that $\ker \sigma^*$ is finitely presented.

Again by lemma 2.2 and condition (1), N_j is finitely presented for each $j \in J$. Hence the group $\mathcal{K}_{\mathbb{F}}$ is finitely presented by theorem 4.1.

Note that condition (1) implies condition (C2), so that the sequences in proposition 3.12 are virtually exact. By the second short sequence, $\mathcal{K}_{\mathbb{B}}$ is finitely presented. Using the first short sequence we get that $\text{Out}^0(G)$ is finitely presented as well. Since we assumed condition (C1) holds $\text{Out}_{\mathcal{H}}(G)$ contains $\text{Out}^0(G)$ as a finite index subgroup, hence $\text{Out}_{\mathcal{H}}(G)$ is finitely presented. \square

Remark. In light of lemma 2.3 theorem 1.1 is implied by its relative version.

5. GROUPS WITH INFINITE $\text{Out}(G)$

Similarly to section 5, we shall prove theorem 1.2 by induction on the number of edges of \mathbb{A} using the following relative version:

Theorem 5.1 (Relative version of theorem 1.2). *Let $(\mathbb{A}, G, \mathcal{H})$ be a triple as in section 3 satisfying condition (C1). Then $\text{Out}_{\mathcal{H}}(G)$ is infinite if and only if one of the following holds:*

- (1) *There is a vertex stabilizer G_v of T_A such that $\text{Out}_{\mathcal{H}_v}(G_v)$ is infinite;*
- (2) *There is a splitting of G as an amalgam $A *_C B$ over a finite group with $B \neq C$ such that the center of A has infinite index in the centralizer of C in A ;*
- (3) *There is a splitting of G as an HNN extension $A *_C \tilde{C}$ over a finite group such that the centralizer of \tilde{C} in A is infinite, where \tilde{C} is one of the two isomorphic copies of C in A given by the HNN extension.*

Proof. One implication is straightforward. Suppose that G satisfies condition (1). Let \mathcal{H}_v^* be the set of edge stabilizers of G contained in G_v . Let $\text{Out}^*(G_v)$ be the finite index subgroup of $\text{Out}_{\mathcal{H}_v}(G_v)$ acting on each element of \mathcal{H}_v^* as a conjugation in G_v . Using lemma 3.2 one can extend any outer automorphism φ of $\text{Out}^*(G_v)$ to an outer automorphism $e(\varphi)$ of G . As $\text{Out}(G_v)$ is infinite, then so is G_v . Hence G_v is its own normalizer in G . Recall the definition of ρ_v from section 3.2. By definition, e is such that $\rho_v(e(\varphi)) = \varphi$, so $\text{Out}(G)$ must be infinite. If G satisfies condition (2) or (3), one easily constructs infinitely many Dehn twists using the given splitting.

Let us show by induction on the the number of edges of \mathbb{A} that any accessible group satisfying the hypotheses of this theorem such that $\text{Out}_{\mathcal{H}}(G)$ is infinite must satisfy one of the conditions (1), (2) or (3). If $|EA| = 0$, this is obvious as \mathbb{A} consists of a single vertex. Suppose that $|EA| > 0$. Thus G is not one-ended and section 3 applies to \mathbb{A} . We use the notation from section 3 until the end of the proof. As we assumed condition (C1) $\text{Out}^0(G)$ has finite index in $\text{Out}_{\mathcal{H}}(G)$ and so $\text{Out}^0(G)$ is infinite. Combining corollaries 3.3 and 3.11 there are four possibilities for $\text{Out}^0(G)$ to be infinite (see figure 2): either one of the vertex groups of \mathbb{B} has infinite relative outer automorphism group; or $\ker \sigma^*$ is infinite; or $\ker \tau^*$ is infinite; or $\mathcal{K}_{\mathbb{F}}$ is infinite.

For any $v \in V$ the graph of group $\mathbb{A}(v)$ has strictly less edges than \mathbb{A} , so we know by induction that for any vertex $v \in VB$ with $\text{Out}_{\mathcal{H}_v}(G_v)$ infinite, G_v satisfies one of (1), (2) or (3). As splittings of G_v over finite groups can be extended to splittings of G , and since maximal elliptic subgroups of G_v are maximal elliptic subgroups of G acting on T_A , conditions (1), (2) and (3) can be lifted to G , so we can assume that all vertex groups of \mathbb{B} have finite relative automorphism group.

Suppose the kernel of σ^* is infinite. By lemma 3.4 only two cases might arise. Either there are $k \neq k' \in K$ such that $v_k = v_{k'}$ and such that $Z_G(N_k)$ is infinite. Thus collapsing all edges except e_k in \mathbb{B} yields an HNN-extension of G over G_0 with $Z_G(G_0)$ infinite and such that no nontrivial power of the stable letter normalizes G_0 , so that G satisfies (3). Or there is some k such that $Z(G_v)$ has infinite index in

$Z_G(N_k)$, yielding either an amalgam satisfying (2) or an HNN extension satisfying (3).

Suppose that the kernel of τ^* is infinite. Since $Z(G_0)$ is finite, lemma 3.5 implies that there is some $j \in J$ such that $Z(N_j)$ has infinite index in $Z_{K_j}(G_0)$. Note that $Z(G_j) \subset Z(N_j)$ and that $Z_{K_j}(G_0) \subset Z_{G_j}(G_0)$, so there is some $j \in J$ such that $Z(G_j)$ has infinite index in $Z_{G_j}(G_0)$. Therefore, collapsing all edges of \mathbb{B} except e_j yields a splitting of G satisfying (2).

Suppose now that $\mathcal{K}_{\mathbb{F}}$ is infinite. If the valence of b_0 is at least 3 in \mathbb{F} , either there is one non-loop edge and collapsing all other edges in \mathbb{B} produces an amalgam satisfying (2), or there are only loop edges and collapsing all but one in \mathbb{B} yields an HNN extension satisfying (3). If $|J| = 2$ and S is empty, then F is a free product $F_j * F_{j'}$ and $\mathcal{K}_{\mathbb{F}} \cong \text{Inn}(F_j) \times \text{Inn}(F_{j'})$. Without loss of generality, the center of F_j has infinite index in F_j , hence so does the center of N_j in N_j and $Z(G_j)$ in $N_{G_j}(G_0)$. Hence, we have the desired amalgam. If either S or J consist of a single element and the other is empty, $\mathcal{K}_{\mathbb{F}}$ is finite, so we have examined all cases. \square

Remark. Theorem 1.2 is implied by its relative version by lemma 2.3.

6. HYPERBOLIC GROUPS

Let us recall some facts about word hyperbolic groups:

Lemma 6.1. *Let Γ be a hyperbolic group. The following holds:*

- (1) Γ is finitely presented.
- (2) If G_0 is a finite subgroup of Γ , then the normalizer of G_0 in Γ is a quasiconvex subgroup of Γ . In particular, it is hyperbolic and thus finitely presented.
- (3) There are finitely many conjugacy classes of finite subgroups of Γ .
- (4) The center of Γ is either finite or virtually infinite cyclic.

Proof. (1) The first assertion is proved in [3, 5.2.3].

(2) By [15, 3.3] the centralizer of an element in Γ is quasi-convex. Hence the centralizer of any finite subgroup G_0 is quasiconvex, as it is the intersection of the centralizer of each element of G_0 , and so $Z_{\Gamma}(G_0)$ is the intersection of finitely many quasiconvex subgroups of Γ . Since $Z_{\Gamma}(G_0)$ is a finite index subgroup of $N_{\Gamma}(G_0)$ the latter is quasiconvex as well. (See [3, Chap. 10] for facts about quasi-convex subgroups of hyperbolic groups).

(3) The third assertion follows from the main theorem of [2].

(4) If there is some $g \in Z(\Gamma)$ of infinite order, then $Z(\Gamma) \subset Z_{\Gamma}(\langle g \rangle)$ is virtually infinite cyclic by [3, 10.7.2]. If not, then $Z(\Gamma)$ is an abelian torsion group. Since there is a bound on the order of a finite subgroup of Γ by assertion (3), any abelian torsion subgroup of Γ is finite. \square

Lemma 6.2. *If $G = A *_C$ (resp. $G = A *_C B$) where C is a finite group, then G is hyperbolic if and only if A is hyperbolic (resp. A and B are hyperbolic).*

Proof. Suppose we have an amalgam $G = A *_C B$. If A and B are hyperbolic and C is finite it can be seen directly from the Cayley graph that G is hyperbolic. It also follows from the (much stronger) combination theorem in [1]. Suppose now that G is hyperbolic. It is not hard to show that both A and B are quasi-convex subgroups of G , so that they are hyperbolic. The case of an HNN extension is identical. \square

Theorem 6.3. *Let Γ be a hyperbolic group. Then $\text{Aut}(\Gamma)$ is finitely presented.*

Proof. Since Γ is finitely presented, so by Dunwoody's accessibility [4] it admits a decomposition \mathbb{A} as a finite reduced graph of groups with vertex groups with at most one end and finite edge groups. Let \mathcal{H} be the family of edge stabilizers of $T_{\mathbb{A}}$.

Let us show that \mathbb{A} satisfies condition 2 of theorem 1.1. Let Γ_v be a vertex stabilizer of T_A . Applying lemma 6.2 repeatedly yields that Γ_v is a hyperbolic group. As Γ_v has at most one end, Levitt showed [10, Theorem 5.1] that $\text{Out}(\Gamma_v)$ is virtually an extension of $\prod_{x \in X} \text{Out}_{\mathcal{S}}(G_x)$ by a group which is virtually \mathbb{Z}^n , where $\text{Out}_{\mathcal{S}}(G_x)$ is the algebraic mapping class group of a hyperbolic 2-orbifold Σ_x , i.e. the group of outer automorphisms of $G_x = \pi_1(\Sigma_x)$ preserving the family \mathcal{S} of subgroups corresponding to boundary components. Following the proof in the orientable case [12], Fujiwara [7] proved in full generality that when Σ_x is a hyperbolic 2-orbifold, the algebraic mapping class group $\text{Out}_{\mathcal{S}}(G_x)$ is isomorphic to the geometric mapping class group $\text{Mod}^{\partial}(\Sigma_x)$ preserving the boundary componentwise. Moreover, he explains how to embed $\text{Mod}^{\partial}(\Sigma_x)$ as a finite index subgroup of the mapping class group of a surface $\text{Mod}^{\partial}(\Sigma'_x)$. Mapping class groups of surfaces are finitely presented by [9, 13]. Therefore $\text{Out}(\Gamma_v)$ is finitely presented. Let \mathcal{H}_v be the set of edge stabilizers of T_A contained in Γ_v . Since any hyperbolic group has only finitely many conjugacy class of finite subgroups, $\text{Out}_{\mathcal{H}_v}(\Gamma_v)$ has finite index in $\text{Out}(\Gamma_v)$ and so $\text{Out}_{\mathcal{H}_v}(\Gamma_v)$ is finitely presented.

\mathbb{A} also satisfies condition 1 of theorem 1.1. Indeed, any vertex stabilizer Γ_v is hyperbolic, and so the normalizer N of any finite group in Γ_v is hyperbolic and finitely presented. Moreover, as N is hyperbolic, its center is either finite or virtually infinite cyclic. In particular, it is finitely generated.

Thus \mathbb{A} satisfies the hypotheses of theorem 1.1, so $\text{Out}(\Gamma)$ is finitely presented. Moreover we have that

$$\text{Out}(\Gamma) \cong \frac{\text{Aut}(\Gamma)}{\Gamma/\mathbf{Z}(\Gamma)}$$

hence $\text{Aut}(\Gamma)$ is finitely presented as well. \square

Combining theorem 1.2 with the characterization of one-ended hyperbolic groups with infinite outer automorphism group in [10, Theorem 1.4], we get the following

Theorem 6.4. *Let Γ be a hyperbolic group. Then $\text{Out}(\Gamma)$ is infinite if and only if one of the following holds:*

- (1) Γ splits as an amalgam of groups with finite center over a virtually cyclic subgroup with infinite center;
- (2) Γ splits as an arbitrary HNN extension over a virtually cyclic group with infinite center;
- (3) Γ splits as an amalgam $A *_C B$ over a finite group with $B \neq C$ such that the center of A has infinite index in the centralizer of C in A ;
- (4) Γ splits as an HNN extension $A *_C$ over a finite group such that the centralizer of \tilde{C} in A is infinite, where \tilde{C} is one of the two isomorphic copies of C in A given by the HNN extension. \square

ACKNOWLEDGEMENTS

The author would like to thank Richard Weidmann for suggesting the problem and for his support during the preparation of this paper. He also profited from several discussions with Vincent Guirardel and Gilbert Levitt.

The author is grateful to Christopher Deninger and Linus Kramer for inviting him as a Marie Curie fellow at the *Universität Münster*, Germany, where part of this research was undertaken.

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