On ternary cubic forms that determine central simple algebras of degree 3

Mélanie Raczek

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Abstract

Fixing a field F of characteristic different from 2 and 3, we consider pairs (A, V) consisting of a degree 3 central simple F-algebra A and a 3-dimensional vector subspace V of the reduced trace zero elements of A which is totally isotropic for the trace quadratic form. Each such pair gives rise to a cubic form mapping an element of V to its cube; therefore we call it a cubic pair over F. Using the Okubo product in the case where F contains a primitive cube root of unity, and extending scalars otherwise, we give an explicit description of all isomorphism classes of such pairs over F. We deduce that a cubic form associated with an algebra in this manner determines the algebra up to (anti-)isomorphism.

Introduction

Consider a field F of characteristic different from 2. Let A be a quaternion algebra over F and let A^0 denote the subspace of reduced trace zero elements of A. Then for all $x \in A^0$ we have $x^2 \in F$. We thus obtain a quadratic form on A^0 mapping x to x^2. Up to the sign, this quadratic form is the norm form of the quaternion algebra restricted to A^0. By Theorem 2.5, p. 57, in [Lam, 2005], this quadratic form determines the quaternion algebra up to isomorphism.

In this paper we shall generalize this construction for algebras of degree 3. Consider a field F of characteristic different from 2 and 3 and let A be a degree 3 central simple algebra over F. Again let A^0 denote the subspace of reduced trace zero elements of A. Then the cube of an arbitrary element $x \in A^0$ need not be in F in general. In fact, it is in F if and only if the reduced trace of x^2 is equal to zero. Let $q\colon A^0 \to F$ be the trace quadratic form on A^0 (mapping x to the reduced trace of x^2). Then the Witt index of q is equal to 4 if F contains a primitive cube root of unity and is equal to 3 otherwise (see Lemma 0.1). In both cases there exist 3-dimensional subspaces of A^0 which are totally isotropic for the trace quadratic form. Each such vector subspace $V \subset A^0$ gives rise to a cubic form. In this paper we shall prove that this cubic form determines the algebra up to isomorphism or anti-isomorphism.

In the first two sections we shall give an explicit description of the pairs (A, V) where A and V are as above. In the first section we assume that the field F contains a primitive cube root of unity and we use the fact that we may write V in terms of the Okubo product. In the second section we assume that F does not contain a primitive cube root of unity, and we shall minimally extend the field F to use the results of the previous case. In the last section we use these descriptions to prove that a cubic form associated with a pair (A, V) determines A up to (anti-)isomorphism.

Throughout the paper, we denote by F a field of characteristic different from 2 and 3, by F_s a separable closure of F, and by Γ the absolute Galois group $\mathsf{Gal}(F_s/F)$. We fix $\omega \in F_s$ a primitive cube root of unity. We say that a pair (A, V) is a *cubic pair* over F if A is a degree 3 central simple F-algebra and V is a 3-dimensional subspace of A^0 (= the subspace of reduced trace zero elements of A) which is totally isotropic for the trace quadratic form. For a cubic pair (A, V) over F we define a cubic form

$$f_{A,V} : V \to F : x \mapsto x^3.$$

We say that $\Theta : (A, V) \to (B, W)$ is an *isomorphism of cubic pairs* over F if $\Theta : A \to B$ is an F-algebra isomorphism such that $\Theta(V) = W$. Note that if (A, V) and (B, W) are isomorphic then $f_{A,V}$ and $f_{B,W}$ are isometric (i.e. there exists an F-vector space isomorphism $\Theta : V \to W$ such that $f_{A,V} = f_{B,W} \circ \Theta$). For a field extension L over F we write A_L (resp. V_L) for $A \otimes_F L$ (resp. $V \otimes_F L$). Further we let Trd_A denote the reduced trace of A, and for an F-algebra K, we denote by Tr_K (resp. N_K) the trace (resp. the norm) of K.

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0. Some results on quadratic forms

Before we start the classification of cubic pairs, we need preliminary results on quadratic forms.

Let A be a degree 3 central simple algebra over F. First we shall compute the Witt index of the trace quadratic form of A.

Lemma 0.1 *Let q be the trace quadratic form on A^0. Then the Witt index of q is equal to 4 if F contains a primitive cube root of unity and is equal to 3 otherwise.*

Proof : There exists a splitting field L of A of odd degree over F. Indeed, we may choose $L := F$ if A is split and we choose a splitting field of degree 3 over F otherwise. Then straightforward computations show that the class of the form $q_L \colon A^0_L \to L$ is equal to the class of $2\langle 1, 3\rangle$ in the Witt ring of L. Hence, by Springer's Theorem about odd degree extensions (see Theorem 2.7, p. 194, in [Lam, 2005]), the Witt index of q is greater than or equal to 3 and it is 4 if and only if F contains a primitive cube root of unity. □

Suppose that F contains a primitive cube root of unity.

Lemma 0.2 *Let V be a 3-dimensional totally isotropic subspace of A^0. Then there exist exactly two maximal totally isotropic subspaces W_1, W_2 of A^0 containing V; thus $V = W_1 \cap W_2$.*

Proof : A more general fact is proved in III.1.11 of [Chevalley, 1997]: in a quadratic space with a Witt index equal to n, all $(n-1)$-dimensional totally isotropic subspaces are contained in exactly two maximal totally isotropic subspaces. □

1. Classification of cubic pairs over a field with a primitive cube root of unity

We assume that F contains a primitive cube root of unity.

1.1. Okubo product

Let A be a degree 3 central simple F-algebra. In [Knus *et al.*, 1998] the Okubo product over A^0 is defined as follows:

$$x \star y := \mu xy + (1-\mu)yx - \frac{1}{3}\mathsf{Trd}_A(xy) = \frac{1}{1-\omega}(yx - \omega xy) - \frac{1}{3}\mathsf{Trd}_A(xy)$$

where $\mu := \frac{1-\omega}{3}$. Let q denote the trace quadratic form on A^0. Because F contains a primitive cube root of unity, the Witt index of q is equal to 4. In [2008], Matzri interprets the results of van der Blij and Springer [1960] on triality, in the language of the Okubo product: he gives a description of the 4-dimensional subspaces of A which are totally isotropic for q, in terms of the Okubo product.

Theorem 1.1 (Matzri) *Let $u \in A^0 \setminus \{0\}$ be such that $\mathsf{Trd}_A(u^2) = 0$. Then $u \star A^0$ and $A^0 \star u$ are 4-dimensional totally isotropic subspaces of A^0. Moreover any 4-dimensional totally isotropic subspace is of this form.*

We may also write the 3-dimensional totally isotropic subspaces of A^0 in terms of the Okubo product. By Lemma 0.2, the 3-dimensional totally isotropic subspaces of A^0 are the intersections of two subspaces of the form $u \star A^0$ or $A^0 \star u$. We can be more precise using the following:

Theorem 1.2 (Matzri) *Let $u, v \in A^0 \setminus \{0\}$ be such that $\mathsf{Trd}_A(u^2) = 0$ and $\mathsf{Trd}_A(v^2) = 0$. Then*

1. *the dimension of $u \star A^0 \cap v \star A^0$ is even;*
2. *if $u \star v \neq 0$, then $\dim(u \star A^0 \cap A^0 \star v) = 1$;*
3. *if $u \star v = 0$, then $\dim(u \star A^0 \cap A^0 \star v) = 3$.*

Note that the Okubo product depends on the choice of the primitive cube root of unity. If we set, for a primitive cube root of unity ρ,

$$x \star_\rho y := \mu_\rho xy + (1 - \mu_\rho)yx - \frac{1}{3}\mathsf{Trd}_A(xy)$$

where $\mu_\rho := \frac{1-\rho}{3}$, then $x \star_\omega y = y \star_{\omega^2} x$. So by Theorem 1.2, the dimension of $A^0 \star u \cap A^0 \star v$ is also even.

Corollary 1.3 *Let (A, V) be a cubic pair over F. Then there exist nonzero $u, v \in A^0$ with $\mathsf{Trd}_A(u^2) = 0$, $\mathsf{Trd}_A(v^2) = 0$ and $u \star v = 0$ such that $V = u \star A^0 \cap A^0 \star v$.* □

The vectors u and v are in fact uniquely determined up to scalars. We shall prove this as a part of a more general statement:

Lemma 1.4 *If (B, W) is another cubic pair over F with $W = r \star B^0 \cap B^0 \star s$ as in Corollary 1.3, then an F-algebra isomorphism $\Theta\colon A \to B$ induces an isomorphism $\Theta\colon (A, V) \to (B, W)$ of cubic pairs if and only if $\Theta(u)F = rF$ and $\Theta(v)F = sF$.*

Proof: Assume that $\Theta\colon A \to B$ is an F-algebra isomorphism. If $\Theta(V) = W$, then $W = \Theta(u) \star B^0 \cap B^0 \star \Theta(v) = r \star B^0 \cap B^0 \star s$. By Lemma 0.2, we have

$$\{\Theta(u) \star B^0, B^0 \star \Theta(v)\} = \{r \star B^0, B^0 \star s\}.$$

If $r \star B^0 = B^0 \star \Theta(v)$, then $W = \Theta(u) \star B^0 \cap r \star B^0$ and by Theorem 1.2, the dimension of W is even. Hence $\Theta(u) \star B^0 = r \star B^0$ and $B^0 \star \Theta(v) = B^0 \star s$. By Theorem 2.10 in [Matzri, 2008], we then have $\Theta(u)F = rF$ and $\Theta(v)F = sF$. The converse is obvious. □

1.2. Classification

We shall describe a cubic pair (A, V) over F up to isomorphism. By Corollary 1.3, there exist nonzero $u, v \in A^0$ such that $\mathsf{Trd}_A(u^2) = 0$, $\mathsf{Trd}_A(v^2) = 0$, $u \star v = 0$ and $V = u \star A^0 \cap A^0 \star v$. Since $\mathsf{Trd}_A(v) = \mathsf{Trd}_A(v^2) = 0$ we have $v^3 \in F$, and similarly $u^3 \in F$. Note that $u \star v = 0$ implies that

$$vu = \frac{1-\omega}{3}\mathsf{Trd}_A(uv) + \omega uv.$$

Set $t := uv - \frac{1}{3}\mathsf{Trd}_A(uv)$. Then

$$tu = uvu - \frac{1}{3}\mathsf{Trd}_A(uv)u = \frac{1-\omega}{3}\mathsf{Trd}_A(uv)u + \omega u^2 v - \frac{1}{3}\mathsf{Trd}_A(uv)u = \omega ut$$

and similarly $vt = \omega tv$. We deduce that $t^3 \in F$. Indeed,

$$\begin{aligned} t^2 &= t\big(uv - \frac{1}{3}\mathsf{Trd}_A(uv)\big) \\ &= \omega utv - \frac{1}{3}\mathsf{Trd}_A(uv)t \\ &= \omega u^2v^2 + \frac{\omega^2}{3}\mathsf{Trd}_A(uv)uv + \frac{1}{9}\mathsf{Trd}_A(uv)^2 \end{aligned}$$

and thus

$$\begin{aligned} t^3 &= t^2\big(uv - \frac{1}{3}\mathsf{Trd}_A(uv)\big) \\ &= \omega^2 ut^2 v - \frac{1}{3}\mathsf{Trd}_A(uv)t^2 \\ &= u^3v^3 - \frac{1}{27}\mathsf{Trd}_A(uv)^3 \in F. \end{aligned} \tag{1}$$

This implies in particular that $\mathsf{Trd}_A(t^2) = 0$, so $\mathsf{Trd}_A(u^2v^2) = \frac{1}{3}\mathsf{Trd}_A(uv)^2$.

We shall prove that $t^2, t^2u, t^2v \in V$. First we observe that

$$u \star A^0 = \{x \in A^0 \mid x \star u = 0\}.$$

Indeed, by Proposition (34.19) in [Knus *et al.*, 1998] we have $(u \star x) \star u = \frac{1}{6}\mathsf{Trd}_A(u^2)x = 0$ for all $x \in A^0$. Hence

$$u \star A^0 \subset \{x \in A^0 \mid x \star u = 0\} = \mathsf{ker}(R_u)$$

where $R_u \colon A^0 \to A^0 \colon x \mapsto x \star u$. But $\dim \mathsf{ker}(R_u) + \dim \mathsf{im}(R_u) = 8$, thus $\dim \mathsf{ker}(R_u) = \dim(u \star A^0) = 4$. Similarly,

$$A^0 \star v = \{x \in A^0 \mid v \star x = 0\}.$$

One can see that $t^2 \in V$ since $ut^2 = \omega t^2 u$ and $t^2 v = \omega v t^2$. Also $u(ut^2) = \omega(ut^2)u$ and $(vt^2)v = \omega v(vt^2)$ imply $(ut^2) \star u = 0$ and $v \star (vt^2) = 0$. Now

$$\begin{aligned}
v \star (ut^2) &= \frac{1}{1-\omega}(ut^2 v - \omega v u t^2) - \frac{1}{3}\mathsf{Trd}_A(ut^2 v) \\
&= \frac{1}{1-\omega}\Big(ut^2 v - \omega \frac{1-\omega}{3}\mathsf{Trd}_A(uv)t^2 - \omega^2 uvt^2\Big) - \frac{1}{3}\mathsf{Trd}_A(ut^2 v) \\
&= ut^2 v - \frac{\omega}{3}\mathsf{Trd}_A(uv)t^2 - \frac{1}{3}\mathsf{Trd}_A(ut^2 v) \\
&= \frac{\omega^2}{27}\mathsf{Trd}_A(uv)^3 - \frac{\omega^2}{9}\mathsf{Trd}_A(uv)\mathsf{Trd}_A(u^2v^2) \\
&= 0.
\end{aligned}$$

Since $(vt^2) \star u = \omega^2 v \star (ut^2) = 0$, we obtain that $ut^2, vt^2 \in V$. So $t^2, t^2u, t^2v \in V$ because $vt = \omega tv$ and $tu = \omega ut$.

To work out the classification of cubic pairs we shall distinguish different situations:

First case: We assume that $u, v, t \in A^\times$. Observe that $t = \frac{1}{1-\omega}(uv - vu) \neq 0$, hence u and v are linearly independent. Since $\mathsf{Trd}_A(u) = \mathsf{Trd}_A(v) = 0$, the vectors $1, u, v$ are also linearly independent. Therefore t^2, t^2u, t^2v span V. Set $\xi := t^2$, $\eta := t^2 v$ and $\lambda := \frac{1}{3t^3}\mathsf{Trd}_A(uv)$. Because $t = uv - \frac{1}{3}\mathsf{Trd}_A(uv)$, we have $u = tv^{-1} + \frac{1}{3}\mathsf{Trd}_A(uv)v^{-1}$. One can check that

$$t^2 u = \frac{\omega}{v^3 t^3}\Big(\xi\eta^2 + \lambda\xi^2\eta^2\Big).$$

Finally A is the symbol algebra $(a,b)_{\omega,F}$ generated by ξ and η such that $\xi^3 = a$, $\eta^3 = b$, $\xi\eta = \omega\eta\xi$, and V is the vector subspace spanned by ξ, η and $\xi\eta^2 + \lambda\xi^2\eta^2$ where $1 + \lambda^3 a \neq 0$ since $u^3 \neq 0$. In this basis of V, the cubic form $f_{A,V}$ takes the generalized Hesse normal form:

$$f_{A,V}\big(x\xi + y\eta + z(\xi\eta^2 + \lambda\xi^2\eta^2)\big) = ax^3 + by^3 + ab^2(1 + \lambda^3 a)z^3 - 3\omega^2 ab\lambda xyz.$$

The form $f_{A,V}$ is nonsingular.

Conversely, suppose that B is the symbol algebra $(a,b)_{\omega,F}$ generated by ξ, η such that $\xi^3 = a$, $\eta^3 = b$, $\xi\eta = \omega\eta\xi$, and W the vector subspace spanned by ξ, η, $\xi\eta^2 + \lambda\xi^2\eta^2$, for some $a, b \in F^\times$ and $\lambda \in F$ such that $1 + \lambda^3 a \neq 0$. Then one can check that (B, W) is a cubic pair over F such that $f_{B,W}$ is nonsingular.

Second case: We suppose that $u, v \in A^\times$ and $t = 0$. Then $uv = \frac{1}{3}\mathsf{Trd}_A(uv) \in F^\times$. Thus we may assume that $u = v^2$. We need the following:

Lemma 1.5 *Let $\xi \in A^0$ be such that $\xi^3 \in F^\times$. Then there exists $\eta \in A$ such that $\eta^3 \in F^\times$ and $\xi\eta = \omega\eta\xi$.*

Proof: Assume that $\xi^3 \notin F^{\times 3}$, then $F(\xi)$ is a subfield of A. Let $\sigma\colon F(\xi) \to F(\xi)$ be the F-automorphism defined by $\sigma(\xi) = \omega^2\xi$. By the Skolem-Noether Theorem, there exists $\eta \in A^\times$ such that $\eta x \eta^{-1} = \sigma(x)$ for all $x \in F(\xi)$. In particular $\xi\eta = \omega\eta\xi$. Because $\eta = \omega\xi^{-1}\eta\xi$ and $\eta^2 = \omega^2\xi^{-1}\eta^2\xi$, we have $\mathsf{Trd}_A(\eta) = 0$, $\mathsf{Trd}_A(\eta^2) = 0$; so $\eta^3 \in F^\times$.

Now we suppose that $\xi^3 \in F^{\times 3}$. Then we may assume that $\xi^3 = 1$ and $A = \mathsf{M}_3(F)$. The minimal polynomial of ξ divides $t^3 - 1$, so ξ is diagonalizable and its eigen values are cube roots of unity. Hence we may assume that

$$\xi = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

with $\lambda_i \in \{1, \omega, \omega^2\}$. Since $\mathsf{tr}(\xi) = 0$ we have $\{\lambda_1, \lambda_2, \lambda_3\} = \{1, \omega, \omega^2\}$. Conjugating by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

if necessary, we may assume that $\lambda_1 = 1$, $\lambda_2 = \omega$, $\lambda_3 = \omega^2$. Then

$$\eta := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

is such that $\eta^3 \in F^\times$ and $\xi\eta = \omega\eta\xi$. $\square$

Let $w \in A$ be such that $w^3 \in F^\times$ and $vw = \omega wv$. Then the subspace of the elements x in A such that $vx = \omega^2 xv$ is spanned by w^2, vw^2, v^2w^2, and it is contained in V; therefore

$$V = \{x \in A \mid vx = \omega^2 xv\}.$$

Set $\xi := w^2$ and $\eta := vw^2$, then A is the symbol algebra $(a, b)_{\omega,F}$ generated by ξ and η such that $\xi^3 = a$, $\eta^3 = b$ and $\xi\eta = \omega\eta\xi$, and V is the vector subspace spanned by ξ, η and $\xi^2\eta^2$. In this basis of V, the form $f_{A,V}$ takes the generalized Hesse normal form:

$$f_{A,V}(x\xi + y\eta + z\xi^2\eta^2) = ax^3 + by^3 + a^2b^2z^3 - 3\omega^2 abxyz.$$

The form $f_{A,V}$ is singular; more precisely, it is *triangular*, i.e., there exist linearly independent forms $\varphi_1, \varphi_2, \varphi_3 \in (V \otimes_F F_s)^\star$ such that $f_{A,V} = \varphi_1\varphi_2\varphi_3$ as a cubic form over $V \otimes_F F_s$.

Conversely, suppose that B is the symbol algebra $(a, b)_{\omega,F}$ generated by ξ, η such that $\xi^3 = a$, $\eta^3 = b$, $\xi\eta = \omega\eta\xi$, and W is the vector subspace spanned by

ξ, η, $\xi^2\eta^2$, for some $a, b \in F^\times$. Then (B, W) is a cubic pair over F and $f_{B,W}$ is triangular.

Third case: We suppose that either $u \notin A^\times$, or $v \notin A^\times$, or $t \notin A^\times$ and $t \neq 0$. Then the algebra A is split, so we may assume that $A = \mathsf{M}_3(F)$. This case is less interesting and thus we shall not give an explicit description of the pair (A, V), but we shall only prove that the cubic form $f_{A,V}$ is singular and not triangular (it is possible to describe V by matrix computations distinguishing several cases; details can be found in [Raczek, 2007]).

To show that $f_{A,V}$ is not triangular, we first prove a more general Lemma on triangular forms.

Lemma 1.6 *Suppose that (B, W) is any cubic pair over F such that $f_{B,W}$ is triangular. Then $W = s^2 \star B^0 \cap B^0 \star s$ for some $s \in B^0$ such that s is invertible.*

Proof: We may assume that $F = F_s$ and $B = \mathsf{M}_3(F)$. Let $e_1, e_2, e_3 \in W$ be such that

$$f_{B,W}(x_1e_1 + x_2e_2 + x_3e_3) = x_1x_2x_3$$

for all $x_1, x_2, x_3 \in F$. Observe that $x^3 = \frac{1}{3}\mathsf{tr}(x^3)$ for all $x \in W$, hence $f_{B,W}(x_1e_1 + x_2e_2 + x_3e_3)$ is equal to

$$\sum_{i=1}^{3} e_i^3 x_i^3 + \sum_{i \neq j} \mathsf{tr}(e_i^2 e_j) x_i^2 x_j + \mathsf{tr}(e_1e_2e_3 + e_2e_1e_3)x_1x_2x_3$$

for all $x_1, x_2, x_3 \in F$. We deduce that $\mathsf{tr}(e_i^2 e_j) = 0$ for all i, j. Set $e_2 = (x_{ij})$ and $e_3 = (y_{ij})$.
Suppose that $e_1^2 \neq 0$. Since $e_1^3 = 0$, we may assume that

$$e_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Because $\mathsf{tr}(e_2) = 0$, $\mathsf{tr}(e_1e_2) = 0$ and $\mathsf{tr}(e_1^2 e_2) = 0$, we have

$$x_{33} = -x_{11} - x_{22}, \quad x_{32} = -x_{21}, \quad x_{31} = 0.$$

From $\mathsf{tr}(e_1e_2^2) = 0$ we deduce that $x_{21}(2x_{11} + x_{22}) = 0$. If $x_{21} = 0$ then $\mathsf{tr}(e_2^2) = 0$ and $e_2^3 = 0$ imply $x_{11} = x_{22} = 0$. Then $e_1e_2 + e_2e_1 = (x_{12} + x_{23})e_1^2$ and it contradicts the fact that $\mathsf{tr}(e_1^2 e_3) = 0$ and $\mathsf{tr}(e_1e_2e_3 + e_2e_1e_3) = 1$. If $x_{22} = -2x_{11}$ then

$$e_1e_2 + e_2e_1 = \begin{pmatrix} x_{21} & -x_{11} & x_{12} + x_{23} \\ 0 & 0 & -x_{11} \\ 0 & 0 & -x_{21} \end{pmatrix}.$$

By symmetry we know that

$$e_3 = \begin{pmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & -2y_{11} & y_{23} \\ 0 & -y_{21} & y_{11} \end{pmatrix},$$

thus $\mathsf{tr}(e_1e_2e_3 + e_2e_1e_3) = 1$ is impossible.
Therefore $e_1^2 = 0$ (by symmetry we also have $e_2^2 = 0$, $e_3^2 = 0$) and we may assume that

$$e_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\mathsf{tr}(e_2) = 0$ and $\mathsf{tr}(e_1e_2) = 0$, we have $x_{33} = -x_{11} - x_{22}$ and $x_{31} = 0$. Then $e_2^2 = 0$ implies $x_{21}x_{32} = 0$. Observe that

$$e_1e_2 + e_2e_1 = \begin{pmatrix} 0 & x_{32} & -x_{22} \\ 0 & 0 & x_{21} \\ 0 & 0 & 0 \end{pmatrix}.$$

Because $\mathsf{tr}(e_1e_2e_3 + e_2e_1e_3) = 1$ we have either $x_{21} = 0 = y_{32}$ and $x_{32}, y_{21} \neq 0$ or $x_{21}, y_{32} \neq 0$ and $x_{32} = 0 = y_{21}$. Thus we may assume that $x_{21} = 0$ and $x_{32} = 1$. From $e_2^2 = 0$ we deduce that

$$e_2 = \begin{pmatrix} 0 & \alpha & -\alpha\beta \\ 0 & \beta & -\beta^2 \\ 0 & 1 & -\beta \end{pmatrix}$$

for some $\alpha, \beta \in F$. We may assume that $\alpha = \beta = 0$ since the invertible matrix

$$m = \begin{pmatrix} 1 & 0 & -\alpha \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{pmatrix}$$

is such that $me_1m^{-1} = e_1$ and

$$me_2m^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Similarly we see that

$$e_3 = \begin{pmatrix} \alpha & -\alpha^2 & 0 \\ 1 & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for some $\alpha \in F$. Again we may assume that $\alpha = 0$ conjugating by

$$\begin{pmatrix} 1 & -\alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

if necessary.
Then one can check that $W = s^2 \star B^0 \cap B^0 \star s$ with

$$s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

$\square$

In our case, the subspace V is equal to $u \star A^0 \cap A^0 \star v$ where either $u \notin A^\times$, or $v \notin A^\times$, or $t \notin A^\times$ and $t \neq 0$. Observe that if $u, v \in A^\times$, then $uF = v^2F$ if and only if $t = 0$. Thus, by the previous Lemma, the form $f_{A,V}$ is not triangular.

To prove that $f_{A,V}$ is singular we shall distinguish different cases.

1. Suppose that $v \notin A^\times$.
If $\mathsf{tr}(uv) \neq 0$ then, by the relation (1), t is invertible; hence V is spanned by t^2, t^2u and t^2v. Since $\mathsf{tr}\big(x(t^2v)^2\big) = 0$ for all $x \in V$, the point t^2vF_s of the projective plane $\mathbb{P}_V(F_s)$ is a singular zero of $f_{A,V}$.
If $v^2 \neq 0$ and $\mathsf{tr}(uv) = 0$, then $v^2 \in V$ and v^2F_s is a singular zero of $f_{A,V}$.
If $v^2 = 0$, then we may assume that

$$v = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \text{ so } u = \begin{pmatrix} \omega^2\alpha_1 & \alpha_2 & \alpha_3 \\ 0 & \omega\alpha_1 & \alpha_4 \\ 0 & 0 & \alpha_1 \end{pmatrix}$$

for some $\alpha_i \in F$. If $\alpha_1 = 0$ then vF_s is a singular zero of $f_{A,V}$. If $\alpha_1 \neq 0$ then V is spanned by

$$\begin{pmatrix} \alpha_1 & 0 & -\alpha_3 \\ 0 & \omega\alpha_1 & 0 \\ 0 & 0 & \omega^2\alpha_1 \end{pmatrix}, \begin{pmatrix} 0 & (\omega-1)\alpha_1 & \alpha_4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & \alpha_2 \\ 0 & 0 & (\omega-\omega^2)\alpha_1 \\ 0 & 0 & 0 \end{pmatrix}$$

and the cubic curve associated with $f_{A,V}$ is a triple line.

2. Suppose that $u \notin A^\times$, then, by symmetry, we deduce that $f_{A,V}$ is also singular.

3. Suppose that $u, v \in A^\times$, $t \notin A^\times$ and $t \neq 0$.
If $t^2 \neq 0$ then t^2F_s is a singular zero of $f_{A,V}$.
If $t^2 = 0$, we shall prove that there exists a nonzero $s \in A$ such that $s^2 = 0$, $vs = \omega^2sv$, $s(tv^{-1}) = (tv^{-1})s = 0$. Since $u = tv^{-1} + \frac{1}{3}\mathsf{tr}(uv)v^{-1}$, we then have $s \in V$ and so sF_s is a singular zero of $f_{A,V}$. Let $w \in A$ be such that $w^3 \in F^\times$ and $vw = \omega wv$. Since $v(tv^{-1}) = \omega(tv^{-1})v$ and $tv^{-1} \neq 0$, there exist $\alpha_i \in F$ not all zero such that $tv^{-1} = \alpha_0w + \alpha_1vw + \alpha_2v^2w$. But $(tv^{-1})^2 = \omega^2t^2v^{-2} = 0$, so $\alpha_0 \neq 0$, $\alpha_2 = \alpha_1^2\alpha_0^{-1}$ and $\alpha_0^3 = v^3\alpha_1^3$. Hence $v^3 \in F^{\times 3}$ and we may assume that

$v^3 = 1$. Replacing w by $\alpha_0^{-1}w$ if necessary, we may assume that $\alpha_0 = 1$. Then $\alpha_1 = 1$, ω or ω^2. Conjugating by w or w^{-1} if necessary, we may assume that $tv^{-1} = w + vw + v^2w$. Then we may choose $s = w^2 + vw^2 + v^2w^2$.

We summarize the above classification in the following Theorem.

Theorem 1.7 *Suppose that F contains a primitive cube root of unity. Let (A, V) be a cubic pair over F.*

1. *If $f_{A,V}$ is nonsingular, then*

$$(A, V) \cong \big((a, b)_{\omega,F}, \mathsf{span}_F\langle \xi, \eta, \xi\eta^2 + \lambda\xi^2\eta^2\rangle\big)$$

for some $a, b \in F^\times$, $\lambda \in F$ such that $1+\lambda^3 a \neq 0$, where ξ, η are generators of the symbol algebra such that $\xi^3 = a$, $\eta^3 = b$, $\xi\eta = \omega\eta\xi$. Conversely, let $a, b \in F^\times$, $\lambda \in F$ be such that $1 + \lambda^3 a \neq 0$. Let B be the symbol algebra $(a, b)_{\omega,F}$ generated by ξ, η such that $\xi^3 = a$, $\eta^3 = b$, $\xi\eta = \omega\eta\xi$, and W the subspace spanned by ξ, η, $\xi\eta^2 + \lambda\xi^2\eta^2$. Then (B, W) is a cubic pair over F and $f_{B,W}$ is nonsingular. In the basis $(\xi, \eta, \xi\eta^2 + \lambda\xi^2\eta^2)$, the form $f_{B,W}$ takes the generalized Hesse normal form:

$$\big(x\xi + y\eta + z(\xi\eta^2 + \lambda\xi^2\eta^2)\big)^3 = ax^3 + by^3 + ab^2(1 + \lambda^3 a)z^3 - 3\omega^2 ab\lambda xyz.$$

2. *If $f_{A,V}$ is triangular, then*

$$(A, V) \cong \big((a, b)_{\omega,F}, \mathsf{span}_F\langle \xi, \eta, \xi^2\eta^2\rangle\big)$$

for some $a, b \in F^\times$, where ξ, η are generators of the symbol algebra such that $\xi^3 = a$, $\eta^3 = b$ and $\xi\eta = \omega\eta\xi$. Conversely, let $a, b \in F^\times$, let B be the symbol algebra $(a, b)_{\omega,F}$ generated by ξ, η such that $\xi^3 = a$, $\eta^3 = b$, $\xi\eta = \omega\eta\xi$, and W the subspace spanned by ξ, η, $\xi^2\eta^2$. Then (B, W) is a cubic pair over F and $f_{B,W}$ is triangular. In the basis $(\xi, \eta, \xi^2\eta^2)$, the form $f_{B,W}$ takes the generalized Hesse normal form:

$$(x\xi + y\eta + z\xi^2\eta^2)^3 = ax^3 + by^3 + a^2b^2z^3 - 3\omega^2 abxyz.$$

3. *If $f_{A,V}$ is singular and not triangular, then A is split.*

2. Classification of cubic pairs over a field without primitive cube root of unity

Suppose that F does not contain a primitive cube root of unity. We shall give the classification of cubic pairs over F in the case where the associated cubic form is nonsingular or triangular. For the remaining cases we know by Theorem 1.7 that the algebra is split and it is just a matter of matrix computations to describe all the possible subspaces up to conjugacy (see [Raczek, 2007] for details). We shall extend the scalars to $F(\omega)$ to use the previous classification. To simplify notations, let T denote the reduced trace of $A_{F(\omega)}$. Throughout this section, we denote by σ the F-automorphism of $F(\omega)$ such that $\sigma(\omega)=\omega^2$.

2.1. Nonsingular form

Let (A,V) be a cubic pair over F such that $f_{A,V}$ is nonsingular. By Subsection 1.2, there exist nonzero $u,v\in A^0_{F(\omega)}$ such that $\mathsf{T}(u^2)=0$, $\mathsf{T}(v^2)=0$, $u\star v=0$ and

$$V_{F(\omega)}=\Big(u\star A^0_{F(\omega)}\Big)\cap\Big(A^0_{F(\omega)}\star v\Big)$$

where $u,v,t:=uv-\frac{1}{3}\mathsf{T}(uv)\in A^\times_{F(\omega)}$. Then $V_{F(\omega)}$ is spanned by t^2, t^2v and t^2u.

We extend σ to an F-automorphism of $A_{F(\omega)}$: for $x\in A_{F(\omega)}$ and $\lambda\in F(\omega)$, define $\sigma(x\otimes\lambda)=x\otimes\sigma(\lambda)$. Then A (resp. V) consists of the elements of $A_{F(\omega)}$ (resp. $V_{F(\omega)}$) which are fixed under σ. Note that $\sigma(x\star y)=\sigma(y)\star\sigma(x)$ for all $x,y\in A_{F(\omega)}$. Since $\sigma(V_{F(\omega)})=V_{F(\omega)}$, we have

$$\Big(\sigma(v)\star A^0_{F(\omega)}\Big)\cap\Big(A^0_{F(\omega)}\star\sigma(u)\Big)=\Big(u\star A^0_{F(\omega)}\Big)\cap\Big(A^0_{F(\omega)}\star v\Big).$$

Hence there exists $\lambda\in F(\omega)^\times$ such that $\sigma(u)=\lambda v$. Replacing v by λv if necessary we may assume that $\sigma(u)=v$ and thus $\sigma(v)=u$. Recall that

$$vu=\frac{1-\omega}{3}\mathsf{T}(uv)+\omega uv,$$

hence we have

$$\begin{aligned}\sigma(t)&=\sigma(uv)-\frac{1}{3}\sigma\big(\mathsf{T}(uv)\big)\\&=\sigma(u)\sigma(v)-\frac{1}{3}\mathsf{T}\big(\sigma(uv)\big)\\&=vu-\frac{1}{3}\mathsf{T}(vu)\\&=\omega t.\end{aligned}$$

Therefore $\omega^2t\in A$ and $\omega t^2\in V$. Set $e:=\omega t^2$, then V is spanned by e, $e(v+u)$ and $e(\omega v+\omega^2u)$.

We shall find a Galois $\mathbb{Z}/3\mathbb{Z}$-algebra (L,ρ) such that $L\subset A$, the vector e and L generate A, and $ex=\rho(x)e$ for all $x\in L$. To do this we first construct an element $\eta\in A_{F(\omega)}$ such that $e\eta=\omega\eta e$, $\eta^3\in F(\omega)^\times$ and $\sigma(\eta)\eta\in F^\times$. Recall that $t^3=u^3v^3-\frac{1}{27}\mathsf{T}(uv)^3$, $vt=\omega tv$, $tu=\omega ut$ and

$$t^2=\omega u^2v^2+\frac{\omega^2}{3}\mathsf{T}(uv)uv+\frac{1}{9}\mathsf{T}(uv)^2.$$

Set

$$\eta := \frac{\mathsf{T}(uv)}{3}v + \frac{1}{t^3}e^2v,$$

then $e\eta = \omega\eta e$, $\eta^3 = u^3v^6$ and

$$\begin{aligned}\sigma(\eta)\eta &= \frac{\mathsf{T}(uv)^2}{9}uv - \frac{\omega\mathsf{T}(uv)}{3}utv + \omega^2ut^2v \\ &= u^3v^3.\end{aligned}$$

We set $\lambda := u^3v^3$, so $\eta + \lambda\eta^{-1}$ is fixed under σ.

If $\eta^3 \notin F(\omega)^{\times 3}$, then set $L := F(\eta + \lambda\eta^{-1})$ and let $\rho\colon L \to L$ be the F-automorphism defined by $\rho(\eta + \lambda\eta^{-1}) = \omega\eta + \omega^2\lambda\eta^{-1}$. Then L is a cyclic extension of degree 3 over F which is contained in A and with a Galois group generated by ρ. Moreover $ex = \rho(x)e$ for all $x \in L$.

If $\eta^3 \in F(\omega)^{\times 3}$, then $\eta^3 = \nu^3$ for some $\nu \in F(\omega)^\times$. Replacing η by $\nu^{-1}\eta$ if necessary, we may assume that $\eta^3 = 1$ and $\sigma(\eta)\eta = 1$. Set

$$L := F \cdot 1 + F \cdot (\eta + \eta^{-1}) + F \cdot (\omega\eta + \omega^2\eta^{-1})$$

and define ρ as the F-automorphism of L such that $\rho(\eta + \eta^{-1}) = \omega\eta + \omega^2\eta^{-1}$ and $\rho(\omega\eta + \omega^2\eta^{-1}) = \omega^2\eta + \omega\eta^{-1}$. Since $(\eta + \eta^{-1})^2 = \eta + \eta^{-1} + 2$, the algebra L is isomorphic to $F \times F \times F$. Again (L, ρ) is a Galois $\mathbb{Z}/3\mathbb{Z}$-algebra such that $L \subset A$ and $ex = \rho(x)e$ for all $x \in L$.

In both cases we obtain that $A = \bigoplus_{i=0}^{2} Le^i$ where the multiplication in A is determined by $e^3 = a \in F^\times$ and $ex = \rho(x)e$ for all $x \in L$.

To finish the description of the pair (A, V), we shall write ev in function of e and η:

$$\begin{aligned}ev &= e\Big(\frac{\mathsf{T}(uv)}{3} + \frac{1}{t^3}e^2\Big)^{-1}\eta \\ &= \frac{1}{u^3v^3}(\alpha + a^{-1}\alpha^2e + e^2)\eta\end{aligned}$$

where $\alpha := -\mathsf{T}(uv)t^3/3$. Hence $V_{F(\omega)}$ is spanned by e, $(\alpha + a^{-1}\alpha^2e + e^2)\eta$ and $(\alpha + a^{-1}\alpha^2e + e^2)\lambda\eta^{-1}$. Note that $\alpha^3 \neq a^2$ since ev is invertible. We obtain that

$$V = \mathsf{span}_F\langle e, (\alpha + a^{-1}\alpha^2e + e^2)\theta, (\alpha + a^{-1}\alpha^2e + e^2)\rho(\theta)\rangle$$

where $\theta = \eta + \lambda\eta^{-1} \in L \setminus \{0\}$ is such that $\theta + \rho(\theta) + \rho^2(\theta) = 0$.

We shall prove that the cubic form $f_{A,V}$ is isometric to the form

$$\mathsf{Tr}_{K,\beta} - 3b\mathsf{N}_K\colon K \to F\colon x \mapsto \mathsf{Tr}_K(\beta x^3) - 3b\mathsf{N}_K(x)$$

for $K = F \times F(\omega)$ and, for some $\beta \in K^\times$ and $b \in F$. We set

$$v_1 := e,\;\; v_2 := (\alpha + a^{-1}\alpha^2e + e^2)\theta,\;\; v_3 := (\alpha + a^{-1}\alpha^2e + e^2)\rho(\theta)$$

so that V is the vector space spanned by v_1, v_2, v_3. Then $f_{A,V}(xv_1 + yv_2 + zv_3)$ is equal to

$$\Big(xu_1 + \big(y + \omega z\big)u_2 + \big(y + \omega^2 z\big)u_3\Big)^3$$

where $u_1 := e$, $u_2 := (\alpha + a^{-1}\alpha^2 e + e^2)\eta$, $u_3 := (\alpha + a^{-1}\alpha^2 e + e^2)\lambda\eta^{-1}$. But $(xu_1 + yu_2 + zu_3)^3$ is equal to

$$ax^3 + \eta^3(a^{-1}\alpha^3 - a)^2 y^3 + \sigma(\eta^3)(a^{-1}\alpha^3 - a)^2 z^3 - 3\lambda\alpha(a^{-1}\alpha^3 - a)xyz.$$

Therefore $f_{A,V}$ is isometric to the form $\mathsf{Tr}_{K,\beta} - 3b\mathsf{N}_K$ where $K := F \times F(\omega)$, $\beta := \big(a, \eta^3(a^{-1}\alpha^3 - a)^2\big)$ and $b := \lambda\alpha(a^{-1}\alpha^3 - a)$.

Observe that if (L, ρ) is a Galois $\mathbb{Z}/3\mathbb{Z}$-algebra, then there exist $\lambda \in F^\times$ and $\phi \in L \otimes_F F(\omega)$ such that $\phi^3 \in F(\omega)^\times$, $\phi \notin F(\omega)$, $\sigma(\phi^3) = \lambda^3\phi^{-3}$ and

$$L = F \cdot 1 + F \cdot (\phi + \lambda\phi^{-1}) + F \cdot (\omega\phi + \omega^2\lambda\phi^{-1}).$$

Indeed, suppose that (L, ρ) is not split, then $L(\omega) \cong L \otimes_F F(\omega)$ is a Galois extension of F with a Galois group isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$. Let $\tilde{\sigma}, \tau$ be F-automorphisms of $L(\omega)$ such that L (resp. $F(\omega)$) is the subfield of $L(\omega)$ which is fixed under $\tilde{\sigma}$ (resp. τ). Let $\phi \in L(\omega)$ be such that $\phi^3 \in F(\omega)^\times$ and $\phi \notin F(\omega)$. Replacing ϕ by ϕ^{-1} if necessary, we may assume that $\tau(\phi) = \omega\phi$. Hence

$$\tau\tilde{\sigma}(\phi) = \tilde{\sigma}\tau(\phi) = \tilde{\sigma}(\omega\phi) = \omega^2\tilde{\sigma}(\phi).$$

Thus $\tilde{\sigma}(\phi) = \lambda\phi^{-1}$ for some $\lambda \in F(\omega)^\times$. But

$$\phi = \tilde{\sigma}(\lambda\phi^{-1}) = \tilde{\sigma}(\lambda)\lambda^{-1}\phi,$$

so $\tilde{\sigma}(\lambda) = \lambda$ and $\lambda \in F^\times$. Then we have $L = F(\phi + \lambda\phi^{-1})$ with

$$\sigma(\phi^3) = \big(\tilde{\sigma}(\phi)\big)^3 = (\lambda\phi^{-1})^3 = \lambda^3\phi^{-3}.$$

Suppose that $L = F \times F \times F$, then we may choose $\lambda = 1$ and $\phi = (1, \omega, \omega^2)$.

Now let (L, ρ) be a Galois $\mathbb{Z}/3\mathbb{Z}$-algebra, $a \in F^\times$ and $\alpha \in F$ such that $\alpha^3 \neq a^2$. Set

$$(B, W) := \Big(\bigoplus_{i=0}^{2} Le^i, \mathsf{span}_F\langle e, (\alpha + a^{-1}\alpha^2 e + e^2)\theta, (\alpha + a^{-1}\alpha^2 e + e^2)\rho(\theta)\rangle\Big),$$

where the multiplication in B is defined by $e^3 = a$, $ex = \rho(x)e$ for all $x \in L$, and $\theta \in L \setminus \{0\}$ is such that $\theta + \rho(\theta) + \rho^2(\theta) = 0$. Then one can check that (B, W) is a cubic pair over F and $f_{B,W}$ is nonsingular.

We summarize this subsection by the following Theorem:

Theorem 2.1 *Suppose that F does not contain a primitive cube root of unity. Let (A,V) be a cubic pair over F such that $f_{A,V}$ is nonsingular. Then (A,V) is isomorphic to*

$$\Big(\bigoplus_{i=0}^{2} Le^i, \mathsf{span}_F\langle e, (\alpha + a^{-1}\alpha^2 e + e^2)\theta, (\alpha + a^{-1}\alpha^2 e + e^2)\rho(\theta)\rangle\Big)$$

for some Galois $\mathbb{Z}/3\mathbb{Z}$-algebra (L,ρ), $a \in F^\times$, $\alpha \in F$ such that $\alpha^3 \neq a^2$, where $e^3 = a$, $ex = \rho(x)e$ for all $x \in L$ and $\theta \in L \setminus \{0\}$ is such that $\theta + \rho(\theta) + \rho^2(\theta) = 0$. Conversely, let (L,ρ) be a Galois $\mathbb{Z}/3\mathbb{Z}$-algebra, $a \in F^\times$, $\alpha \in F$ such that $\alpha^3 \neq a^2$. Let $B = \bigoplus_{i=0}^{2} Le^i$ be the algebra with multiplication defined by $e^3 = a$, $ex = \rho(x)e$ for all $x \in L$, and let W be the subspace spanned by e, $(\alpha + a^{-1}\alpha^2 e + e^2)\theta$, $(\alpha + a^{-1}\alpha^2 e + e^2)\rho(\theta)$, where $\theta \in L \setminus \{0\}$ is such that $\theta + \rho(\theta) + \rho^2(\theta) = 0$. Then (B,W) is a cubic pair over F and $f_{B,W}$ is nonsingular. Let $\phi \in L \otimes_F F(\omega)$ and $\lambda \in F^\times$ be such that $\phi^3 \in F(\omega)^\times$, $\phi \notin F(\omega)$, $\sigma(\phi^3) = \lambda\phi^{-3}$ and $1, \phi + \lambda\phi^{-1}, \omega\phi + \omega^2\lambda\phi^{-1}$ span L, where σ is the nontrivial F-automorphism of $F(\omega)$. Then $f_{B,W}$ is isometric to $\mathsf{Tr}_{K,\beta} - 3b\mathsf{N}_K$ where $K = F \times F(\omega)$, $\beta = \big(a, \phi^3(a^{-1}\alpha^3 - a)^2\big)$ and $b = \lambda\alpha(a^{-1}\alpha^3 - a)$.

2.2. Triangular form

Let (A,V) be a cubic pair over F such that $f_{A,V}$ is triangular. By Subsection 1.2, there exists $v \in A^0_{F(\omega)}$ such that $\mathsf{T}(v^2) = 0$, $v \in A^\times_{F(\omega)}$ and

$$V = \big(v^2 \star A^0_{F(\omega)}\big) \cap \big(A^0_{F(\omega)} \star v\big);$$

then $V = \{x \in A_{F(\omega)} \mid vx = \omega^2 xv\}$. Fix $e \in V$, then $ev = \omega ve$. We extend the F-automorphism σ of $F(\omega)$ to $A_{F(\omega)}$. Then $\sigma(v) = \lambda v^2$ for some $\lambda \in F(\omega)^\times$. Since

$$v = \sigma(\lambda v^2) = \sigma(\lambda)\lambda^2 v^4,$$

we deduce that $\sigma(\lambda)\lambda^2 v^3 = 1$. Hence

$$\sigma(\lambda v) = \sigma(\lambda)\sigma(v) = \lambda^{-2}v^{-3}\lambda v^2 = (\lambda v)^{-1};$$

so we may assume that $\sigma(v) = v^{-1}$. Set

$$L := F \cdot 1 + F \cdot (v + v^{-1}) + F \cdot (\omega v + \omega^2 v^{-1})$$

and define ρ as the F-automorphism of L such that $\rho(v + v^{-1}) = \omega v + \omega^2 v^{-1}$ and $\rho(\omega v + \omega^2 v^{-1}) = \omega^2 v + \omega v^{-1}$. Then (L,ρ) is a Galois $\mathbb{Z}/3\mathbb{Z}$-algebra (note that (L,ρ) is split if and only if $v^3 \in F(\omega)^{\times 3}$). Moreover $L \subset A$, $ex = \rho(x)e$ for all $x \in L$, $A = \bigoplus_{i=0}^{2} Le^i$ and $V = eL$. It is easy to check that $f_{A,V}$ is isometric to $a\mathsf{N}_L$, where $a := e^3$.

Conversely, let (L,ρ) be a Galois $\mathbb{Z}/3\mathbb{Z}$-algebra and $a \in F^\times$. Set

$$(B,W) := \Big(\bigoplus_{i=0}^{2} Le^i, eL\Big)$$

where the multiplication in B is defined by $e^3 = a$ and $ex = \rho(x)e$ for all $x \in L$. Then (B,W) is a cubic pair over F and $f_{B,W}$ is triangular.

Thus we obtain:

Theorem 2.2 *Suppose that F does not contain a primitive cube root of unity. Let (A,V) be a cubic pair over F such that $f_{A,V}$ is triangular. Then (A,V) is isomorphic to*

$$\Big(\bigoplus_{i=0}^{2} Le^i, eL\Big)$$

for some Galois $\mathbb{Z}/3\mathbb{Z}$-algebra (L,ρ) and $a \in F^\times$, where $e^3 = a$ and $ex = \rho(x)e$ for all $x \in L$. Conversely, let (L,ρ) be a Galois $\mathbb{Z}/3\mathbb{Z}$-algebra and $a \in F^\times$. Let $B = \bigoplus_{i=0}^{2} Le^i$ be the algebra with multiplication defined by $e^3 = a$, $ex = \rho(x)e$ for all $x \in L$, and set $W := eL$. Then (B,W) is a cubic pair over F and $f_{B,W}$ is triangular. Moreover $f_{B,W}$ is isometric to $a\mathsf{N}_L$.

3. The form determines the algebra

Let (A,V) and (A',V') be cubic pairs over F and suppose that $f_{A,V}$ and $f_{A',V'}$ are isometric. In this section we shall prove that A and A' are either isomorphic or anti-isomorphic.

We may assume that F contains a primitive cube root of unity. Indeed, if $A \otimes_F F(\omega) \cong A' \otimes_F F(\omega)$, then $A \cong A'$ since A and A' are central simple algebras of degree 3 and $F(\omega)/F$ is an extension of degree at most 2. We may also assume that A is division, because there is nothing to prove if A and A' are split. Therefore, by Theorem 1.7, the cubic form $f_{A,V}$ is either nonsingular or triangular.

First case: Suppose that $f_{A,V}$ is nonsingular, then so is $f_{A',V'}$. By Theorem 1.7, there exist $a_i, a_i' \in F^\times$, $\lambda, \lambda' \in F$ such that $1+\lambda^3 a_1, 1+\lambda'^3 a_1' \neq 0$ and

$$\begin{aligned} A &= (a_1,a_2)_{\omega,F}, \quad & V &= \mathsf{span}_F\langle \xi_1, \xi_2, \xi_1\xi_2^2 + \lambda\xi_1^2\xi_2^2\rangle \\ A' &= (a_1',a_2')_{\omega,F}, \quad & V' &= \mathsf{span}_F\langle \xi_1', \xi_2', \xi_1'\xi_2'^2 + \lambda'\xi_1'^2\xi_2'^2\rangle \end{aligned}$$

where A (resp. A') is generated by ξ_1, ξ_2 such that $\xi_i^3 = a_i$ and $\xi_1\xi_2 = \omega\xi_2\xi_1$ (resp. ξ_1', ξ_2' such that $\xi_i'^3 = a_i'$ and $\xi_1'\xi_2' = \omega\xi_2'\xi_1'$). Set

$$\xi_3 := \xi_1\xi_2^2 + \lambda\xi_1^2\xi_2^2, \quad a_3 := \xi_3^3, \quad \xi_3' := \xi_1'\xi_2'^2 + \lambda'\xi_1'^2\xi_2'^2, \quad a_3' := \xi_3'^3.$$

We recall properties of nonsingular cubic forms and we refer to [Brieskorn and Knörrer, 1986] or [Hirschfeld, 1979] for more details. A nonsingular cubic form f on a 3-dimensional vector space V has 9 inflexion points in the projective plane $\mathbb{P}_V(F_s)$. There are four triangles (i.e. cubic curves associated with triangular cubic forms) in $\mathbb{P}_V(F_s)$ with the property that each inflexion point is incident with one and only one line of the triangle and each line of the triangle passes through exactly 3 inflexion points. These triangles are called *inflexional triangles* of f. For a triangular cubic form $g = \varphi_1\varphi_2\varphi_3$ over V, we denote by $g = 0$ the triangle formed by the zeros of the linear forms φ_i in $\mathbb{P}_V(F_s)$.

The map $V \to K := F \times F \times F$ which sends ξ_1, ξ_2, ξ_3 on the canonical basis of K is an F-vector space isomorphism. Under this isomorphism, $f_{A,V}$ is isometric to the form

$$\mathsf{Tr}_{K,\alpha} - 3b\mathsf{N}_K \colon K \to F \colon x \mapsto \mathsf{Tr}_K(\alpha x^3) - 3b\mathsf{N}_K(x)$$

where $\alpha = (a_1, a_2, a_3)$ and $b = \omega^2 a_1 a_2 \lambda$. The inflexional triangles of the form $\mathsf{Tr}_{K,\alpha} - 3b\mathsf{N}_K$ are $\mathsf{N}_K = 0$ and $\mathsf{Tr}_{K,\alpha} - 3\theta\mathsf{N}_K = 0$ for all $\theta \in F_s$ such that $\theta^3 = \mathsf{N}_K(\alpha)$. Let $(\varphi_1, \varphi_2, \varphi_3)$ denote the dual basis of (ξ_1, ξ_2, ξ_3), then under the previous isomorphism $V \to K$, the form $\varphi_1\varphi_2\varphi_3$ is isometric to N_K. Hence Γ acts trivially on the lines of the corresponding inflexional triangle. In fact, we have the following:

Lemma 3.1 *There exists a unique inflexional triangle of $f_{A,V}$ whose lines are defined over F.*

Proof: Suppose that Γ acts trivially on the lines of $\mathsf{Tr}_{K,\alpha} - 3\theta\mathsf{N}_K = 0$, for some $\theta \in F_s$ such that $\theta^3 = \mathsf{N}_K(\alpha)$. For $x = (x_1, x_2, x_3) \in K$, $\mathsf{Tr}_K(\alpha x^3) - 3\theta\mathsf{N}_K(x)$ is equal to

$$(\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3)(\theta_1 x_1 + \omega\theta_2 x_2 + \omega^2\theta_3 x_3)(\theta_1 x_1 + \omega^2\theta_2 x_2 + \omega\theta_3 x_3)$$

for some $\theta_i \in F_s$ such that $\theta_i^3 = a_i$ and $\theta_1\theta_2\theta_3 = \theta$. Since Γ acts trivially on the line $\theta_1 x_1 + \theta_2 x_2 + \theta_3 x_3 = 0$, there exists a nonzero $u \in F$ such that $\theta_2 = u\theta_1$. This implies that $a_2 = u^3 a_1$, which contradicts the assumption that A is division. □

Let $(\varphi_1', \varphi_2', \varphi_3')$ be the dual basis of (ξ_1', ξ_2', ξ_3'). Then $\varphi_1'\varphi_2'\varphi_3' = 0$ is an inflexional triangle of $f_{A',V'}$ whose lines are defined over F. Let $\Theta \colon V \to V'$ be an F-vector space isomorphism such that $f_{A,V} = f_{A',V'} \circ \Theta$, then

$$\varphi_1\varphi_2\varphi_3 F = (\varphi_1'\varphi_2'\varphi_3' \circ \Theta)F.$$

Thus there exist $\lambda_i \in F^\times$ and a permutation π of $\{1, 2, 3\}$ such that

$$\varphi_i' \circ \Theta = \lambda_{\pi(i)}\varphi_{\pi(i)} \qquad \text{for all } i \in \{1, 2, 3\}.$$

For all $i,j \in \{1,2,3\}$, we have

$$\varphi'_i \circ \Theta(\xi_{\pi(j)}) = \lambda_{\pi(i)}\varphi_{\pi(i)}(\xi_{\pi(j)}) = \delta_{ij}\lambda_{\pi(i)},$$

hence $\Theta(\xi_{\pi(j)}) = \lambda_{\pi(j)}\xi'_j$. We obtain that $a_{\pi(j)} = \lambda^3_{\pi(j)}a'_j$ and $b = \lambda_1\lambda_2\lambda_3 b'$ where $b' = \omega^2 a'_1 a'_2 \lambda'$. But a_1 is the only scalar among the a_i's such that

$$\frac{a_1a_2a_3 - b^3}{a_i^2} \in F^{\times 3}.$$

Indeed $a_1a_2a_3 - b^3 = a_1^2a_2^3$, thus $a_1^{-2}(a_1a_2a_3 - b^3) \in F^{\times 3}$; if $a_2^{-2}(a_1a_2a_3 - b^3) \in F^{\times 3}$, then $a_1F^{\times 3} = a_2F^{\times 3}$ and it contradicts the assumption that A is division; similarly, $a_3^{-2}(a_1a_2a_3 - b^3) \notin F^{\times 3}$. On the other hand, we have $a_1'^{-2}(a'_1a'_2a'_3 - b'^3) \in F^{\times 3}$ and

$$\frac{a_1a_2a_3 - b^3}{a_{\pi(1)^2}} = \left(\frac{\lambda_1\lambda_2\lambda_3}{\lambda^2_{\pi(1)}}\right)^3 \frac{a'_1a'_2a'_3 - b'^3}{a_1'^2} \in F^{\times 3};$$

therefore $\pi(1) = 1$. If $\pi(2) = 2$, then

$$(\lambda_i\xi'_i)^3 = a_i \qquad \text{and} \qquad (\lambda_1\xi'_1)(\lambda_2\xi'_2) = \omega(\lambda_2\xi'_2)(\lambda_1\xi'_1),$$

thus $A' \cong A$. If $\pi(2) = 3$, then

$$(\lambda_1\xi'_1)^3 = a_1, \;\; (\lambda_2\xi'_3)^3 = a_2 \quad \text{and} \quad (\lambda_1\xi'_1)(\lambda_2\xi'_3) = \omega^2(\lambda_2\xi'_3)(\lambda_1\xi'_1),$$

thus $A' \cong A^{\mathsf{op}}$.

Second case: Suppose that $f_{A,V}$ is triangular. Then there exist $a_i, a'_i \in F^\times$ such that

$$\begin{aligned} A &= (a_1,a_2)_{\omega,F}, \quad V = \mathsf{span}_F\langle \xi_1, \xi_2, \xi_1^2\xi_2^2\rangle \\ A' &= (a'_1,a'_2)_{\omega,F}, \quad V' = \mathsf{span}_F\langle \xi'_1, \xi'_2, \xi_1'^2\xi_2'^2\rangle \end{aligned}$$

where A (resp. A') is generated by ξ_1, ξ_2 such that $\xi_i^3 = a_i$ and $\xi_1\xi_2 = \omega\xi_2\xi_1$ (resp. ξ'_1, ξ'_2 such that $\xi_i'^3 = a'_i$ and $\xi'_1\xi'_2 = \omega\xi'_2\xi'_1$). Let $\theta \in F_s$ be a cube root of $a_1^{-1}a_2$. Since A is division, $\theta \notin F$. We have

$$f_{A,V}(x_1\xi_1 + x_2\xi_2 + x_3\xi_1^2\xi_2^2) = a_1\mathsf{N}_{F(\theta)}(x_1 + x_2\theta + \omega^2a_1x_3\theta^2).$$

Let $\theta' \in F_s$ be a cube root of $a_1'^{-1}a'_2$. Similarly, $f_{A',V'}(x_1\xi'_1 + x_2\xi'_2 + x_3\xi_1'^2\xi_2'^2)$ is equal to

$$a'_1(x_1 + x_2\theta' + \omega^2a'_1x_3\theta'^2)(x_1 + \omega x_2\theta' + \omega a'_1x_3\theta'^2)(x_1 + \omega^2x_2\theta' + a'_1x_3\theta'^2).$$

Since $f_{A,V} \cong f_{A',V'}$, we have $\theta' \notin F$ and, by Proposition 8 in [Raczek and Tignol, 2008], the fields $F(\theta)$ and $F(\theta')$ are isomorphic. We deduce that either

$\theta' F = \theta F$ or $\theta' F = \theta^2 F$. Identifying θ with $\xi_1^{-1}\xi_2$ (resp. θ' with $\xi_1'^{-1}\xi_2'$), we have

$$A = \bigoplus_{i=0}^{2} F(\theta)\xi_1^i \quad \text{ and } \quad A' = \bigoplus_{i=0}^{2} F(\theta')\xi_1'^i$$

where $\xi_1\theta = \omega\theta\xi_1$ and $\xi_1'\theta' = \omega\theta'\xi_1'$. Because $f_{A,V}$ is isometric to $f_{A',V'}$, there exist $u_i \in F$ such that

$$a_1 \mathsf{N}_{F(\theta)}(u_1 + u_2\theta + u_3\theta^2) = a_1'.$$

Set $\eta_1 := \xi_1(u_1 + u_2\theta + u_3\theta^2)$, then

$$\eta_1^3 = a_1 \mathsf{N}_{F(\theta)}(u_1 + u_2\theta + u_3\theta^2) = a_1' \quad \text{ and } \quad \eta_1\theta = \omega\theta\eta_1.$$

Hence $A = \bigoplus_{i=0}^{2} F(\theta)\eta_1^i$ with $\eta_1^3 = a_1'$ and $\eta_1\theta = \omega\theta\eta_1$. So

$$A \cong \begin{cases} A' & \text{if } \theta' F = \theta F, \\ A'^{\mathsf{op}} & \text{if } \theta' F = \theta^2 F. \end{cases}$$

We thus obtain the following:

Theorem 3.2 *Let (A, V) and (A', V') be cubic pairs over F. Suppose that $f_{A,V}$ and $f_{A',V'}$ are isometric, then the algebras A and A' are either isomorphic or anti-isomorphic.*

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