Classifying subspaces of cube-central order 3 matrices using the geometry of cubic curves

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Abstract

Let F be a separably closed field of characteristic different from 2 and 3. We consider 3-dimensional subspaces V of trace zero cube-central order 3 matrices with coefficients in F. Each such vector space gives rise to a ternary cubic form, mapping an element of V to its cube. We use geometric properties of cubic curves to determine V up to conjugacy in the case where the cubic form is nonsingular.

1. Introduction

Consider a separably closed field F of characteristic different from 2 and 3. Let $\mathsf{M}_3(F)$ denote the algebra of order 3 matrices with coefficients in F and let $\mathsf{M}_3(F)^0$ the subspace of $\mathsf{M}_3(F)$ of trace zero elements. We say that a matrix $x \in \mathsf{M}_3(F)$ is *cube-central* if $x^3 \in F$. Let V be a 3-dimensional subspace of $\mathsf{M}_3(F)^0$ of cube-central elements. Then for all $x \in V$, we have $x^3 = \det(x) \in F$. Hence V gives rise to a ternary cubic form on V

$$f_V : V \to F : x \mapsto x^3.$$

We call V a *cubic subspace* of $\mathsf{M}_3(F)$.

It is now a natural question to try to classify up to conjugacy the cubic subspaces V of $\mathsf{M}_3(F)$ for which the cubic form f_V is nonsingular, i.e to give, for each class of conjugacy, vectors which span a representative of the class. In this paper, we shall prove the following explicit answer:

Theorem 1.1 *Each nonsingular cubic subspace of $\mathsf{M}_3(F)$ is conjugate to the F-subspace of $\mathsf{M}_3(F)$ spanned by*

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & -\frac{1}{2} & 1 \\ 3\alpha^2 & -2\alpha & \frac{1}{2} \\ 0 & -3\alpha^2 & \alpha \end{pmatrix}$$

for some $\alpha \in F \setminus \{0, 1/8, 1/9\}$.

Our proof of this theorem crucially depends on particular geometric properties of nonsingular cubic curves. Therefore, in Section 2 of this paper, we first recall well-known definitions and results on cubic curves, and we define and study particular lines and points associated to a nonsingular cubic curve. Then in Section 3 we classify the cubic subspaces which induce a nonsingular cubic form using precisely their particular lines and points.

In a previous publication [Raczek, 2009] we already classified cubic subspaces of arbitrary degree 3 central simple algebras over a field of characteristic different from 2 and 3 (not necessarily separably closed). For that classification we used a quite different method, depending solely on algebraic manipulations using the so-called Okubo product. In this paper, on the contrary, we show that we can use geometric arguments to obtain algebraic results. Restricting moreover our attention to matrix algebras, we end up with the explicit classification of cubic subspaces as spelled out in Theorem 1.1. But we should remark that, in fact, the general classification of cubic subspaces of arbitrary degree 3 central simple algebras follows from this theorem: in [Raczek, 2007] we achieved this using Galois descent. It is then a corollary, as explained in [Raczek, 2007, 2009], that a cubic form associated to an algebra determines that algebra up to isomorphism or anti-isomorphism.

Throughout the paper, F denotes a separably closed field of characteristic different from 2 and 3. Let $\overline{F}$ denote an algebraic closure of F. For an F-vector space V, we write $\overline{V}$ for $V \otimes_F \overline{F}$ and $\mathbb{P}_V(F)$ (resp. $\mathbb{P}_V(\overline{F})$) for the projective space of V (resp. $\overline{V}$).

2. Geometry of plane cubic curves

2.1. Preliminaries

For the reader's convenience and to fix notation we first recall well-known results on plane cubic curves, using an algebraic presentation. For background information on cubic curves, we refer to [Bretagnolle-Nathan, 1958], [Brieskorn and Knörrer, 1986] and [Hirschfeld, 1979].

Let V be a 3-dimensional vector space over F and let f be a cubic form on V. Let $t\colon V \times V \times V \to F$ denote the symmetric trilinear form such that

$$t(x, x, x) = f(x)$$

for all $x \in V$. Let $\bar{t}\colon \overline{V} \times \overline{V} \times \overline{V} \to \overline{F}$ be the symmetric trilinear form on $\overline{V}$ that we obtain when we extend the scalars to $\overline{F}$. Let $\overline{f}\colon \overline{V} \to \overline{F}$ be defined by $\overline{f}(x) = \bar{t}(x, x, x)$ for all $x \in \overline{V}$. We say that f is *singular* if there exists a nonzero $u \in \overline{V}$ such that $\bar{t}(u, u, x) = 0$ for all $x \in \overline{V}$. In that case, the point

$u\overline{F} \in \mathbb{P}_V(\overline{F})$ is a *singular* zero of f. If $u\overline{F}$ is a nonsingular zero of f, then the set

$$\{x\overline{F} \in \mathbb{P}_V(\overline{F}) \mid \overline{t}(u,u,x) = 0\}$$

(which is a line in $\mathbb{P}_V(\overline{F})$) is called the *tangent* to f at $u\overline{F}$. The tangent at $u\overline{F}$ is the only line such that the intersection multiplicity (as defined in [Hirschfeld, 1979, page 52]) with the cubic curve $f = 0$ at $u\overline{F}$ is greater than or equal to 2. If (e_1, e_2, e_3) is a basis of $\overline{V}$ and $P(X_1, X_2, X_3) \in \overline{F}[X_1, X_2, X_3]$ is the degree 3 homogeneous polynomial such that

$$P(x_1, x_2, x_3) = \overline{f}(x_1e_1 + x_2e_2 + x_3e_3)$$

for all $x_1, x_2, x_3 \in \overline{F}$, then

$$3\overline{t}(u, u, e_i) = \frac{\partial P}{\partial X_i}(a_1, a_2, a_3)$$

for all $i \in \{1, 2, 3\}$ and $u = a_1e_1 + a_2e_2 + a_3e_3 \in \overline{V}$. If f is a nonsingular cubic form, then the intersection multiplicity of the curve $f = 0$ with a line at a point is less than or equal to 3; moreover, there are exactly three intersection points of the cubic curve $f = 0$ with a line, counting multiplicities.

The set

$$\{u\overline{F} \in \mathbb{P}_V(\overline{F}) \mid \text{the form } \overline{V} \times \overline{V} \to \overline{F} \colon (x, y) \mapsto \overline{t}(u, x, y) \text{ is singular}\}$$

is called the *Hessian curve* of f and is denoted by H_f. Suppose that (e_1, e_2, e_3) is a basis of $\overline{V}$, then a point $u\overline{F}$ is on the Hessian curve if and only if

$$h(u) := \det \begin{pmatrix} \overline{t}(u, e_1, e_1) & \overline{t}(u, e_1, e_2) & \overline{t}(u, e_1, e_3) \\ \overline{t}(u, e_2, e_1) & \overline{t}(u, e_2, e_2) & \overline{t}(u, e_2, e_3) \\ \overline{t}(u, e_3, e_1) & \overline{t}(u, e_3, e_2) & \overline{t}(u, e_3, e_3) \end{pmatrix}$$

is equal to zero. Hence the Hessian curve is the cubic curve formed by the zeros of the cubic form $h \colon \overline{V} \to \overline{F} \colon x \mapsto h(x)$. Observe that if $P(X_1, X_2, X_3) \in \overline{F}[X_1, X_2, X_3]$ is such that

$$P(x_1, x_2, x_3) = \overline{f}(x_1e_1 + x_2e_2 + x_3e_3)$$

for all $x_1, x_2, x_3 \in \overline{F}$, then

$$6\overline{t}(u, e_i, e_j) = \frac{\partial^2 P}{\partial X_i \partial X_j}(a_1, a_2, a_3)$$

for all $i, j \in \{1, 2, 3\}$, where a_1, a_2, a_3 are the coordinates of $u \in \overline{V}$ in (e_1, e_2, e_3). We say that $u\overline{F}$ is an *inflection point* of f if $u\overline{F}$ is a nonsingular zero of f which is on the Hessian curve of f. It is equivalent to say that $u\overline{F}$ is a nonsingular zero of f such that the intersection multiplicity of its tangent with the curve with equation $f = 0$ at $u\overline{F}$ is greater than or equal to 3 (see [Walker, 1950, page 71, Theorem 6.3]). Since

$$\overline{f}(u + \lambda x) = \overline{f}(u) + 3\lambda\overline{t}(u, u, x) + 3\lambda^2\overline{t}(u, x, x) + \lambda^3\overline{f}(x)$$

for all $x \in \overline{V}$, one can also say that $u\overline{F}$ is an inflection point if and only if $u\overline{F}$ is nonsingular and $\overline{t}(u, u, x) = 0$ implies $\overline{t}(u, x, x) = 0$ for all $x \in \overline{V}$.

If f is a nonsingular cubic form on V, then there exist exactly 9 distinct inflection points of f in $\mathbb{P}_V(F) \subset \mathbb{P}_V(\overline{F})$ with a special configuration: any line through two inflection points passes through a third inflection point. The 9 inflection points and the 12 lines through them form the configuration of the points and lines of the affine plane over the field with 3 elements (see [Hirschfeld, 1979, §11.1]).

2.2. Special points and special lines

Let V be a 3-dimensional vector space over F and f a nonsingular cubic form on V. Let $u\overline{F}$ be an inflection point of f. Then the conic

$$\{x\overline{F} \in \mathbb{P}_V(\overline{F}) \mid \overline{t}(u, x, x) = 0\}$$

consists of two lines. We shall prove that the lines are distinct, i.e. the conic has a unique singular point.

Lemma 2.1 *Let f be a nonsingular cubic form on V and $u\overline{F}$ an inflection point of f. There exists a unique $u'\overline{F} \in \mathbb{P}_V(\overline{F})$ such that $\overline{t}(u, u', x) = 0$ for all $x \in \overline{V}$. This point $u'\overline{F}$ is distinct from $u\overline{F}$.*

Proof : Since the form $\overline{V} \times \overline{V} \to \overline{F}\colon (x, y) \mapsto \overline{t}(u, x, y)$ is singular, there exists a nonzero $u' \in \overline{V}$ such that $\overline{t}(u, u', x) = 0$ for all $x \in \overline{V}$. Clearly $u'\overline{F} \neq u\overline{F}$ because $u\overline{F}$ is nonsingular; hence the tangent at $u\overline{F}$ is the line through $u\overline{F}$ and $u'\overline{F}$. Suppose that $u'' \in \overline{V}$ is nonzero such that $\overline{t}(u, u'', x) = 0$ for all $x \in \overline{V}$, then $u''\overline{F} \in \mathbb{P}_V(\overline{F})$ is on the tangent at $u\overline{F}$. So there exist $\alpha, \beta \in \overline{F}$ such that $u'' = \alpha u + \beta u'$. If $\alpha \neq 0$, then $\overline{t}(u, u, x) = 0$ for all $x \in \overline{V}$ which contradicts the fact that $u\overline{F}$ is nonsingular. Thus $\alpha = 0$ and $u'\overline{F} = u''\overline{F}$. □

The unique point $u'\overline{F} \in \mathbb{P}_V(\overline{F})$ is in particular a point of the Hessian curve of f. We call it the *Hessian point* of the inflection point $u\overline{F}$.

Since the Hessian point of $u\overline{F}$ is the only singular point of the conic

$$\{x\overline{F} \in \mathbb{P}_V(\overline{F}) \mid \overline{t}(u, x, x) = 0\},$$

this conic consists of two distinct lines which intersect at the Hessian point. One of these lines is the tangent at $u\overline{F}$, because $\overline{t}(u, u, x) = 0$ implies $\overline{t}(u, x, x) = 0$ for all x. The second line is called the *harmonic polar* at $u\overline{F}$.

Proposition 2.2 *Let f be a nonsingular cubic form on V and $u\overline{F}$ an inflection point of f. There are exactly three distinct zeros of f which are on the harmonic polar at $u\overline{F}$.*

Proof: Suppose that there are less than three distinct zeros of f on the harmonic polar at $u\overline{F}$. There are exactly three intersection points between the cubic curve $f = 0$ and the harmonic polar at $u\overline{F}$ counting multiplicities. Hence, there exists a zero $v\overline{F}$ of f such that the tangent to f at $v\overline{F}$ is the harmonic polar at $u\overline{F}$. Since the harmonic polar at $u\overline{F}$ is contained in the conic

$$\{x\overline{F} \in \mathbb{P}_V(\overline{F}) \mid \bar{t}(u, x, x) = 0\}$$

we have in particular $\bar{t}(u, v, v) = 0$. So $u\overline{F}$ is on the tangent to f at $v\overline{F}$. But this line is the harmonic polar at $u\overline{F}$. Thus $u\overline{F}$ is an intersection point of the tangent at $u\overline{F}$ and the harmonic polar at $u\overline{F}$. This contradicts Lemma 2.1. Therefore, there are exactly three distinct zeros of f which are on the harmonic polar at $u\overline{F}$. $\square$

We call these points the *harmonic points* of $u\overline{F}$.

2.3. Remarkable properties

As we recalled earlier, the inflection points of a nonsingular cubic form have the configuration of the affine plane over the field with three elements. We shall prove that the harmonic points and the harmonic polars too verify remarkable geometric properties. For the classification of cubic subspaces, only Proposition 2.3 will be needed (for the construction of a particular matrix). We include Proposition 2.7 here as well for it is interesting in its own right: it shows that, quite surprisingly, the configuration of the harmonic polars is dual to that of the inflection points.

First we recall that fixing a zero o of a nonsingular cubic form, we may define a group law on the zeros of f: for zeros a, b of f, $a +_o b$ is the third zero of f on the line through o and c, where c is the third zero of f on the line through a and b. It is a commutative group law with o as neutral element. In the particular case where o is an inflection point of f, then zeros a_1, a_2, a_3 of f are the intersection points of the curve $f = 0$ with some line, counting multiplicities, if and only if $a_1 +_o a_2 +_o a_3 = o$ (see [Walker, 1950, page 192, Theorem 9.2]).

Proposition 2.3 *Let f be a nonsingular cubic form on V, p_1, p_2, p_3 distinct collinear inflection points of f and q_1 a harmonic point of p_1. Then the third zero of f on the line through q_1 and p_2 is a harmonic point of p_3.*

Proof : We use the group law on the zeros of f with $o := p_1$. We set $q_3 := q_1 +_o p_3$. We shall prove that q_3 is a harmonic point of p_3 and that it is the third zero of f on the line through q_1 and p_2. Let $u_1, v_1, u_3, v_3 \in \overline{V}$ be such that $p_i = u_i\overline{F}$ and $q_i = v_i\overline{F}$. Since q_1 is a harmonic point of p_1, we have in particular $\bar{t}(u_1, v_1, v_1) = 0$. Thus p_1 is on the tangent at q_1 and $2q_1 = o$. We have $p_2 +_o p_3 = o$ because p_1, p_2, p_3 are distinct collinear zeros of f. The tangent at p_3 intersects the curve $f = 0$ with a multiplicity equal to 3 (it is not greater than 3 because f is nonsingular), so $3p_3 = o$. Since

$$q_1 +_o p_2 +_o q_3 = q_1 +_o p_2 +_o q_1 +_o p_3 = o,$$

q_3 is the third zero of f on the line through q_1 and p_2. We have

$$2q_3 +_o p_3 = 2q_1 +_o 2p_3 +_o p_3 = o,$$

so p_3 is on the tangent at q_3 and $\bar{t}(v_3, v_3, u_3) = 0$. Thus q_3 is on the conic

$$\{x\overline{F} \in \mathbb{P}_V(\overline{F}) \mid \bar{t}(u_3, x, x) = 0\}$$

which consists of the tangent at p_3 and the harmonic polar at p_3. The only zero of f on the tangent at p_3 is p_3. Since $q_1 \neq p_1$, we have $q_3 \neq p_3$ and q_3 is on the harmonic polar at p_3. Hence the third zero of f on the line through q_1 and p_2 is a harmonic point of p_3. □

We shall prove that the configuration of the harmonic polars is dual to the configuration of the inflection points of a nonsingular cubic form. First we need three lemmas.

Lemma 2.4 *Let f be a nonsingular cubic form on V and p, q distinct inflection points of f. Then the harmonic polar at p is distinct from the harmonic polar at q.*

Proof : Let $u, v \in \overline{V}$ be such that $p = u\overline{F}$ and $q = v\overline{F}$. Suppose that the harmonic polar at p is equal to the harmonic polar at q. Then these two lines contain a common zero $w\overline{F}$ of f. In particular, we have

$$\bar{t}(u, w, w) = 0 = \bar{t}(v, w, w).$$

Then p and q are on the tangent at $w\overline{F}$. Observe that $w\overline{F}$ is distinct from p and q. We obtain at least four zeros of f on the tangent at $w\overline{F}$, counting multiplicities, which is impossible. Hence the harmonic polar at p is distinct from the harmonic polar at q. □

Lemma 2.5 *Let f be a nonsingular cubic form on V, $u\overline{F}$ an inflection point of f and $u'\overline{F}$ its Hessian point. Then $u\overline{F}$ and $u'\overline{F}$ are the unique intersection points of the Hessian curve and the tangent to f at $u\overline{F}$, and the intersection multiplicity at $u'\overline{F}$ is equal to 2.*

Proof : Let $v \in V$ be such that u, u', v are linearly independent. Then $x\overline{F} \in \mathbb{P}_V(\overline{F})$ is on the Hessian curve of f if and only if

$$h(x) := \det \begin{pmatrix} \bar{t}(u,u,x) & \bar{t}(u,u',x) & \bar{t}(u,v,x) \\ \bar{t}(u,u',x) & \bar{t}(u',u',x) & \bar{t}(u',v,x) \\ \bar{t}(u,v,x) & \bar{t}(u',v,x) & \bar{t}(v,v,x) \end{pmatrix}$$

is equal to zero. The tangent to f at $u\overline{F}$ is the line through $u\overline{F}$ and $u'\overline{F}$. Thus, the intersection points of the Hessian curve and the tangent at $u\overline{F}$ are the points $(\lambda u + \mu u')\overline{F} \in \mathbb{P}_V(\overline{F})$ such that

$$h(\lambda u + \mu u') = -\lambda^2 \mu \overline{f}(u') \bar{t}(u,u,v)^2 = 0.$$

If the Hessian point $u'\overline{F}$ of $u\overline{F}$ is a zero of f, then, counting multiplicities, there would be at least 4 zeros of f on the tangent at $u\overline{F}$; hence $f(u') \neq 0$. Since u, u', v are linearly independent, the point $v\overline{F}$ is not on the tangent to f at $u\overline{F}$ and so $\bar{t}(u,u,v) \neq 0$. Therefore, we obtain that the only intersection points of the Hessian curve and the tangent at $u\overline{F}$ are $u\overline{F}$ and $u'\overline{F}$, and the intersection multiplicity is equal to 2 at $u'\overline{F}$. □

We say that a cubic curve is a *triangle* if it consists of 3 non-concurrent lines. If the Hessian curve of a nonsingular cubic form f is a triangle then each line of the triangle passes through exactly 3 distinct inflection points of f. Indeed, the Hessian curve contains all the inflection points of f and there are at most 3 zeros of f on each line of the triangle.

Lemma 2.6 *Let f be a nonsingular cubic form on V. Then the Hessian curve is singular if and only if there exist distinct inflection points of f with the same Hessian point. In this case, the Hessian curve is a triangle with the property that, for each line of the triangle, the 3 inflection points on this line have the same Hessian point, namely the intersection point of the other two lines of the triangle.*

Proof : For an inflection point $u\overline{F}$ of f, we denote by $u'\overline{F}$ its Hessian point. Suppose that the Hessian curve is singular. Then by [Brieskorn and Knörrer, 1986, pages 293-294], it is a triangle. Let $u_1\overline{F}, u_2\overline{F}, u_3\overline{F}$ be distinct inflection points of f which are on the same line of the Hessian curve. By Lemma 2.5, the intersection multiplicity of the tangent at $u_i\overline{F}$ and the Hessian curve is equal to 2 at $u_i'\overline{F}$. Thus $u_i'\overline{F}$ is a singular zero of the Hessian curve and it is the intersection point of two lines of the triangle. If $u_i\overline{F}$ and $u_i'\overline{F}$ are on the same line of the Hessian curve, then the tangent at $u_i\overline{F}$ is contained in the Hessian curve; this contradicts Lemma 2.5. Thus, for all $i \in \{1,2,3\}$, $u_i'\overline{F}$ is the intersection point of the two lines of the Hessian curve which do not pass through $u_i\overline{F}$; in particular $u_1'\overline{F} = u_2'\overline{F} = u_3'\overline{F}$.

Conversely, assume that there exist distinct inflection points $u\overline{F}, v\overline{F}$ of f with the same Hessian point $u'\overline{F}$. The point $v\overline{F}$ does not lie on the tangent at $u\overline{F}$ because $u\overline{F}$ is the only zero of f on the tangent at $u\overline{F}$. Therefore u, u', v are linearly independent. Let $h\colon \overline{V} \to \overline{F}$ be defined by

$$h(x) := \det \begin{pmatrix} \bar{t}(u,u,x) & \bar{t}(u,u',x) & \bar{t}(u,v,x) \\ \bar{t}(u,u',x) & \bar{t}(u',u',x) & \bar{t}(u',v,x) \\ \bar{t}(u,v,x) & \bar{t}(u',v,x) & \bar{t}(v,v,x) \end{pmatrix}$$

so that the Hessian curve of f consists of the zeros of h. The coefficient of λ in $h(u' + \lambda x)$ is equal to

$$\bar{t}(u,u,x)\big(\overline{f}(u')\bar{t}(u',v,v) - \bar{t}(u',u',v)^2\big) = 0.$$

Hence $u'\overline{F}$ is a singular zero of h and the Hessian curve is singular. □

To simplify notation, we write $p^\star$ for the harmonic polar at an inflection point p.

Proposition 2.7 *Let f be a nonsingular cubic form on V and p, q, r distinct inflection points of f. Then the harmonic polars $p^\star$, $q^\star$ and $r^\star$ are concurrent if and only if the inflection points p, q and r are collinear.*

Proof: Suppose that p, q, r are collinear. Let p', q', r' denote the Hessian points of p, q, r and $\mathsf{T}_p, \mathsf{T}_q, \mathsf{T}_r$ the tangents at p, q, r respectively. Let $u, u', v \in \overline{V}$ be such that $p = u\overline{F}$, $q = v\overline{F}$ and $p' = u'\overline{F}$. Observe that $\bar{t}(u,u,v) \neq 0$ and $\bar{t}(u,v,v) \neq 0$ because q is not on T_p and p is not on T_q. Since p, q and r are collinear, we have $r = w\overline{F}$ with $w = \bar{t}(u,v,v)u - \bar{t}(u,u,v)v$. Let $h\colon \overline{V} \to \overline{F}$ be defined by

$$h(x) := \det \begin{pmatrix} \bar{t}(u,u,x) & \bar{t}(u,u',x) & \bar{t}(u,v,x) \\ \bar{t}(u,u',x) & \bar{t}(u',u',x) & \bar{t}(u',v,x) \\ \bar{t}(u,v,x) & \bar{t}(u',v,x) & \bar{t}(v,v,x) \end{pmatrix}.$$

Since

$$h(v) = -\bar{t}(u,v,v)^2\bar{t}(u',u',v) - \bar{t}(u,u,v)\bar{t}(u',v,v)^2 = 0,$$

the nonzero vector

$$v' = -\bar{t}(u,v,v)\bar{t}(u',v,v)u + \bar{t}(u,v,v)^2u' + \bar{t}(u,u,v)\bar{t}(u',v,v)v$$

is such that $\bar{t}(v,v',x) = 0$ for all $x \in \overline{V}$, so $q' = v'\overline{F}$. If the Hessian curve is singular, then by Lemma 2.6, we have $p' = q' = r'$ and the harmonic polars $p^\star$, $q^\star$ and $r^\star$ intersect at p'. Now we assume that the Hessian curve is nonsingular, then p', q', r' are pairwise distinct. Let $x_0\overline{F}$ be the intersection point of $p^\star$ and $q^\star$. Then we have in particular

$$\bar{t}(u,x_0,x_0) = 0 = \bar{t}(v,x_0,x_0),$$

so $t(w,x_0,x_0)=0$ and $x_0\overline{F}$ is either on T_r or on $r^\star$. Suppose that $x_0\overline{F}\in\mathsf{T}_r$ and let $\alpha,\beta,\gamma\in\overline{F}$ be such that $x_0=\alpha u+\beta u'+\gamma v$. If $x_0\overline{F}\in\mathsf{T}_p$, then $x_0\overline{F}\in\mathsf{T}_p\cap p^\star=\{p'\}$; then we would have $p'=x_0\overline{F}\in\mathsf{H}_f\cap\mathsf{T}_r=\{r,r'\}$ which is impossible. Hence $x_0\overline{F}\notin\mathsf{T}_p$ and $\gamma\neq 0$. Similarly $x_0\overline{F}\notin\mathsf{T}_q$. Furthermore, $\bar{t}(u',v,v)\neq 0$ because otherwise $p'\in\mathsf{H}_f\cap\mathsf{T}_q=\{q,q'\}$ which is impossible. Since $\bar{t}(u,x_0,x_0)=0$, $\bar{t}(w,w,x_0)=0$ and $\bar{t}(u,u,v)\neq 0$, we have

$$\begin{aligned} 2\alpha\gamma\bar{t}(u,u,v)+\gamma^2\bar{t}(u,v,v) &= 0,\\ -\alpha\bar{t}(u,v,v)\bar{t}(u,u,v)+\beta\bar{t}(u,u,v)\bar{t}(u',v,v)-\gamma\bar{t}(u,v,v)^2 &= 0. \end{aligned}$$

It implies that $x_0\overline{F}$ is equal to

$$\big(-\bar{t}(u',v,v)\bar{t}(u,v,v)u+\bar{t}(u,v,v)^2u'+2\bar{t}(u,u,v)\bar{t}(u',v,v)v\big)\overline{F}$$

because $\gamma\neq 0$. We obtain that $x_0\overline{F}=\big(v'+\bar{t}(u,u,v)\bar{t}(u',v,v)v\big)\overline{F}$ is on T_q which is impossible. Therefore $p^\star$, $q^\star$ and $r^\star$ are concurrent at $x_0\overline{F}$.

Conversely, suppose that $p^\star$, $q^\star$ and $r^\star$ are concurrent at a point $x_0\overline{F}$ and p,q and r are non-collinear. Let $u,v,w\in\overline{V}$ be such that $p=u\overline{F}$, $q=v\overline{F}$ and $r=w\overline{F}$. Then

$$\bar{t}(u,x_0,x_0)=\bar{t}(v,x_0,x_0)=\bar{t}(w,x_0,x_0)=0.$$

Since p,q,r are non-collinear, the vectors u,v,w are linearly independent. Thus $\bar{t}(x,x_0,x_0)=0$ for all $x\in\overline{V}$ and $x_0\overline{F}$ is a singular zero of f. This is impossible because f is nonsingular. □

Let us summarize the properties which we obtained on the harmonic polars. For a nonsingular cubic form on V, there are exactly 9 distinct harmonic polars of f. Through the intersection point of two harmonic polars passes a third one; through any given point pass at most 3 harmonic polars. There are 4 triples of points which satisfy the following property: a harmonic polar passes through one and only one point of the triple. Hence the configuration of the harmonic polars is dual to the configuration of the 9 inflection points. Moreover the two configurations are connected: the harmonic polars at 3 inflection points are concurrent if and only if the inflection points are collinear.

3. Classification of cubic subspaces

3.1. Definition

We say that V is a *cubic subspace* of $\mathsf{M}_3(F)$ if V is a 3-dimensional subspace of $\mathsf{M}_3(F)^0$ of cube-central elements. Such a subspace V induces a ternary cubic form

$$f_V\colon V\to F\colon x\mapsto x^3=\det(x).$$

Observe that if V, V' are cubic subspaces of $\mathsf{M}_3(F)$ and $\Theta\colon \mathsf{M}_3(F) \to \mathsf{M}_3(F)$ is an F-algebra automorphism such that $\Theta(V) = V'$ then the cubic forms f_V and $f_{V'}$ are isometric (two cubic forms $f\colon V \to F$ and $g\colon W \to F$ are *isometric* if there exists a vector space isomorphism $\Phi\colon V \to W$ such that $f = g \circ \Phi$). Moreover, by the Skolem-Noether Theorem, an algebra automorphism of $\mathsf{M}_3(F)$ is an inner automorphism

$$\mathsf{int}(m)\colon \mathsf{M}_3(F) \to \mathsf{M}_3(F)\colon x \mapsto mxm^{-1}.$$

We say that a cubic subspace V of $\mathsf{M}_3(F)$ is *nonsingular* if f_V is nonsingular. If V is a nonsingular cubic subspace of $\mathsf{M}_3(F)$, then every conjugate of V (i.e. every mVm^{-1} for some $m \in \mathsf{GL}_3(F)$) is also a nonsingular cubic subspace of $\mathsf{M}_3(F)$.

We shall classify the nonsingular cubic subspaces of $\mathsf{M}_3(F)$ up to conjugacy, by describing explicitly a representative of each conjugacy class of nonsingular cubic subspaces, as in Theorem 1.1. However, before we proceed with the proof of that theorem, we shall first exhibit some simple but useful properties of nonsingular cubic subspaces.

3.2. Preliminary lemmas

Let V be a nonsingular cubic subspace of $\mathsf{M}_3(F)$ and let $t_V : V \times V \times V \to F$ be the symmetric trilinear form on V such that $t_V(x, x, x) = f_V(x)$ for all $x \in V$: we have

$$t_V(x, y, z) = \frac{1}{6}\mathsf{tr}(xyz + xzy).$$

Lemma 3.1 *For all $x, y \in V$, we have* $\mathsf{tr}(xy) = 0$.

Proof: By Cayley-Hamilton Theorem,

$$x^3 - \mathsf{tr}(x)x^2 + \frac{1}{2}\big(\mathsf{tr}(x)^2 - \mathsf{tr}(x^2)\big)x - \det(x) = 0$$

for all $x \in \mathsf{M}_3(F)$. Hence $\mathsf{tr}(x^2) = 0$ for all $x \in V$. It implies that $\mathsf{tr}(xy) = 0$ for all $x, y \in V$. □

Lemma 3.2 *For all nonzero $x \in V$, we have $x^2 \neq 0$.*

Proof: Let $x \in V$ be nonzero. The form f_V is nonsingular, so there exists $y \in V$ such that

$$t_V(x, x, y) = \frac{1}{6}\mathsf{tr}(2x^2y) \neq 0;$$

hence $x^2 \neq 0$. □

Next we give equivalent conditions for $x\overline{F}$ to be a zero of f_V.

Lemma 3.3 *Let $x \in V \setminus \{0\}$. Then the following statements are equivalent:*

1. $f_V(x) = 0$,
2. *the rank of x is equal to 2,*
3. $\mathsf{im}(x) = \mathsf{ker}(x^2)$.

Proof: Suppose that $f_V(x) = 0$. Because $x^3 = 0$ and $x^2 \neq 0$, the Jordan normal form of x is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus (1) implies (2) and (3). Conversely, it is easy to see that (2) and (3) each implies $\det(x) = 0$. $\square$

We need another preliminary result.

Lemma 3.4 *Let $x, y \in V \setminus \{0\}$ be determinant zero matrices. Suppose that $\mathsf{tr}(xy^2) = 0$ and $\mathsf{tr}(x^2y) \neq 0$, then $\mathsf{ker}(y) \not\subset \mathsf{ker}(x^2)$.*

Proof: We have $x^3 = 0$ and $x^2 \neq 0$. So, replacing x and y by conjugates in $\mathsf{M}_3(F)$ if necessary, we may assume that

$$x = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Suppose that $\mathsf{ker}(y) = \mathsf{ker}(x)$, then

$$y = \begin{pmatrix} 0 & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & x_{32} & x_{33} \end{pmatrix}$$

for some $x_{ij} \in F$ and $\mathsf{tr}(x^2y) = 0$ which contradicts the hypothesis; hence $\mathsf{ker}(y) \neq \mathsf{ker}(x)$. Suppose that $\mathsf{ker}(y) \subset \mathsf{ker}(x^2)$, then by Lemma 3.3, there exists a nonzero $a \in F^3$ such that

$$\mathsf{ker}(y) = xa \cdot F.$$

Because $x^3 = 0$ and $\mathsf{ker}(y) \neq \mathsf{ker}(x)$, we have $x^3a = 0$ and $x^2a \neq 0$. So $x^2a \in \mathsf{ker}(x) \setminus \{0\}$, $xa \in \mathsf{ker}(x^2) \setminus \mathsf{ker}(x)$ and $a \notin \mathsf{ker}(x^2)$. Thus (x^2a, xa, a) is a basis of F^3. Let m be the matrix with column x^2a, xa, a. Then

$$m^{-1}xm = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad m^{-1}ym = \begin{pmatrix} y_{11} & 0 & y_{13} \\ y_{21} & 0 & y_{23} \\ y_{31} & 0 & y_{33} \end{pmatrix}$$

for some $y_{ij} \in F$ because $\mathsf{ker}(y) = xa \cdot F$. Since $\mathsf{tr}(y) = 0$ and $\mathsf{tr}(xy) = 0$, we have $y_{33} = -y_{11}$ and $y_{21} = 0$. Thus

$$m^{-1}y^2m = \begin{pmatrix} y_{11}^2 + y_{13}y_{31} & 0 & 0 \\ y_{23}y_{31} & 0 & -y_{23}y_{11} \\ 0 & 0 & y_{31}y_{13} + y_{11}^2 \end{pmatrix}.$$

Then we have $y_{11}^2 + y_{13}y_{31} = 0$ and $y_{23} = 0$ because $\mathsf{tr}(y^2) = 0$, $\mathsf{tr}(xy^2) = 0$ and $\mathsf{tr}(x^2y) \neq 0$. We obtain that $y^2 = 0$ which is impossible. Therefore $\mathsf{ker}(y) \not\subset \mathsf{ker}(x^2)$. □

3.3. Proof of Theorem 1.1

We are now in a position to prove the classification of nonsingular cubic subspaces of $\mathsf{M}_3(F)$ as stated in Theorem 1.1.

Let V be a nonsingular cubic subspace of $\mathsf{M}_3(F)$ and let t_V be the symmetric trilinear form on V such that $t_V(x,x,x) = f_V(x)$ for all $x \in V$. We shall describe V up to conjugacy. We use particular zeros of the nonsingular cubic form f_V. Let $\tilde{u}, \tilde{v}, \tilde{w} \in \overline{V}$ be such that $\tilde{u}\overline{F}$ is an inflection point of f_V, $\tilde{w}\overline{F}$ its Hessian point and $\tilde{v}\overline{F}$ a harmonic point of $\tilde{u}\overline{F}$. Let $\pi\colon \mathbb{P}_V(F) \to \mathbb{P}_V(\overline{F})$ be the inclusion defined by $\pi(xF) = x\overline{F}$. We say that a point $x\overline{F} \in \mathbb{P}_V(\overline{F})$, (resp. a line $\mathcal{L}$ of $\mathbb{P}_V(\overline{F})$, resp. a conic $\mathcal{C}$ of $\mathbb{P}_V(\overline{F})$) is *defined over* F if there exists a point $x'F \in \mathbb{P}_V(F)$ (resp. a line $\mathcal{L}'$ of $\mathbb{P}_V(F)$, resp. a conic $\mathcal{C}'$ of $\mathbb{P}_V(F)$) such that $\pi(x'F) = x\overline{F}$ (resp. $\pi(\mathcal{L}') = \mathcal{L}$, resp. $\pi(\mathcal{C}') = \mathcal{C}$). Since the inflection points of a nonsingular cubic form are defined over F, the point $\tilde{u}\overline{F}$ is defined over F. Then the tangent to f_V at $\tilde{u}\overline{F}$ and the conic

$$\{x\overline{F} \in \mathbb{P}_V(\overline{F}) \mid \overline{t_V}(\tilde{u}, x, x) = 0\}$$

are defined over F; hence so is the harmonic polar at $\tilde{u}\overline{F}$. Therefore $\tilde{v}\overline{F}$ and $\tilde{w}\overline{F}$ are defined over F (the fact that $\tilde{v}\overline{F}$ is defined over F follows from Proposition 2.2). We may thus assume that $\tilde{u}, \tilde{v}, \tilde{w} \in V$.

Since $\tilde{u}\overline{F}$ is an inflection point of f_V and $\tilde{v}\overline{F}$ is a harmonic point of $\tilde{u}\overline{F}$, the points $\tilde{u}\overline{F}$ and $\tilde{v}\overline{F}$ are zeros of f such that $\tilde{v}\overline{F}$ is on the conic

$$\{x\overline{F} \in \mathbb{P}_V(\overline{F}) \mid \overline{t_V}(\tilde{u}, x, x) = 0\}$$

and $\tilde{v}\overline{F}$ is not on the tangent to f at $\tilde{u}\overline{F}$ (because the unique zero of f on the tangent at $\tilde{u}\overline{F}$ is $\tilde{u}\overline{F}$). Hence $\tilde{u}, \tilde{v} \in V \setminus \{0\}$ are determinant zero matrices such that $\mathsf{tr}(\tilde{u}\tilde{v}^2) = 0$ and $\mathsf{tr}(\tilde{u}^2\tilde{v}) \neq 0$. The next proposition determines $\tilde{u}$ and $\tilde{v}$ up to conjugacy.

Proposition 3.5 *There exists an $m \in \mathsf{GL}_3(F)$ and $\lambda, \mu \in F^\times$ such that*

$$m\tilde{u}m^{-1} = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad m\tilde{v}m^{-1} = \mu \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

Moreover if $m' \in \mathsf{GL}_3(F)$ and $\lambda', \mu' \in F^\times$ satisfy the same conditions, then $mF^\times = m'F^\times$, $\lambda = \lambda'$ and $\mu = \mu'$.

Proof : Let $a \in F^3$ be such that $\mathsf{ker}(\tilde{v}) = aF$. By Lemma 3.4, we have $a \notin \mathsf{ker}(\tilde{u}^2)$. Thus $(\tilde{u}^2 a, \tilde{u}a, a)$ is a basis of F^3. Let m_1 be the matrix with columns $\tilde{u}^2 a, \tilde{u}a, a$, then

$$m_1^{-1}\tilde{u}m_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad m_1^{-1}\tilde{v}m_1 = \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & 0 \end{pmatrix}$$

for some $x_{ij} \in F$. Since $\mathsf{tr}(\tilde{v}) = 0$ and $\mathsf{tr}(\tilde{u}\tilde{v}) = 0$, we have $x_{22} = -x_{11}$ and $x_{32} = -x_{21}$. Hence

$$m_1^{-1}\tilde{v}^2 m_1 = \begin{pmatrix} x_{11}^2 + x_{12}x_{21} & 0 & 0 \\ 0 & x_{21}x_{12} + x_{11}^2 & 0 \\ x_{31}x_{11} - x_{21}^2 & x_{31}x_{12} + x_{21}x_{11} & 0 \end{pmatrix}.$$

Then $\mathsf{tr}(\tilde{v}^2) = 0$ and $\mathsf{tr}(\tilde{u}\tilde{v}^2) = 0$ imply

$$\begin{aligned} x_{11}^2 + x_{12}x_{21} &= 0, \\ x_{31}x_{12} + x_{21}x_{11} &= 0. \end{aligned}$$

Therefore, we have $x_{31}x_{11} - x_{21}^2 \neq 0$ because $\tilde{v}^2 \neq 0$. Since

$$x_{12}(x_{31}x_{11} - x_{21}^2) = x_{11}(x_{31}x_{12} + x_{21}x_{11}) - x_{21}(x_{11}^2 + x_{12}x_{21}) = 0$$

we have $x_{12} = 0$, so $x_{11} = 0$ and $x_{21} \neq 0$. Hence

$$m_1^{-1}\tilde{v}m_1 = \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & 0 & 0 \\ x_{31} & -x_{21} & 0 \end{pmatrix}$$

with $x_{31} \neq 0$ because $\mathsf{tr}(\tilde{u}^2\tilde{v}) \neq 0$. Choosing

$$m := \begin{pmatrix} x_{21}^{-2}x_{31}^2 & 0 & 0 \\ 0 & x_{21}^{-1}x_{31} & 0 \\ 0 & 0 & 1 \end{pmatrix} m_1^{-1}$$

we get

$$m\tilde{u}m^{-1} = \frac{x_{31}}{x_{21}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } m\tilde{v}m^{-1} = \frac{x_{21}^2}{x_{31}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

For the uniqueness, it is easy to check that if $n \in \mathsf{GL}_3(F)$ and $\alpha, \beta \in F^\times$ are such that

$$n \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \alpha \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} n$$

and

$$n \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} = \beta \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} n$$

then $n \in F^\times$ and $\alpha = \beta = 1$. $\square$

Before describing V up to conjugacy, we fix some notation. Let ω denote a primitive cube root of unity in F. We set

$$u := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad v := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \quad w_1(\alpha) := \begin{pmatrix} \alpha & -\frac{1}{2} & 1 \\ 3\alpha^2 & -2\alpha & \frac{1}{2} \\ 0 & -3\alpha^2 & \alpha \end{pmatrix}$$

$$\begin{aligned} w_2(\alpha) &:= \begin{pmatrix} \alpha & \frac{1}{2}\big((\omega^2-1)\alpha - 1\big) & 1 \\ 0 & \omega\alpha & \frac{1}{2}\big((\omega^2-1)\alpha + 1\big) \\ 0 & 0 & \omega^2\alpha \end{pmatrix}, \\ w_3(\alpha) &:= \begin{pmatrix} \alpha & \frac{1}{2}\big((\omega-1)\alpha - 1\big) & 1 \\ 0 & \omega^2\alpha & \frac{1}{2}\big((\omega-1)\alpha + 1\big) \\ 0 & 0 & \omega\alpha \end{pmatrix}, \end{aligned}$$

for $\alpha \in F$.

Proposition 3.6 *Let V be a nonsingular cubic subspace of $\mathsf{M}_3(F)$. Let $\tilde{u}, \tilde{v}, \tilde{w} \in V$ be such that $\tilde{u}\overline{F}$ is an inflection point of f_V, $\tilde{w}\overline{F}$ its Hessian point and $\tilde{v}\overline{F}$ a harmonic point of $\tilde{u}\overline{F}$. Let $mF^\times$ be the unique element of $\mathsf{PGL}_3(F)$ such that*

$$m\tilde{u}m^{-1}F = uF \qquad \text{and} \qquad m\tilde{v}m^{-1}F = vF.$$

Then $m\tilde{w}m^{-1}F = w_i(\alpha)F$ for some $\alpha \in F$ and $i \in \{1,2,3\}$. In particular, V is conjugate to the F-vector subspace of $\mathsf{M}_3(F)$ spanned by $u, v, w_i(\alpha)$.

Proof : The point $\tilde{v}\overline{F}$ is not on the tangent at $\tilde{u}\overline{F}$ which is the line through the distinct points $\tilde{u}\overline{F}$ and $\tilde{w}\overline{F}$. Hence $\tilde{u}\overline{F}$, $\tilde{v}\overline{F}$ and $\tilde{w}\overline{F}$ are not collinear and $(\tilde{u}, \tilde{v}, \tilde{w})$ is a basis of V.

By Proposition 3.5, we may assume that $\tilde{u} = u$ and $\tilde{v} = v$. Because $\tilde{w} \in V$, we have $\mathsf{tr}(\tilde{w}) = 0$, $\mathsf{tr}(u\tilde{w}) = 0$ and $\mathsf{tr}(v\tilde{w}) = 0$; thus

$$\tilde{w} = \begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{12} + x_{13} \\ x_{31} & -x_{21} & -x_{11} - x_{22} \end{pmatrix}$$

for some $x_{ij} \in F$. Observe that $x_{13} \neq 0$ because otherwise $\overline{t_V}(v,v,x) = 0$ for all $x \in \overline{V}$ and $v\overline{F}$ would be a singular zero of f_V. Replacing $\tilde{w}$ by a multiple if necessary, we may assume that $x_{13} = 1$. Because $\mathsf{tr}(\tilde{w}^2) = 0$, we have

$$x_{21} = x_{11}^2 + x_{22}^2 + x_{11}x_{22} + x_{31}.$$

Since $\tilde{w}\overline{F}$ is the Hessian point of $u\overline{F}$, we have $\overline{t_V}(u, \tilde{w}, x) = 0$ for all $x \in V$. Replacing x by u, v and $\tilde{w}$ successively, we get

$$\begin{gathered} x_{31} = 0, \\ x_{12} = -\tfrac{1}{2}(2x_{11} + x_{22} + 1), \\ (2x_{11} + x_{22})(x_{11}^2 + x_{22}^2 + x_{11}x_{22}) = 0. \end{gathered}$$

If $x_{22} = -2x_{11}$, then $\tilde{w} = w_1(x_{11})$. Otherwise we have $x_{11}^2 + x_{22}^2 + x_{11}x_{22} = 0$ which implies $x_{22} = \omega x_{11}$ or $x_{22} = \omega^2 x_{11}$; hence $\tilde{w} = w_2(x_{11})$ or $\tilde{w} = w_3(x_{11})$. □

We call a nonsingular cubic subspace of $\mathsf{M}_3(F)$ which is spanned by u, v and $w_i(\alpha)$ for some $\alpha \in F$ and some $i \in \{1,2,3\}$, a *special subspace* of $\mathsf{M}_3(F)$. To improve the previous proposition, we give a geometric condition for special subspaces of $\mathsf{M}_3(F)$ to be conjugate.

Proposition 3.7 *Let V be a special subspace of $\mathsf{M}_3(F)$. Suppose that mVm^{-1} is special for some $m \in \mathsf{GL}_3(F)$. Then $m^{-1}um\overline{F}$ is an inflection point of f_V and $m^{-1}vm\overline{F}$ is a harmonic point of $m^{-1}um\overline{F}$.*

Proof: Let $\alpha \in F$ and $i \in \{1,2,3\}$ be such that $w_i(\alpha) \in mVm^{-1}$, then $w_i(\alpha)\overline{F}$ is the Hessian point of $u\overline{F}$. Set $\tilde{u} := m^{-1}um$, $\tilde{v} := m^{-1}vm$ and $\tilde{w} := m^{-1}w_i(\alpha)m$. Then $\tilde{u}^3 = m^{-1}u^3m = 0$ and

$$\mathsf{tr}\big((\tilde{u}\tilde{w} + \tilde{w}\tilde{u})x\big) = \mathsf{tr}\Big(\big(uw_i(\alpha) + w_i(\alpha)u\big)mxm^{-1}\Big) = 0$$

for all $x \in \overline{V}$. Hence $\tilde{u}\overline{F}$ is an inflection point of f_V and $\tilde{w}\overline{F}$ is its Hessian point. We can prove similarly that $\tilde{v}^3 = 0$, $\mathsf{tr}(\tilde{u}\tilde{v}^2) = 0$ and $\mathsf{tr}(\tilde{u}^2\tilde{v}) \neq 0$, so $\tilde{v}\overline{F}$ is a harmonic point of $\tilde{u}\overline{F}$. □

Hence, to find the conjugates of V which are special subspaces of $\mathsf{M}_3(F)$, it suffices to find the inflection points of f_V and their harmonic points and, for each couple $(\tilde{u}\overline{F}, \tilde{v}\overline{F})$ which consists of an inflection point and one of its harmonic points, to find the $mF^\times \in \mathsf{PGL}_3(F)$ such that

$$\begin{cases} m\tilde{u}m^{-1}F = uF, \\ m\tilde{v}m^{-1}F = vF. \end{cases}$$

Then, by Proposition 3.6, mVm^{-1} is a special subspace which is a conjugate of V.

To simplify notation we write $V_{i,\alpha}$ for the subspace of $\mathsf{M}_3(F)$ spanned by u, v and $w_i(\alpha)$. We shall work out the case where V is equal to $V_{1,\alpha}$ and prove that it is conjugate to $V_{2,\beta}$ for some $\beta \in F$. Set

$$P(X_1, X_2, X_3) := \alpha^3(1-9\alpha)X_3^3 + X_1^2X_2 - X_2^2X_3 - (6\alpha^2 - 3\alpha + 1/4)X_2X_3^2$$

and let $H(X_1, X_2, X_3)$ be equal to

$$\det \begin{pmatrix} \frac{\partial^2 P}{\partial X_1^2}(X_1, X_2, X_3) & \frac{\partial^2 P}{\partial X_1 \partial X_2}(X_1, X_2, X_3) & \frac{\partial^2 P}{\partial X_1 \partial X_3}(X_1, X_2, X_3) \\ \frac{\partial^2 P}{\partial X_1 \partial X_2}(X_1, X_2, X_3) & \frac{\partial^2 P}{\partial X_2^2}(X_1, X_2, X_3) & \frac{\partial^2 P}{\partial X_2 \partial X_3}(X_1, X_2, X_3) \\ \frac{\partial^2 P}{\partial X_1 \partial X_3}(X_1, X_2, X_3) & \frac{\partial^2 P}{\partial X_2 \partial X_3}(X_1, X_2, X_3) & \frac{\partial^2 P}{\partial X_3^2}(X_1, X_2, X_3) \end{pmatrix}$$

so that $P(x_1, x_2, x_3) = \overline{f_V}\big(x_1u + x_2v + x_3w_1(\alpha)\big)$ for all $x_1, x_2, x_3 \in \overline{F}$ and a point $x\overline{F} = \big(x_1u + x_2v + x_3w_1(\alpha)\big)\overline{F} \in \mathbb{P}_V(\overline{F})$ is on the Hessian curve if and only if $H(x_1, x_2, x_3) = 0$. The cubic form f_V is nonsingular if and only if the system of equations

$$\frac{\partial P}{\partial X_i}(X_1, X_2, X_3) = 0, \qquad \text{for all } i = 1, 2, 3$$

has no nontrivial solutions; thus $\alpha \neq 0, 1/8, 1/9$.

In the next proposition, we prove that in the couple $(\tilde{u}\overline{F}, \tilde{v}\overline{F})$, the choice of $\tilde{v}\overline{F}$ among the harmonic points of $\tilde{u}\overline{F}$ is not important to find an invertible matrix m such that $mV_{1,\alpha}m^{-1} = V_{2,\beta}$.

Proposition 3.8 *Let V be a nonsingular cubic subspace of $\mathsf{M}_3(F)$. Suppose that $\tilde{u}\overline{F}$ is an inflection point of f_V, $\tilde{v}\overline{F}, \tilde{v}'\overline{F}$ are harmonic points of $\tilde{u}\overline{F}$ and $m, m' \in \mathsf{GL}_3(F)$ are such that*

$$m\tilde{u}m^{-1}F = uF,\ m\tilde{v}m^{-1}F = vF, \qquad m'\tilde{u}m'^{-1}F = uF,\ m'\tilde{v}'m'^{-1}F = vF,$$
$$mVm^{-1} = V_{j,\beta}, \qquad m'Vm'^{-1} = V_{j',\beta'}.$$

Then we have $j = j'$.

Proof : We have $m'm^{-1}u(m'm^{-1})^{-1}F = uF$ and $m'm^{-1}V_{j,\beta}(m'm^{-1})^{-1} = V_{j',\beta'}$. Set $n := m'm^{-1}$, then $nun^{-1} = \lambda u$ implies that

$$nF^\times = \begin{pmatrix} \lambda^2 & \lambda a & b \\ 0 & \lambda & a \\ 0 & 0 & 1 \end{pmatrix} F^\times$$

for some $a, b \in F$. Observe that $nw_j(\beta)n^{-1}\overline{F}$ is the Hessian point of $u\overline{F}$ in $\mathbb{P}_{V_{j',\beta'}}(\overline{F})$; hence $nw_j(\beta)n^{-1}\overline{F} = w_{j'}(\beta')\overline{F}$. If $j = 1$, then $j' = 1$, because otherwise $n^{-1}w_{j'}(\beta')n$ is an upper triangular matrix and $n^{-1}w_{j'}(\beta')n\overline{F} \neq w_1(\beta)\overline{F}$

($\beta \neq 0$ because $V_{1,\beta}$ is nonsingular). If $j = 2$, then

$$nw_j(\beta)n^{-1} = \begin{pmatrix} \beta & \star & \star \\ 0 & \omega\beta & \star \\ 0 & 0 & \omega^2\beta \end{pmatrix},$$

so $nw_j(\beta)n^{-1}\overline{F} = w_j(\beta')\overline{F}$ for some β', and $j = j'$. Clearly we obtain the same conclusion if $j = 3$. $\square$

Suppose that $\tilde{u}\overline{F}$ is an inflection point of $f_{V_{1,\alpha}}$ with $\tilde{u} = a_1u + a_2v + a_3w_1(\alpha)$, then $\tilde{u}\overline{F} = u\overline{F}$ if and only if $a_3 = 0$. By Proposition 3.8, for all harmonic points $\tilde{v}\overline{F}$ of $u\overline{F}$, the matrix $m \in \mathsf{GL}_3(F)$ such that $mum^{-1}F = uF$ and $m\tilde{v}m^{-1}F = vF$, conjugates V into $V_{1,\alpha'}$ for some $\alpha' \in F$. Therefore we may assume that $\tilde{u}\overline{F} \neq u\overline{F}$ and $a_3 = 1$. Observe that $a_2 \neq 0$. Since

$$a_1^2a_2 = \alpha^3(9\alpha - 1) + a_2^2 + (6\alpha^2 - 3\alpha + \frac{1}{4})a_2 \tag{1}$$

and $a_2H(a_1, a_2, a_3) = 0$, we have

$$\big(a_2 - \alpha(9\alpha - 1)\big)\big(a_2^3 + \alpha(9\alpha - 1)a_2^2 + \alpha^2(3\alpha - 1)(9\alpha - 1)a_2 + 3\alpha^5(9\alpha - 1)\big) = 0.$$

Thus, either $a_2 = \alpha(9\alpha - 1)$ or $a_2 = \alpha/3(-2\theta^2 + \theta + 1 - 9\alpha)$ for some $\theta \in F$ such that $\theta^3 = 1 - 9\alpha$. Let $\theta \in F$ be a cube root of $1 - 9\alpha$ in F and set $a_2 := \alpha/3(-2\theta^2 + \theta + 1 - 9\alpha)$. Then the relation (1) implies that

$$a_1^2 = \Big(\frac{\omega - \omega^2}{18}(-4\theta^2 + 2\theta - 1)\Big)^2.$$

We set $a_1 := (\omega - \omega^2)/18(-4\theta^2 + 2\theta - 1)$. Observe that the third inflection point on the line through $u\overline{F}$ and $\tilde{u}\overline{F}$ is the point $\big(-a_1u + a_2v + w_1(\alpha)\big)\overline{F}$. By Proposition 2.3, the third zero of f on the line through $\big(-a_1u + a_2v + w_1(\alpha)\big)\overline{F}$ and $v\overline{F}$ is a harmonic point of $\tilde{u}\overline{F}$; let $\tilde{v}\overline{F}$ be this point. The points on the line through $\big(-a_1u+a_2v+w_1(\alpha)\big)\overline{F}$ and $v\overline{F}$ are of the form $\big(-a_1\lambda u+\mu v+\lambda w_1(\alpha)\big)\overline{F}$. Because

$$P(-a_1\lambda, \mu, \lambda) = -\lambda(\mu - a_2\lambda)(\mu - b_2\lambda)$$

with $b_2 := 1/9\big((6\alpha - 1)\theta^2 + (9\alpha - 1)\theta - (9\alpha - 1)(3\alpha - 1)\big)$, we obtain that $\tilde{v}\overline{F} = \big(-a_1u + b_2v + w_1(\alpha)\big)\overline{F}$. With the help of the software Mathematica, we find that the matrix

$$m := \begin{pmatrix} 1 & \frac{\omega\theta^2+\omega^2\theta+1-9\alpha}{3\alpha} & \frac{(1-\omega^2)(\theta^2+9\alpha-1)}{3\alpha} \\ \frac{(\omega-1)(\theta-1)}{3} & \omega^2\theta & \frac{-\omega^2\theta^2+(9\alpha-1)\theta+\omega(9\alpha-1)}{3\alpha} \\ \frac{-\omega^2\theta^2-\theta+3(\omega-\omega^2)\alpha-\omega}{3} & \frac{\theta^2+9\alpha-1}{\omega-\omega^2} & -\omega^2\theta^2 + \omega(9\alpha - 1) \end{pmatrix}$$

is invertible and is such that $m^{-1}\tilde{u}m\overline{F} = u\overline{F}$ and $m^{-1}\tilde{v}m\overline{F} = v\overline{F}$. Now we find the Hessian point $\tilde{w}\overline{F}$ of $\tilde{u}\overline{F}$. We have $\tilde{w}\overline{F} = \big(c_1u + c_2v + c_3w_1(\alpha)\big)\overline{F}$ with

$c_1 = -a_1 a_2^{-1}\big(3\alpha^3(1-9\alpha)+a_2^2\big)$, $c_2 = 3\alpha^3(1-9\alpha)+a_2^2$ and $c_3 = \alpha^3(1-9\alpha)a_2^{-1}-a_2$ because it satisfies the relations $\overline{t_V}(\tilde{u}, \tilde{w}, x) = 0$ for all $x \in \overline{V}$. We obtain that $m^{-1}\tilde{w}m\overline{F} = w_2(\beta)\overline{F}$ with $\beta = -\omega^2\alpha(9\alpha-1)^{-1}$. Hence $V_{1,\alpha}$ is conjugate to $V_{2,-\omega^2\alpha(9\alpha-1)^{-1}}$ for all $\alpha \neq 0, 1/8, 1/9$.

Observe that $f_{V_{2,\beta}}$ is singular if and only if $\beta \in \{0, -\omega^2/3, -\omega^2/9\}$ and $(9\alpha-1)\beta = -\omega^2\alpha$ if and only if $(9\beta+\omega^2)\alpha = \beta$. Therefore, for all $\beta \notin \{0, -\omega^2/3, -\omega^2/9\}$, $V_{2,\beta}$ is conjugate to $V_{1,\beta(9\beta+\omega^2)^{-1}}$. Replacing ω by ω^2, we obtain that, for all $\beta \notin \{0, -\omega/3, -\omega/9\}$, $V_{3,\beta}$ is conjugate to $V_{1,\beta(9\beta+\omega)^{-1}}$. Hence we can improve the classification of nonsingular cubic subspaces of $\mathsf{M}_3(F)$: every nonsingular cubic subspace of $\mathsf{M}_3(F)$ is conjugate to $V_{1,\alpha}$ for some $\alpha \in F \setminus \{0, 1/8, 1/9\}$. This proves Theorem 1.1 as stated in the introduction.

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