Ramification sequences of central simple algebras

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Abstract

In this paper, we study central simple algebras over function fields in one variable through their ramification sequence. We determine a sufficient condition under which the Faddeev index of algebras with a given degree four ramification sequence is two. Further, we study through several examples other related conditions concerning Faddeev index of central simple algebras over function fields.

1 Introduction

Let F be a field with characteristic zero and $K = F(t)$, the function field in one variable over F. Let $x \in \mathbb{P}^1_F$ be a closed point and F_x be the residue of F at x. Let ${}_2\mathsf{Br}(K)$ denote the 2-torsion part of the Brauer group $\mathsf{Br}(K)$ of K. By [10, II.§3] there is a map $\delta_x : \mathsf{Br}(K) \to H^1(F_x, \mathbb{Z}/2) \cong F_x^\times/F_x^{\times 2}$ called the *residue map*. In 1956, Faddeev [2] gave the following description of ${}_2\mathsf{Br}(K)$:

Theorem 1 *With the notation as above, the following sequence is exact:*

$$0 \longrightarrow {}_2\mathsf{Br}(F) \longrightarrow {}_2\mathsf{Br}(K) \xrightarrow{\oplus\delta_x} \bigoplus_{x\in\mathbb{P}_F^{1(1)}} H^1(F_x, \mathbb{Z}/2) \xrightarrow{cor} H^1(F, \mathbb{Z}/2) \longrightarrow 0$$

where ${}_2\mathsf{Br}(F)$ is the 2-torsion part of the Brauer group of F, $\mathbb{P}_F^{1(1)}$ is the set of closed point of $\mathbb{P}^1_F$ and cor is the sum of corestriction maps $cor_x : H^1(F_x, \mathbb{Z}/2) \to H^1(F, \mathbb{Z}/2)$. □

Under the isomorphisms $H^1(F_x, \mathbb{Z}/2) \simeq F_x^*/F_x^{*2}$ and $H^1(F, \mathbb{Z}/2) \simeq F^*/F^{*2}$, the map *cor* corresponds to $\prod_{x\in\mathbb{P}^1_F} N_{F_x/F} : F_x^*/F_x^{*2} \to F^*/F^{*2}$, where $N_{F_x/F}$ are norm maps from finite extensions F_x/F to F.

Let A be a central simple algebra of exponent two over K which represents $[A] \in {}_2\mathsf{Br}(K)$. The residue $\delta_x([A])$ is trivial for all but finitely many $x \in \mathbb{P}^1_F$. The sequence $\{(x, \delta_x([A]))\}_{\delta_x([A])\neq 0}$ is called the *ramification sequence* of A. The following statements follows immediately from Theorem 1:

(i) For arbitrary $a_x \in H^1(F_x, \mathbb{Z}/2)$, $a_x \neq 0$; the sequence $\{(x, a_x)\}$ is a ramification sequence for some 2-torsion central simple algebra if and only if $\sum_x cor_x(a_x) = 0 \in H^1(F, \mathbb{Z}/2)$. This allows us to talk of ramification sequences without explicit reference to the central simple algebra. We say therefore that the elements in the kernal of *cor* are *ramification sequences.*

(ii) Two central simple algebras A and B of exponent two over K have the same ramification sequence if and only if there exists an exponent two central simple algebra $\mathcal{C}$ over F such that $A \otimes_F \mathcal{C} \simeq B$.

For a central simple algebra A of exponent two, the notion of *Faddeev index* $fi(A)$ is defined as follows:

$$fi(A) = \min\{\mathsf{index}(A \otimes_F \mathcal{C}) : \mathcal{C} \text{ is an exponent two algebra over } F\}$$

It follows in view of (ii) above that if two algebras over K have same ramification sequence, then their Faddeev indices are equal.

Let $\rho = \{(x, a_x)\}$, $a_x \neq 0 \in H^1(F_x)$ be an arbitrary ramification sequence. Then we define the *Faddeev index of* ρ to be the Faddeev index of a K-algebra whose ramification sequence is $\{(x, a_x)\}$. This index is independent of the choice of the K-algebra.

Let $\mathsf{deg}(x)$ denote the degree of the closed point x. Then $\sum_{\delta_x([A])\neq 0} \mathsf{deg}(x)$ is called the *degree of the ramification sequence of* A. In view of (i) above, it follows that the degree of ramification sequences is at least two.

It was proved in [5, Cor. 2.4] and [8, §4, §5] that the ramification sequences ρ of degree two or three have Faddeev index two. Also, it was shown in [8, Cor. 4.2, Cor. 4.3] that the quaternion algebras A with ρ of degree two as their ramification sequence forms 1-parameter family while for the case when $\mathsf{deg}(\rho) = 3$, such algebra is unique upto isomorphism [8, Cor 5.2].

It is therefore interesting to study similar questions when $\mathsf{deg}(\rho) = 4$. There are examples of ramification sequences with degree four and Faddeev index four. Thus we ask the following question:

Question 2 *Which ramification sequences of degree four have Faddeev index two?*

In section 2, we give a sufficient condition on degree four linear ramification sequences to have Faddeev index two. In section 3, we give a necessary condition on degree four linear ramification sequence over $\mathbb{Q}$, with two identical ramifications, to have Faddeev index two.

2 Ramification sequence of degree four over an arbitrary field

In this section, F is an arbitrary field of characteristic zero. We give several recalls. The notation $\mathbb{P}^1_F$ means $\mathsf{Proj}(F[u,v])$ which is the set of homogeneous prime ideals of $F[u,v]$ which do not contain the ideal generated by u and v. There is a bijection between

$$\mathbb{P}^1_F \leftrightarrow \{\text{monic irreducible polynomials of } F[t]\} \cup \{\infty\}$$

sending the ideal generated by $p(u,v)$, where $p(u,v)$ is a homogeneous irreducible polynomial such that v does not divide $p(u,v)$ and $p(t,1)$ is a monic polynomial, to $p(t,1)$, and the ideal generated by v to ∞. We identify an element of $\mathbb{P}^1_F$ with its image by this bijection. We say that $x \in \mathbb{P}^1_F$ is a rational point if $x = p(t) \in F[t]$ and $\mathsf{deg} p = 1$ or if $x = \infty$. The rational points are in one to one correspondance with $F \cup \{\infty\}$. We have other isomorphisms $\mathsf{H}^1(F_x, \mathbb{Z}/2) \cong F_x^\times / F_x^{\times 2}$ and $\mathsf{H}^1(F, \mathbb{Z}/2) \cong F^\times / F^{\times 2}$ and we also identify elements with their image by these isomorphisms. Let $x = p(t)$ be in $\mathbb{P}^1_F$, then we denote by v_x the discrete valuation over $F(t)$ define by

$$v_x(f(t)) = \text{the biggest integer } n \text{ such that } p(t)^n \text{ divides } f(t),$$

for $f(t) \in F[t]$. We denote by v_∞ the discrete valuation over $F(t)$ define by $v_\infty(f(t)) = -\mathsf{deg} f(t)$, for $f(t) \in F[t]$. We may describe, for $x \in \mathbb{P}^1_F$, the residue field F_x and the map δ_x. Let $x \in \mathbb{P}^1_F$, then F_x is the residue field of the valuation v_x. If $x = p(t)$, then $F_x = F[t]/(p(t))$. Let θ be a root of $p(t)$ (in the algebraic closure of F), then there is an F-isomorphism between F_x and $F(\theta)$ define by $t \mapsto \theta$. If $x = \infty$, then

$$F_x = \{f(t)/g(t) \mid \mathsf{deg} f(t) \le \mathsf{deg} g(t)\} / \{f(t)/g(t) \mid \mathsf{deg} f(t) < \mathsf{deg} g(t)\},$$

which is isomorphic to F sending the class of $f(t)/g(t)$, where $\mathsf{deg} f(t) = \mathsf{deg} g(t)$, $f(t) = a_n t^n + \ldots + a_0$ and $g(t) = b_n t^n + \ldots + b_0$, to a_n/b_n. Let $x \in \mathbb{P}^1_F$ and $f(t), g(t) \in F(t)$, then

$$\delta_x(f(t), g(t))_{F(t)} = (-1)^{v_x(f(t)) v_x(g(t))} f(t)^{v_x(g(t))} g(t)^{-v_x(f(t))} F_x^{\times 2}.$$

These definitions do not depend on the choice of homogeneous coordinates of $\mathbb{P}^1_F$. A transformation of $\mathbb{P}^1_F$ (i.e. a linear change of homogeneous coordinates) is dertermined by its restriction on the rational points which is a homography of $F \cup \{\infty\}$. So given two triples of distinct rational points, there exists an unique transformation of $\mathbb{P}^1_F$ sending a triple on the second one (see [Berger, 1994], 4.6.9). The cross ratio of a quadruple of distinct points $\{a, b, c, d\}$ of $F \cup \{\infty\}$, denoted by $[a, b, c, d]$, is $f(d)$ where f is the unique automorphism of

$F \cup \{\infty\}$ such that $f(a) = \infty$, $f(b) = 0$ and $f(c) = 1$ (see [Berger, 1994], 6.1.2). So given two quadruples $\{a_i\}$ and $\{a'_i\}$ of distinct rational points, there exists an unique transformation f of $\mathbb{P}^1_F$ such that $f(a_i) = a'_i$ for all i if and only if $[a_1, a_2, a_3, a_4] = [a'_1, a'_2, a'_3, a'_4]$. Suppose the homogeneous coordinates of a_i in an arbitrary basis are $(\lambda_i : \mu_i)$ then the cross ratio is

$$[a_1, a_2, a_3, a_4] = \frac{\begin{vmatrix} \lambda_3 & \lambda_1 \\ \mu_3 & \mu_1 \end{vmatrix} \begin{vmatrix} \lambda_4 & \lambda_2 \\ \mu_4 & \mu_2 \end{vmatrix}}{\begin{vmatrix} \lambda_3 & \lambda_2 \\ \mu_3 & \mu_2 \end{vmatrix} \begin{vmatrix} \lambda_4 & \lambda_1 \\ \mu_4 & \mu_1 \end{vmatrix}}$$

(see [Berger, 1994], 6.2.3). Given a quadruple of distinct rational points, we may find homogeneous coordinates such that the quadruple is $\{\infty, t, t-1, t-c\}$, for some $c \in F$.

Proposition 3 *Let* $\rho = \{(\infty, \alpha F^{\times 2}), (t, \beta F^{\times 2}), (t-1, \gamma F^{\times 2}), (t-c, \alpha\beta\gamma F^{\times 2})\}$ *be a ramification sequence of degree four. Suppose there exists* $x, y, z, w \in F^\times$ *such that*

$$c = \frac{(\alpha x^2 - \gamma z^2)(\beta y^2 - \alpha\beta\gamma w^2)}{(\beta y^2 - \gamma z^2)(\alpha x^2 - \alpha\beta\gamma w^2)}. \quad (1)$$

Then the Faddeev index of ρ *is two.*

Proof: Let $x, y, z, w \in F^\times$ such that

$$c = \frac{(\alpha x^2 - \gamma z^2)(\beta y^2 - \alpha\beta\gamma w^2)}{(\beta y^2 - \gamma z^2)(\alpha x^2 - \alpha\beta\gamma w^2)}.$$

Then

$$[\infty, 0, 1, c] = c = \frac{(\alpha x^2 - \gamma z^2)(\beta y^2 - \alpha\beta\gamma w^2)}{(\beta y^2 - \gamma z^2)(\alpha x^2 - \alpha\beta\gamma w^2)} = [\alpha x^2, \beta y^2, \gamma z^2, \alpha\beta\gamma w^2],$$

and so there exist homogeneous variables u' and v' of $\mathbb{P}^1_F$ such that

$$\begin{aligned} vF &= (u' - \alpha x^2 v')F, \\ uF &= (u' - \beta y^2 v')F, \\ (u - v)F &= (u' - \gamma z^2 v')F, \\ (u - cv)F &= (u' - \alpha\beta\gamma w^2 v')F. \end{aligned}$$

Hence

$$\rho = \{(t' - \alpha x^2, \alpha F^{\times 2}), (t' - \beta y^2, \beta F^{\times 2}), (t' - \gamma z^2, \gamma F^{\times 2}), (t' - \alpha\beta\gamma w^2, \alpha\beta\gamma F^{\times 2})\}.$$

We consider the quaternion algebra

$$Q = \Big(t', (t' - \alpha x^2)(t' - \beta y^2)(t' - \gamma z^2)(t' - \alpha\beta\gamma w^2)\Big)_{F(t')}.$$

Then

$$\delta_{t'-\alpha x^2}(Q) = t'(F_{t'-\alpha x^2})^{\times 2} = \alpha x^2 F^{\times 2} = \alpha F^{\times 2},$$

and, in the same way,

$$\begin{aligned}
\delta_{t'-\beta y^2}(Q) &= \beta F^{\times 2}, \\
\delta_{t'-\gamma z^2}(Q) &= \gamma F^{\times 2}, \\
\delta_{t'-\alpha\beta\gamma w^2}(Q) &= \alpha\beta\gamma F^{\times 2}.
\end{aligned}$$

Also

$$\begin{aligned}
\delta_{t'}(Q) &= (t'-\alpha x^2)(t'-\beta y^2)(t'-\gamma z^2)(t'-\alpha\beta\gamma w^2)(F_{t'})^{\times 2} \\
&= \alpha^2\beta^2\gamma^2 x^2 y^2 z^2 w^2 F^{\times 2} \\
&= F^{\times 2}
\end{aligned}$$

and

$$\begin{aligned}
\delta_{\infty}(Q) &= (-1)^4 \frac{(t'-\alpha x^2)(t'-\beta y^2)(t'-\gamma z^2)(t'-\gamma z^2)(t'-\alpha\beta\gamma w^2)}{t'^4}(F_{\infty})^{\times 2} \\
&= F^{\times 2}.
\end{aligned}$$

So the ramification sequence of Q is ρ and the Faddeev index of ρ is equal to two. $\square$

We remark that (1) implies

$$c = \frac{\left(x^2 - \alpha\gamma(z/\gamma)^2\right)\left(y^2 - \alpha\gamma w^2\right)}{\left(y^2 - \beta\gamma(z/\beta)^2\right)\left(x^2 - \beta\gamma w^2\right)}.$$

In particular, if $\alpha F^{\times 2} = \beta F^{\times 2}$ and $\alpha F^{\times 2} \neq \gamma F^{\times 2}$, then (1) implies that c is a norm over $F(\sqrt{\alpha\gamma})/F$. By the proposition 2.7 in [Kunyavskii et al, 2006], if one of the quaternion algebras $(\alpha, c)_{F(\sqrt{\gamma})}$ and $(\gamma, c)_{F(\sqrt{\alpha})}$ is not trivial ($\alpha F^{\times 2} = \beta F^{\times 2}$ and $\alpha F^{\times 2} \neq \gamma F^{\times 2}$ with the notation of the proposition 3), then the Faddeev index of ρ is equal to four. We remark that if c is a norme over $F(\sqrt{\alpha\gamma})$ i.e. the quaternion algebra $(\alpha\gamma, c)_F$ is trivial, then both the quaternion algebras $(\alpha, c)_{F(\sqrt{\gamma})}$ and $(\gamma, c)_{F(\sqrt{\alpha})}$ are trivial. Using the proposition 3, we may give examples of ramification sequences of degree four with Faddeev index two and construct a quaternion algebra having ρ as its ramification sequence.

Example 4 *Let $\alpha = 3$, $\beta = -1$, $\gamma = 2$ and $c = -5/27$. Then (1) holds with $x = y = z = w = 1$ and the Faddeev index of*

$$\rho = \{(\infty, \alpha F^{\times 2}), (t, \beta F^{\times 2}), (t-1, \gamma F^{\times 2}), (t-c, \alpha\beta\gamma F^{\times 2})\}$$

is equal to two. Let $u' = 9u - v$, $v' = 3u + v$ and $t' = u'/v'$. Then

$$\rho = \{(t' - 3, 3F^{\times 2}), (t' + 1, -F^{\times 2}), (t' - 2, 2F^{\times 2}), (t' + 6, -6F^{\times 2})\}.$$

By the proposition, the ramification sequence of

$$Q = \Big(t', (t' - 3)(t' + 1)(t' - 2)(t' + 6)\Big)_{F(t')}$$

is ρ. Replacing t' by $\frac{9t-1}{3t+1}$ we obtain

$$Q = \Big((9t - 1)(3t + 1), -t(t - 1)(27t + 5)\Big)_{F(t)}.$$

Lemma 5 *The quaternion algebra $(\alpha, c)_{F(\sqrt{\beta})}$ is trivial if and only if the quadratic form $\langle \beta, -\alpha, -c, \alpha c\rangle$ is isotropic.*

Proof : The quaternion algebra $(\alpha, c)_{F(\sqrt{\beta})}$ is trivial if and only if $(\alpha, c)_F$ is Brauer equivalent to (β, d) for some $d \in F^{\times}$ (see [Knus et. al., 1998], (16.29)) i.e. the quadratic forms $\langle 1, -\alpha, -c, \alpha c\rangle$ and $\langle 1, -\beta, -d, \beta d\rangle$ are isometric. So $(\alpha, c)_{F(\sqrt{\beta})}$ is trivial if and only if $\langle \beta, -\alpha, -c, \alpha c\rangle$ is isometric to $\langle \beta, -\beta, -d, \beta d\rangle$ which is the case if and only if $\langle \beta, -\alpha, -c, \alpha c\rangle$ is isotropic. □

Using this lemma, we get a condition equivalent to the condition in the proposition 2.7 in [Kunyavskii et al., 2006] to have Faddeev index four.

Proposition 6 *Let*

$$\rho = \{(\infty, \alpha F^{\times 2}), (t, \alpha F^{\times 2}), (t - 1, \beta F^{\times 2}), (t - c, \beta F^{\times 2})\}.$$

Suppose that one of the quadratic forms $\langle \beta, -\alpha, -c, \alpha c\rangle$ and $\langle \alpha, -\beta, -c, \beta c\rangle$ is anisotropic. Then the Faddeev index of ρ is equal to four. □

3 Ramification sequence of degree four over $\mathbb{Q}$

In this section, we assume $F = \mathbb{Q}$. We need some lemmas.

Lemma 7 *Let A be an exponent 2 central simple algebra over $\mathbb{Q}$. Then A is Brauer equivalent to a quaternion algebra.*

Proof : By Merkurjev in [1981], A is Brauer equivalent to a tensor product of quaternion algebras. By Albert's theorem (see [Knus et. al., 1998], (16.5)), a biquaternion algebra (over any field) is a division algebra if and only if its Albert form is anisotropic. But, the Hasse-Minkowski Principle ([Lam, 1973], VI.3.1) says that a quadratic form over $\mathbb{Q}$ is isotropic if and only if it is isotropic over $\mathbb{R}$ and over $\mathbb{Q}_p$, for all prime number p. On one hand, as an Albert form is 6-dimensional, it isotropic over all $\mathbb{Q}_p$ (see [Lam, 1973], VI.2.12). On the other hand, an Albert form is always isotropic over $\mathbb{R}$. So, there is no division biquaternion algebra over $\mathbb{Q}$ and any exponent two central simple algebra over $\mathbb{Q}$ is Brauer equivalent to a quaternion algebra. $\square$

Lemma 8 *Let $\alpha, \beta, c \in \mathbb{Q}$ and*

$$\rho = \{(\infty, \alpha F^{\times 2}), (t, \alpha F^{\times 2}), (t-1, \beta F^{\times 2}), (t-c, \beta F^{\times 2})\}.$$

Then

$$fi(A) = \min\Big\{\mathsf{index}\Big((\alpha, \lambda t)_K \otimes_K (\beta, (t-1)(t-c))_K\Big) \mid \lambda \in \mathbb{Z} \text{ square-free}\Big\}.$$

In consequence, the Faddeev index of ρ is two if there exists $\lambda \in \mathbb{Z}$ such that the quadratic form

$$\langle -\alpha, \beta, -\lambda t, \alpha\lambda t, (t-1)(t-c), -\beta(t-1)(t-c)\rangle$$

is isotropic and the Faddeev index of ρ is four otherwise.

Proof: Let $A = (\alpha, t)_K \otimes_K (\beta, (t-1)(t-c))_K$. Then the ramification sequence of A is equal to ρ. Let C be an exponent 2 F-algebra. By the lemma 7, we may assume that C is a quaternion algebra. If $C \otimes_F F(\sqrt{\alpha})$ is not trivial, then, by the proof of the proposition 2.7 in [Kunyavskii et. al., 2006], the index of $A \otimes_K C$ is greater or equal to four. So, we may assume that $C \otimes_F F(\sqrt{\alpha})$ is trivial. Using (16.29) in [Knus et. al., 1998], we obtain that $C = (\alpha, \lambda)_F$ for some $\lambda \in \mathbb{Q}$. We can assume that $\lambda \in \mathbb{Z}$ and is square-free. As

$$A \otimes_K C = (\alpha, \lambda t)_K \otimes_K (\beta, (t-1)(t-c))_K,$$

we have the result. $\square$

Lemma 9 *Suppose $\alpha, \beta, p, x, y \in \mathbb{Z}$ are such that $p \neq 2$ is prime, $p \nmid \alpha\beta$, $\left(\frac{\alpha\beta}{p}\right) = -1$ and $p \mid -\alpha x^2 + \beta y^2$. Then $p \mid x$ and $p \mid y$.*

Proof: Suppose $p \nmid x$. Then $\alpha\beta = (\beta y x^{-1})^2$ in $\mathbb{Z}/p$ which contradicts $\left(\frac{\alpha\beta}{p}\right) = -1$. If $p \nmid y$, we also have a contradiction. $\square$

We introduce some notations. For a field k with a valuation v, we denote by $\overline{k}^v$ the residue field of k with respect to the valuation v:

$$\overline{k}^v = \mathcal{O}_v / \mathcal{M}_v,$$

where $\mathcal{O}_v = \{x \in k \mid v(x) \geq 0\}$ is the local ring of the units of k and $\mathcal{M}_v = \{x \in k \mid v(x) > 0\}$ is the unique maximal ideal of $\mathcal{O}_v$. So, we have a surjective morphism $\mathcal{O}_v \to \overline{k}^v$. It induces a map from the Witt group of $\mathcal{O}_v$ to the Witt group of $\overline{k}^v$. For a quadratic form q over $\mathcal{O}_v$, we denote by $\overline{q}^v$ the image of q by this map.

Proposition 10 *Let $\alpha, \beta, c \in \mathbb{Z}$ be square-free. Let*

$$\rho = \{(\infty, \alpha F^{\times 2}), (t, \alpha F^{\times 2}), (t-1, \beta F^{\times 2}), (t-c, \beta F^{\times 2})\}.$$

Suppose there exists a prime $p \neq 2$ such that $p \nmid \alpha c$, $p \mid \beta$, $\left(\frac{\alpha}{p}\right) = -1$ and $\left(\frac{c}{p}\right) = -1$ or there exists a prime $p \neq 2$ such that $p \nmid \beta c$, $p \mid \alpha$, $\left(\frac{\beta}{p}\right) = -1$ and $\left(\frac{c}{p}\right) = -1$. Then the Faddeev index of ρ is four.

Proof : Suppose there exists a prime $p \neq 2$ such that $p \nmid \alpha c$, $p \mid \beta$, $\left(\frac{\alpha}{p}\right) = -1$ and $\left(\frac{c}{p}\right) = -1$. Let $A = (\alpha, t)_K \otimes_K (\beta, (t-1)(t-c))_K$. For $\lambda \in \mathbb{Z}$ square-free, we denote by q_λ the Albert form associated to $A \otimes_K (\alpha, \lambda)_K$, so

$$q_\lambda = \langle -\alpha, \beta, -\lambda t, \alpha\lambda t, (t-1)(t-c), -\beta(t-1)(t-c)\rangle.$$

By the lemma 8, it is enough to show that q_λ is anisotropic, for all $\lambda \in \mathbb{Z}$, square-free. First we suppose that p divides λ. We may write $q = q_1 \perp p \cdot q_2$ over $\mathbb{Q}(t)$ with

$$q_1 = \langle -\alpha, (t-1)(t-c)\rangle$$

and

$$q_2 = \langle \frac{\beta}{p}, -\frac{\lambda}{p}t, \alpha\frac{\lambda}{p}t, -\frac{\beta}{p}(t-1)(t-c)\rangle,$$

q_1 and q_2 being defined over the units with respect to the valutation $\widehat{v_p}$ defined on $\mathbb{Q}(t)$ by

$$\widehat{v_p}(\sum_i a_i t^i) = \min_i v_p(a_i).$$

Over $\mathbb{F}_p(t)$,

$$\overline{q_1}^{\widehat{v_p}} = \langle -\alpha\rangle \perp (t-1) \cdot \langle t-c\rangle,$$

$\overline{\langle -\alpha\rangle}^{v_{t-1}} = \langle -\alpha\rangle$ and $\overline{\langle t-c\rangle}^{v_{t-1}} = \langle 1-c\rangle$ are anisotropic over $\mathbb{F}_p$, so, using the Spinger's theorem with respect to the valuation v_{t-1}, $\overline{q_1}^{\widehat{v_p}}$ is anisotropic over $\mathbb{F}_p(t)$. Over $\mathbb{F}_p(t)$,

$$\overline{q_2}^{\widehat{v_p}} = \langle \frac{\beta}{p}, -\frac{\beta}{p}(t-1)(t-c)\rangle \perp t \cdot \langle -\frac{\lambda}{p}, \alpha\frac{\lambda}{p}\rangle$$

with

$$\overline{\langle \frac{\beta}{p}, -\frac{\beta}{p}(t-1)(t-c)\rangle}^{v_t} = \frac{\beta}{p}\langle 1, -c\rangle$$

and

$$\overline{\langle -\frac{\lambda}{p}, \alpha\frac{\lambda}{p}\rangle}^{v_t} = \frac{\lambda}{p}\langle -1, \alpha\rangle$$

anisotropic over $\mathbb{F}_p$ since $\left(\frac{c}{p}\right) = -1$ and $\left(\frac{\alpha}{p}\right) = -1$. Hence, using the Springer's theorem with respect to the valuation v_t, $\overline{q_2}^{\widehat{v_p}}$ is anisotropic. Finally, by the same theorem used on q with repect to the valuation $\widehat{v_p}$, q is anisotropic. Now we assume that p does not divide λ. Then, over $\mathbb{Q}(t)$, we write $q = q_1 \perp p \cdot q_2$ with

$$q_1 = \langle -\alpha, -\lambda t, \alpha\lambda t, (t-1)(t-c)\rangle$$

and

$$q_2 = \langle \frac{\beta}{p}, -\frac{\beta}{p}(t-1)(t-c)\rangle$$

defined over the units of $\mathbb{Q}(t)$ with respect to the valuation $\widehat{v_p}$. Since

$$\overline{q_1}^{\widehat{v_p}} = \langle -\alpha, 1\rangle \perp \frac{1}{t} \cdot \langle -\lambda, \alpha\lambda\rangle$$

and $\left(\frac{\alpha}{p}\right) = -1$, by the Springer's theorem, $\overline{q_1}^{\widehat{v_p}}$ is anisotripic over $\mathbb{F}_p(t)$. We also prove that $\overline{q_2}^{\widehat{v_p}}$ is anisotropic over $\mathbb{F}_p(t)$ using the Springer's theorem with respect to the valuation v_{t-1}. So we conlude that q is anisotropic over $\mathbb{Q}(t)$. We proved that, if there exists a prime number $p \neq 2$, such that $p \nmid \alpha c$, $p \mid \beta$ and $\left(\frac{\alpha}{p}\right) = -1 = \left(\frac{c}{p}\right)$, then the Faddeev index of ρ is equal to four. Now we assume that there exists a prime number $p \neq 2$ such that $p \nmid \beta c$, $p \mid \alpha$ and $\left(\frac{\beta}{p}\right) = -1 = \left(\frac{c}{p}\right)$. Then we may change the homogeneous coordinates so that

$$\rho = \{(t'-1, \alpha F^{\times 2}), (t'-c, \alpha F^{\times 2}), (\infty, \beta F^{\times 2}), (t', \beta F^{\times 2})\}$$

since

$$[\infty, 0, 1, c] = c = [1, c, \infty, 0].$$

Using the first part of the proof, we also show that the Faddeev index of ρ is equal to four. $\square$

In particular, the implication in the proposition 6 is not an equivalence. Indeed, if $\alpha = 2$, $\beta = 3$ and $c = -1$, then $p = 3$ is such that $p \nmid \alpha c$, $p \mid \beta$, $\left(\frac{\alpha}{p}\right) = -1$ and $\left(\frac{c}{p}\right) = -1$, so by the proposition 10, the Faddeev index of ρ is equal to four. But the quaternion algebras $(\alpha, c)_{F(\sqrt{\beta})}$ and $(\beta, c)_{F(\sqrt{\alpha})}$ are trivial ($\alpha = (1)^2 - c(1)^2$ and $\beta = (\sqrt{\alpha})^2 - c(1)^2$).

Proposition 11 *Let*

$$\rho = \{(\infty, \alpha F^{\times 2}), (t, \alpha F^{\times 2}), (t-1, \beta F^{\times 2}), (t-c, \beta F^{\times 2})\}$$

with $\alpha, \beta, c \in \mathbb{Z}$ square-free. Suppose there exists a prime number $p \neq 2$ such that $p \mid \beta, c$, $p \nmid \alpha$, $\left(\frac{-\alpha\beta c/p^2}{p}\right) = -1$ and $\left(\frac{\alpha}{p}\right) = -1$ or there exists a prime number $p \neq 2$ such that $p \mid \alpha, c$, $p \nmid \beta$, $\left(\frac{-\alpha\beta c/p^2}{p}\right) = -1$ and $\left(\frac{\beta}{p}\right) = -1$. Then the Faddeev index of ρ is equal to four.

Proof: Let $p \neq 2$ be a prime number such that $p \mid \beta, c$, $p \nmid \alpha$, $\left(\frac{-\alpha\beta c/p^2}{p}\right) = -1$ and $\left(\frac{\alpha}{p}\right) = -1$. Let

$$q_\lambda = \langle -\alpha, \beta, -\lambda t, \alpha\lambda t, (t-1)(t-c), -\beta(t-1)(t-c)\rangle.$$

We want to prove that q_λ is anisotropic, for all $\lambda \in \mathbb{Z}$ square-free. Suppose there exist $a(t), b(t), c(t), d(t), e(t), f(t) \in F[t]$ such that

$$-\alpha a(t)^2 + \beta b(t)^2 - \lambda t c(t)^2 + \alpha\lambda t d(t)^2 + (t-1)(t-c)e(t)^2 - \beta(t-1)(t-c)f(t)^2 = 0.$$

We may assume that

$$a(t) = a_n t^n + \ldots + a_0, \quad b(t) = b_n t^n + \ldots + b_0, \quad c(t) = c_{n-1}t^{n-1} + \ldots + c_0,$$

$$d(t) = d_{n-1}t^{n-1} + \ldots + d_0, \quad e(t) = e_{n-1}t^{n-1} + \ldots + e_0, \quad f(t) = f_{n-1}t^{n-1} + \ldots + f_0,$$

where $a_i, b_i, c_i, d_i, e_i, f_i \in \mathbb{Z}$ and the $\mathtt{gcd}(\{a_i, b_i, c_i, d_i, e_i, f_i \mid i = 1, \ldots, n\}) = 1$. For all $k = 0, \ldots, 2n$, the coefficient of t^k in the isotropy relation is equal to zero, so

$$\begin{aligned} -\alpha \sum_{i=0}^{k} a_i a_{k-i} + \beta \sum_{i=0}^{k} b_k b_{k-i} - \lambda \sum_{i=0}^{k-1} c_i c_{k-1-i} + \alpha\lambda \sum_{i=0}^{k-1} d_i d_{k-1-i} \\ + \sum_{i=0}^{k-2} e_i e_{k-2-i} - (c+1) \sum_{i=0}^{k-1} e_i e_{k-1-i} + c \sum_{i=0}^{k} e_i e_{k-i} \\ -\beta \sum_{i=0}^{k-2} f_i f_{k-2-i} + \beta(c+1) \sum_{i=0}^{k-1} f_i f_{k-1-i} - \beta c \sum_{i=0}^{k} f_i f_{k-i} = 0 \end{aligned}$$

(we let $a_i, b_i = 0$ for all $i > n$ and $c_i = d_i = e_i = f_i = 0$ for all $i \geq n$). We denote by C_k the coefficient of t^k in the isotropy relation. First we assume that $p \mid \lambda$. As $C_{2n} = 0$ and $p \mid \beta$, we have $p \mid -\alpha a_n^2 + e_{n-1}^2$. By lemma 9, $p \mid a_n, e_{n-1}$, since $\left(\frac{\alpha}{p}\right) = -1$. We assume that $p \mid a_i, e_{i-1}$ for all $i \in \{n, \ldots, k+1\}$. Then, as p divides C_{2k}, $p \mid -\alpha a_k^2 + e_{k-1}^2$ and $p \mid a_k, e_{k-1}$. Hence $p \mid a_i, e_i$ for all i. Since $\left(\frac{-\alpha\beta c/p^2}{p}\right) = -1$ and $\left(\frac{\alpha}{p}\right) = -1$, we have

$$p^2 \mid C_0 \Rightarrow p^2 \mid \beta b_0^2 \Rightarrow p \mid b_0.$$

We assume that $p^{k+1-i} \mid a_i, b_i, e_i$ for all $i \in \{0, \dots, k\}$, $p^{k-i} \mid f_i, c_i, d_i$ for all $i \in \{0, \dots, k-1\}$ and we will prove that $p^{k+2-i} \mid a_i, b_i, e_i$ for all $i \in \{0, \dots, k+1\}$ and $p^{k+1-i} \mid f_i, c_i, d_i$ for all $i \in \{0, \dots, k\}$. We have

$$p^{2k+3} \mid C_0 \Rightarrow p^{2k+3} \mid -\alpha a_0^2 - \beta c f_0^2 \Rightarrow p^{k+2} \mid a_0, p^{k+1} \mid f_0,$$

$$p^{2k+2} \mid C_1 \Rightarrow p^{2k+2} \mid -\lambda c_0^2 + \alpha\lambda d_0^2 \Rightarrow p^{k+1} \mid c_0, d_0.$$

We assume that $p^{k+2-i} \mid a_i$ for all $i \in \{0, \dots, l\}$ and $p^{k+1-i} \mid f_i, c_i, d_i$ for all $i \in \{0, \dots, l\}$ (where $0 \leq l < k-1$). Then $p^{2k-2l+1} \mid a_i a_{2l+2-i}$ for all $i \in \{0, \dots, l+1\}$. Indeed, let $i \in \{0, \dots, l+1\}$, then $p^{k+2-i} \mid a_i$. We have

$$2l+2-i \leq k \Rightarrow p^{k+1-(2l+2-i)} \mid a_{2l+2-i} \Rightarrow p^{2k-2l+1} \mid a_i a_{2l+2-i}$$

$$2l+2-i \geq k+1 \Rightarrow 2k-2l+1 \leq k+2-i \Rightarrow p^{2k-2l+1} \mid a_i a_{2l+2-i}.$$

In the same way, we prove that, $\forall i \in \{0, \dots, l+1\}$,

$$p^{2k-2l+1} \mid \beta b_i b_{2l+2-i}, \quad c e_i e_{2l+2-i},$$

and, $\forall i \in \{0, \dots, l\}$,

$$p^{2k-2l+1} \mid \lambda c_i c_{2l+1-i}, \quad \lambda d_i d_{2l+1-i}, \quad e_i e_{2l-i}, \quad e_i e_{2l+1-i},$$

$$p^{2k-2l+1} \mid f_i f_{2l-i}, \quad \beta f_i f_{2l+1-i}, \quad \beta c f_i f_{2l+2-i}.$$

So $p^{2k-2l+1}$ divides C_{2l+2} implies $p^{2k-2l+1} \mid -\alpha a_{l+1}^2 - \beta c f_{l+1}^2$ and, by the lemma 9, $p^{k+1-l} \mid a_{l+1}, p^{k-l} \mid f_{l+1}$. In the same way, we prove that p^{2k-2l} divides C_{2l+3} implies that $p^{2k-2l} \mid -\lambda c_{l+1}^2 + \alpha\lambda d_{l+1}^2$ and $p^{k-l} \mid c_{l+1}, d_{l+1}$. By induction, we proved that $p^{k+2-i} \mid a_i$, for all $i \in \{0, \dots, k+1\}$ and $p^{k+1-i} \mid f_i, c_i, d_i$ for all $i \in \{0, \dots, k\}$. We have that $p \mid e_{k+1}$ and $p \mid b_{k+1}$ since p^2 divides C_{2k+2}. We assume that $p^{k+2-i} \mid b_i, e_i$ for all $i \in \{l+1, \dots, k+1\}$ (where $0 \leq l \leq k$). Using the same kind of trick, we prove that

$$p^{2k-2l+3} \mid C_{2l+1} \Rightarrow p^{2k-2l+3} \mid -(c+1)e_l^2 \Rightarrow p^{k+2-l} \mid e_l,$$

$$p^{2k-2l+4} \mid C_{2l} \Rightarrow p^{2k-2l+4} \mid \beta t^{2l} \Rightarrow p^{k+2-l} \mid b_l.$$

Hence, we proved that $p^{k+2-i} \mid b_i, e_i$ for all $i \in \{0, \dots, k+1\}$. By induction, we proved that $p^{n+1-i} \mid a_i, b_i, e_i$ for all $i \in \{0, \dots, n\}$ and $p^{n-i} \mid f_i, c_i, d_i$ for all $i \in \{0, \dots, n-1\}$. In particular, $p \mid a_i, b_i, c_i, d_i, e_i, f_i$ for all i and so we get a contradiction. Hence if $p \mid \lambda$, the quadratic form q_λ is anisotropic. Now we assume that $p \nmid \lambda$. Over $\mathbb{Q}(t)$, $q_\lambda = q_1 \perp p \cdot q_2$, with

$$q_1 = \langle -\alpha, -\lambda t, \alpha\lambda t, (t-1)(t-c) \rangle$$

and

$$q_2 = \langle \frac{\beta}{p}, -\frac{\beta}{p}(t-1)(t-c) \rangle$$

defined over the units of $\mathbb{Q}(t)$ with respect to the valuation $\widehat{v_p}$. Over $\mathbb{F}_p(t)$,

$$\overline{q_1}^{\widehat{v_p}} = \langle -\alpha, -\lambda t, \alpha\lambda t, t(t-1)\rangle = \langle -\alpha, 1\rangle \perp \frac{1}{t}\langle -\lambda, \alpha\lambda\rangle$$

with $\overline{\langle -\alpha, 1\rangle}^{v_{\frac{1}{t}}}$ and $\overline{\langle -\lambda, \alpha\lambda\rangle}^{v_{\frac{1}{t}}}$ anisotropic over $\mathbb{F}_p$, since $\left(\frac{\alpha}{p}\right) = -1$. So, by Springer's theorem with respect to the valuation $v_{\frac{1}{t}}$, $\overline{q_1}^{\widehat{v_p}}$ is anisotropic over $\mathbb{F}_p(t)$. We also prove, using Springer's theorem with respect to the valuation v_{t-1}, that $\overline{q_2}^{\widehat{v_p}}$ is anisotropic over $\mathbb{F}_p(t)$. So q_λ is anisotropic over $\mathbb{Q}(t)$. Hence, if $p \nmid \lambda$, then q_λ is anisotropic. So, if there exists a prime number $p \neq 2$ such that $p \mid \beta, c$, $p \nmid \alpha$ and $\left(\frac{-\alpha\beta c/p^2}{p}\right) = -1 = \left(\frac{\alpha}{p}\right)$, then the Faddeev index of ρ is equal to four. If there exists a prime number $p \neq 2$ such that $p \mid \alpha, c$, $p \nmid \beta$, $\left(\frac{-\alpha\beta c/p^2}{p}\right) = -1$ and $\left(\frac{\beta}{p}\right) = -1$, we change the homogeneous coordinates as in the proof of the proposition 10 and, using the first part of the proof, we obtain that the Faddeev index of ρ is equal to four. □

Proposition 12 *Let*

$$\rho = \{(\infty, \alpha F^{\times 2}), (t, \alpha F^{\times 2}), (t-1, \beta F^{\times 2}), (t-c, \beta F^{\times 2})\}$$

with $\alpha, \beta, c \in \mathbb{Z}$ square-free. Suppose there exists a prime number p, $p \neq 2$, such that $p \mid \alpha, \beta, c$ and $\left(\frac{\alpha\beta/p^2}{p}\right) = -1$, then the Faddeev index of ρ is equal to four.

Proof : Let

$$q_\lambda = \langle -\alpha, \beta, -\lambda t, \alpha\lambda t, (t-1)(t-c), -\beta(t-1)(t-c)\rangle.$$

We want to prove that q_λ is anisotropic, for all $\lambda \in \mathbb{Z}$ square-free. We may assume that p divides λ. Indeed, if p does not divide λ, let $\lambda' = -\alpha\lambda$. Then p divides λ' and

$$q_\lambda = \langle -\alpha, \beta, \alpha\lambda' t, -\lambda' t, (t-1)(t-c), -\beta(t-1)(t-c)\rangle.$$

Suppose that $\left(\frac{\beta\lambda/p^2}{p}\right) = -1$, then $q_\lambda = q_1 \perp p \cdot q_2$ with

$$q_1 = \langle \frac{\alpha\lambda}{p^2} t, (t-1)(t-c)\rangle$$

and

$$q_2 = \langle -\frac{\alpha}{p}, \frac{\beta}{p}, -\frac{\lambda}{p} t, -\frac{\beta}{p}(t-1)(t-c)\rangle$$

defined over the units of $\mathbb{Q}(t)$ with respect to the valuation $\widehat{v_p}$. By the Springer's theorem with respect to the valutation v_{t-1}, $\overline{q_1}^{\widehat{v_p}}$ is anisotropic over $\mathbb{F}_p(t)$. We may write over $\mathbb{F}_p(t)$

$$\begin{aligned}\widehat{\overline{q_2}^{v_p}} &= \langle -\frac{\alpha}{p}, \frac{\beta}{p}, -\frac{\lambda}{p}t, -\frac{\beta}{p}(t-1)t\rangle \\ &= \langle -\frac{\alpha}{p}, \frac{\beta}{p}\rangle \perp t\cdot\langle -\frac{\lambda}{p}, -\frac{\beta}{p}(t-1)\rangle.\end{aligned}$$

As $\left(\frac{\alpha\beta/p^2}{p}\right) = -1$ and $\left(\frac{\beta\lambda/p^2}{p}\right) = -1$,

$$\overline{\langle -\frac{\alpha}{p}, \frac{\beta}{p}\rangle}^{v_t} = \langle -\frac{\alpha}{p}, \frac{\beta}{p}\rangle \text{ and } \overline{\langle -\frac{\lambda}{p}, -\frac{\beta}{p}(t-1)\rangle}^{v_t} = \langle -\frac{\lambda}{p}, \frac{\beta}{p}\rangle$$

are anisotropic over $\mathbb{F}_p$. So q is anisotropic over $\mathbb{Q}(t)$. Now we assume that $\left(\frac{\beta\lambda/p^2}{p}\right) = 1$. Suppose that q_λ is isotropic. Then there exist $a(t)$, $b(t)$, $c(t)$, $d(t)$, $e(t)$, $f(t)$ in $F[t]$ such that

$$-\alpha a(t)^2 + \beta b(t)^2 - \lambda t c(t)^2 + \alpha\lambda t d(t)^2 + (t-1)(t-c)e(t)^2 - \beta(t-1)(t-c)f(t)^2 = 0.$$

We may assume that

$$a(t) = a_n t^n + \ldots + a_0,\ \ b(t) = b_n t^n + \ldots + b_0,\ \ c(t) = c_{n-1}t^{n-1} + \ldots + c_0,$$

$$d(t) = d_{n-1}t^{n-1} + \ldots + d_0,\ \ e(t) = e_{n-1}t^{n-1} + \ldots + e_0,\ \ f(t) = f_{n-1}t^{n-1} + \ldots + f_0,$$

where $a_i, b_i, c_i, d_i, e_i, f_i \in \mathbb{Z}$ and the $\mathsf{gcd}(\{a_i, b_i, c_i, d_i, e_i, f_i \mid i = 1, \ldots, n\}) = 1$. For all $k = 0, \ldots, 2n$, the coefficient C_k of t^k in the isotropy relation is equal to zero:

$$\begin{gathered}-\alpha\sum_{i=0}^{k} a_i a_{k-i} + \beta\sum_{i=0}^{k} b_k b_{k-i} - \lambda\sum_{i=0}^{k-1} c_i c_{k-1-i} + \alpha\lambda\sum_{i=0}^{k-1} d_i d_{k-1-i} \\ + \sum_{i=0}^{k-2} e_i e_{k-2-i} - (c+1)\sum_{i=0}^{k-1} e_i e_{k-1-i} + c\sum_{i=0}^{k} e_i e_{k-i} \\ -\beta\sum_{i=0}^{k-2} f_i f_{k-2-i} + \beta(c+1)\sum_{i=0}^{k-1} f_i f_{k-1-i} - \beta c\sum_{i=0}^{k} f_i f_{k-i} = 0\end{gathered}$$

(we let $a_i, b_i = 0$ for all $i > n$ and $c_i = d_i = e_i = f_i = 0$ for all $i \geq n$). As p divides C_{2n}, $p \mid e_{n-1}^2$ and so $p \mid e_{n-1}$. As p divides C_{2n-1}, $p \mid \frac{\alpha\lambda}{p^2} d_{n-1}^2$ and so $p \mid d_{n-1}$. We assume that p divides e_i, d_i for all $i \in \{n-1, \ldots, k\}$. Then

$$p \mid C_{2k} \Rightarrow p \mid e_{k-1}^2 \Rightarrow p \mid e_{k-1},$$

$$p \mid C_{2k-1} \Rightarrow p \mid \frac{\alpha\lambda}{p^2} d_{k-1}^2 \Rightarrow p \mid d_{k-1}^2.$$

So $p \mid e_i, d_i$ for all i. We have

$$p^2 \mid C_0 \Rightarrow p^2 \mid -\alpha a_0^2 + \beta b_0^2 \Rightarrow p \mid a_0, b_0,$$
$$p^3 \mid C_0 \Rightarrow p^3 \mid -\beta c f_0^2 \Rightarrow p \mid f_0,$$
$$p^2 \mid C_1 \Rightarrow p^2 \mid -\lambda c_0^2 \Rightarrow p \mid c_0.$$

Let $k \in \mathbb{N}$. We assume that p^{k+1-i}divides $a_i, b_i, c_i, d_i, e_i, f_i$ for all $i \in \{0, \dots, k\}$. Then we know that $p \mid d_{k+1}, e_{k+1}$ and we have

$$p^2 \mid C_{2k+2} \Rightarrow p^2 \mid -\alpha a_{k+1}^2 + \beta b_{k+1}^2 \Rightarrow p \mid a_{k+1}, b_{k+1}.$$

Let $l \in \mathbb{N}$ such that $0 \leq l \leq k$ and suppose that $p^{k+2-i} \mid a_i, b_i, d_i, e_i$ for all $i \in \{l+1, \dots, k+1\}$. Then

$$p^{2k-2l+3} \mid C_{2l+1} \Rightarrow p^{2k-2l+3} \mid \frac{\alpha\lambda}{p^2} d_l^2 - (c+1)e_l^2.$$

As
$$\left(\frac{\alpha\lambda(c+1)/p^2}{p}\right) = \left(\frac{\alpha\beta/p^2}{p}\right)\left(\frac{\beta\lambda/p^2}{p}\right)\left(\frac{1}{p}\right) = -1,$$
$p^{k+2-l} \mid d_l, e_l$. Also,

$$p^{2k-2l+4} \mid C_{2l} \Rightarrow p^{2k-2l+4} \mid -\alpha a_l^2 + \beta b_l^2 \Rightarrow p^{k+2-l} \mid a_l, b_l.$$

Hence we proved by induction that $p^{k+2-i} \mid a_i, b_i, d_i, e_i$ for all $i \in \{0, \dots, k+1\}$. We have

$$p^{2k+5} \mid C_0 \Rightarrow p^{2k+5} \mid -\beta c f_0^2 \Rightarrow p^{k+2} \mid f_0,$$
$$p^{2k+4} \mid C_1 \Rightarrow p^{2k+4} \mid -\lambda c_0^2 \Rightarrow c_0.$$

Let $l \in \mathbb{N}$, $1 \leq l \leq k+1$, and assume that $p^{k+2-i} \mid c_i, f_i$ for all $i \in \{0, \dots, l-1\}$. Then

$$p^{2k-2l+5} \mid C_{2l} \Rightarrow p^{2k-2l+5} \mid -\beta c f_l^2 \Rightarrow p^{k+2-l} \mid f_l,$$
$$p^{2k-2l+4} \mid C_{2l+1} \Rightarrow p^{2k-2l+4} \mid -\lambda c_l^2 \Rightarrow p^{k+2-l} \mid c_l.$$

Hence, we obtain that $p^{k+2-i} \mid a_i, b_i, c_i, d_i, e_i, f_i$ for all $i \in \{0, \dots, k+1\}$. We proved by induction that, $p^{n+1-i} \mid a_i, b_i, c_i, d_i, e_i, f_i$, for all $i \in \{0, \dots, n\}$. So, in particular, $p \mid a_i, \dots, f_i$ for all i. We have a contradiction, thus q_λ is anisotropic. □

Suppose that $\alpha, \beta, c \in \mathbb{Z}$ are square-free. One of the quadratic forms

$$\langle \beta, -\alpha, -c, \alpha c\rangle \text{ and } \langle \alpha, -\beta, -c, \beta c\rangle$$

is anisotropic over $\mathbb{Q}$ if and only if one of them is anisotropic over $\mathbb{Q}_p$, for some prime number p, or over $\mathbb{R}$ i.e.

$$\alpha\beta < 0 \text{ and } c < 0 \text{ or}$$

$$\exists p \neq 2 \text{ prime such that } p \mid \alpha, p \nmid \beta c, \left(\frac{\beta}{p}\right) = 1, \left(\frac{c}{p}\right) = -1 \text{ or}$$

$$\exists p \neq 2 \text{ prime such that } p \mid \beta, p \nmid \alpha c, \left(\frac{\alpha}{p}\right) = 1, \left(\frac{c}{p}\right) = -1 \text{ or}$$

$$\exists p \neq 2 \text{ prime such that } p \mid c, p \nmid \alpha\beta, \left(\frac{\alpha\beta}{p}\right) = -1 \text{ or}$$

$$\exists p \neq 2 \text{ prime such that } p \mid \alpha, c, \quad p \nmid \beta, \left(\frac{-\alpha\beta c/p^2}{p}\right) = -1, \left(\frac{\beta}{p}\right) = 1 \text{ or}$$

$$\exists p \neq 2 \text{ prime such that } p \mid \beta, c, \quad p \nmid \alpha, \left(\frac{-\alpha\beta c/p^2}{p}\right) = -1, \left(\frac{\alpha}{p}\right) = 1 \text{ or}$$

$$\langle \beta, -\alpha, -c, \alpha c\rangle \text{ is anisotropic over } \mathbb{Q}_2 \text{ or}$$

$$\langle \alpha, -\beta, -c, \beta c\rangle \text{ is anisotropic over } \mathbb{Q}_2.$$

So, until now, we have the following result: if

$$\begin{cases} \alpha\beta < 0 \text{ and } c < 0 \text{ or} \\ \exists p \neq 2 \text{ prime such that } p \mid \alpha, p \nmid \beta c, \left(\frac{c}{p}\right) = -1 \text{ or} \\ \exists p \neq 2 \text{ prime such that } p \mid \beta, p \nmid \alpha c, \left(\frac{c}{p}\right) = -1 \text{ or} \\ \exists p \neq 2 \text{ prime such that } p \mid c, p \nmid \alpha\beta, \left(\frac{\alpha\beta}{p}\right) = -1 \text{ or} \\ \exists p \neq 2 \text{ prime such that } p \mid \alpha, c, \quad p \nmid \beta, \left(\frac{-\alpha\beta c/p^2}{p}\right) = -1 \text{ or} \\ \exists p \neq 2 \text{ prime such that } p \mid \beta, c, \quad p \nmid \alpha, \left(\frac{-\alpha\beta c/p^2}{p}\right) = -1 \text{ or} \\ \exists p \neq 2 \text{ prime such that } p \mid \alpha, \beta, c, \left(\frac{\alpha\beta/p^2}{p^2}\right) = -1 \text{ or} \\ \langle \beta, -\alpha, -c, \alpha c\rangle \text{ is anisotropic over } \mathbb{Q}_2 \text{ or} \\ \langle \alpha, -\beta, -c, \beta c\rangle \text{ is anisotropic over } \mathbb{Q}_2, \end{cases} \tag{2}$$

then the Faddeev index of

$$\rho = \{(\infty, \alpha F^{\times 2}), (t, \alpha F^{\times 2}), (t-1, \beta F^{\times 2}), (t-c, \beta F^{\times 2})\}$$

is equal to four. The conditions in 2 are equivalent to the condition $(\alpha\beta, c)_F \neq 1$. So we have the following theorem

Theorem 13 *We assume that $F = \mathbb{Q}$. Let*

$$\rho = \{(\infty, \alpha F^{\times 2}), (t, \alpha F^{\times 2}), (t-1, \beta F^{\times 2}), (t-c, \beta F^{\times 2})\}$$

with $\alpha, \beta, c \in \mathbb{Z}$, square-free. If $(\alpha\beta, c)_F$ is not trivial, then the Faddeev index of ρ is equal to four.

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