Ternary cubic forms and central simple algebras of degree 3

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A ma famille

Contents

Introduction

Statement of the problem

There is a well-known relation between quaternion algebras and quadratic forms. Suppose that F is a field of characteristic not 2, for $a, b \in F^\times$, let $(a, b)_F$ be the quaternion algebra generated by i and j such that $i^2 = a$, $j^2 = b$ and $ij = -ji$. We may define a quadratic form over $(a, b)_F$, called the norm form, as follows:

$$N\colon (a, b)_F \to F\colon \xi \mapsto \xi\overline{\xi}$$

where $\overline{x_0 + x_1 i + x_2 j + x_3 ij} = x_0 - x_1 i - x_2 j - x_3 ij$. The norm form has interesting properties: two quaternion algebras are isomorphic if and only if their respective norm forms are isometric (see Proposition 2.5, page 57 in [Lam, 1973]); a quaternion algebra is isomorphic to $\mathsf{M}_2(F)$ if and only if its norm form is isotropic (see Theorem 2.7, page 58, in *op. cit.*); and the Clifford algebra of the norm form of a quaternion algebra A is isomorphic to $A \otimes_F \mathsf{M}_2(F)$ (see Theorem 1.8, page 106, and Corollary 3.3, page 116, in *op. cit*). We observe that

$$N(\xi) = -\xi^2$$

for all ξ in the subspace of $(a, b)_F$ spanned by i, j and ij, i.e. the subspace of reduced trace zero elements of $(a, b)_F$. A natural question is then: *Is it possible to extend these results and establish a similar relation between central simple algebras of degree* 3 *and cubic forms?*

One way to proceed, as did D. Haile in [1984], is to study the Clifford algebra of a binary cubic form. Considering a field F of characteristic neither 2 nor 3 and a binary cubic form with a non-zero discriminant over F, Haile proves that the Clifford algebra of this cubic form is always an Azumaya algebra. Moreover each homomorphic image of the Clifford algebra is a degree 3 simple algebra over its center, and there exists a correspondence between the simple homomorphic images and the points of the curve $y^2 = x^3 - 27D$ in an algebraic closure of F, where D denotes the discriminant of the binary cubic form.

We favor however a different approach. Suppose that F is a field of characteristic neither 2 nor 3 and let A be a central simple algebra of degree 3 over F. By Cayley-Hamilton's Theorem

$$\xi^3 - \mathsf{Trd}_A(\xi)\xi^2 + \frac{1}{2}\big(\mathsf{Trd}_A(\xi)^2 - \mathsf{Trd}_A(\xi^2)\big)\xi - \mathsf{Nrd}_A(\xi) = 0$$

for all $\xi \in A$, where Trd_A and Nrd_A denote respectively the reduced trace and the reduced norm of A. So the cube of a reduced trace zero element ξ of A is not necessarily in F; it is in F if and only if $\mathsf{Trd}_A(\xi^2) = 0$. Let

$$q_A \colon A \to F \colon \xi \mapsto \mathsf{Trd}_A(\xi^2)$$

denote the so-called trace quadratic form of A. The restriction of q_A to the subspace A° of reduced trace zero elements of A is isometric to

$$\langle 1, -1, 1, -1, 1, -1, 1, 3\rangle$$

over an extension of degree 3 over F which splits A. Therefore there exist 3-dimensional subspaces[1] of A° which are totally isotropic for q_A. Such a subspace V gives rise to a cubic form

$$f_{A,V} \colon V \to F \colon \xi \mapsto \xi^3.$$

So a more precise formulation of the problem of generalizing the situation for quaternion algebras and quadratic norm forms is: *Does the cubic form $f_{A,V}$ as above determine the algebra A?*

D. Haile and J.-P. Tignol partly answered this question in a handwritten note, dated April 2002. They consider a central division algebra A of degree 3 over F and assume that F contains a primitive cube root of unity ω. Then they prove that if $f_{A,V}$ is semi-diagonal, i.e.

$$f_{A,V} = a_1\varphi_1^3 + a_2\varphi_2^3 + a_3\varphi_3^3 - 3\lambda\varphi_1\varphi_2\varphi_3$$

for some $a_1, a_2, a_3, \lambda \in F$ and linearly independent $\varphi_1, \varphi_2, \varphi_3 \in V^\star$, then either $a_1a_2a_3 = \lambda^3$ or there exists one and only one $i \in \{1, 2, 3\}$ such that

$$\frac{a_1a_2a_3 - \lambda^3}{a_i^2} \in F^{\times 3}.$$

[1] There exist 4-dimensional totally isotropic subspaces of A° if and only if F contains a primitive cube root of unity.

If $a_1a_2a_3 = \lambda^3$ then A is isomorphic to

$$(a_1, a_2)_{\omega^{\pm 1}, F} \cong (a_1, a_3)_{\omega^{\pm 1}, F} \cong (a_2, a_3)_{\omega^{\pm 1}, F}.$$

If $a_i^{-2}(a_1a_2a_3 - \lambda^3) \in F^{\times 3}$ for some $i \in \{1, 2, 3\}$ then A is isomorphic to $(a_i, a_j)_{\omega^{\pm 1}, F}$ for all $j \in \{1, 2, 3\}$ such that $j \neq i$.

Observe that for a semi-diagonal cubic form $f_{A,V}$ as above we can write

$$f_{A,V}(\xi) = \mathsf{Tr}_{F^3/F}\big(a\Theta(\xi)^3\big) - 3\lambda \mathsf{N}_{F^3/F}\big(\Theta(\xi)\big)$$

where $a = (a_1, a_2, a_3) \in F^3$, $\Theta\colon V \to F^3$ is the F-vector space isomorphism defined by

$$\Theta(\xi) = \big(\varphi_1(\xi), \varphi_2(\xi), \varphi_3(\xi)\big),$$

and $\mathsf{Tr}_{F^3/F}$, respectively $\mathsf{N}_{F^3/F}$, denotes the trace, respectively norm, of the F-algebra F^3. The result of Haile and Tignol can then be reformulated as follows: let A be a central division algebra of degree 3 over F and suppose that there exists a subspace V of A° such that

$$f_{A,V}(\xi) = \mathsf{Tr}_{F^3/F}\big(a\Theta(\xi)^3\big) - 3\lambda \mathsf{N}_{F^3/F}\big(\Theta(\xi)\big)$$

for some $a = (a_1, a_2, a_3) \in F^3$, then either $\mathsf{N}_{F^3/F}(a) = \lambda^3$ or there exists one and only one $i \in \{1, 2, 3\}$ such that $a_i^{-2}\big(\mathsf{N}_{F^3/F}(a) - \lambda^3\big)$ is a non-zero cube in F. This suggests the following generalization of semi-diagonal forms: say that $f\colon V \to F$ is a semi-trace form if

$$f(\xi) = \mathsf{Tr}_{K/F}\big(a\Theta(\xi)^3\big) - 3\lambda \mathsf{N}_{K/F}\big(\Theta(\xi)\big)$$

for some cubic étale F-algebra K, elements $a \in K$ and $\lambda \in F$, and an F-vector space isomorphism $\Theta\colon V \to K$. Our problem then becomes: *Is it possible to extend the preceding result on semi-diagonal forms to semi-trace forms?*

However, note that $a_i^{-2}\big(\mathsf{N}_{K/F}(a) - \lambda^3\big)$ does not make sense in an arbitrary cubic étale F-algebra: indeed, we cannot talk about the coordinates a_1, a_2, a_3 of $a \in K$ because there does not exist a "canonical" basis of K. But since a cubic form $f_{A,V}$ is completely determined by A and V, we can reformulate once more our problem, this time avoiding the cubic form $f_{A,V}$: *Is it possible to classify the pairs (A, V) where A is a degree 3 central simple algebra over F and V is a 3-dimensional subspace of A° which is totally isotropic for the trace quadratic form?*

Classification of cubic pairs

In this thesis we answer this question in the affirmative: we give a complete classification of those so-called cubic pairs (A, V) over F (up to isomorphism).

The next two theorems summarize the classification of non-singular cubic pairs[2]. We let F be a field of characteristic neither 2 nor 3 such that F either contains a primitive cube root of unity or is infinite; let F_{sep} denote a separable closure of F. We prove that the automorphism group of a non-singular cubic pair over F_{sep} is either $\mathbb{Z}/3$ or $\mathbb{Z}/3\times\mathbb{Z}/3$ as an abstract group (see p. 66), and we say that a non-singular cubic pair (A, V) is of the first kind if $(A\otimes_F F_{\mathsf{sep}}, V\otimes_F F_{\mathsf{sep}})$ has automorphism group $\mathbb{Z}/3$ and it is of the second kind otherwise. Fix $\omega\in F_{\mathsf{sep}}$ a primitive cube root of unity.

Theorem I (cf. p. 131) *Suppose that F contains a primitive cube root of unity. Up to F-isomorphism, the non-singular cubic pairs over F are the pairs*

$$\big((a,b)_{\omega,F}, \mathsf{span}_F\langle\xi_0,\eta_0,\xi_0\eta_0^2+\alpha\xi_0^2\eta_0^2\rangle\big)$$

for all $a, b\in F^\times$ and $\alpha\in F$ with $\alpha^3\neq -a^{-1}$, where ξ_0,η_0 are generators of the symbol algebra such that $\xi_0^3=a$, $\eta_0^3=b$ and $\xi_0\eta_0=\omega\eta_0\xi_0$. Such a cubic pair is of the second kind if and only if $\alpha=0$. The associated cubic form is always semi-diagonal and it is diagonal if the pair is of the second kind:

$$\big(x\xi_0+y\eta_0+z(\xi_0\eta_0^2+\alpha\xi_0^2\eta_0^2)\big)^3=ax^3+by^3+cz^3-3\lambda xyz$$

where $c=ab^2+\alpha^3a^2b^2$, $\lambda=\omega^2\alpha ab$ and $a^{-2}(abc-\lambda^3)=b^3$ is a non-zero cube in F.

Theorem II (cf. p. 133) *Suppose that F does not contain a primitive cube root of unity and is infinite. Up to F-isomorphism, the non-singular cubic pairs over F are the pairs*

$$\Big(\bigoplus_{i=0}^{2}Le^i, \mathsf{span}_F\langle\xi_0,\eta_0,\zeta_0\rangle\Big)$$

for all Galois $\mathbb{Z}/3$-algebras (L,ρ) and $a,\alpha\in F$ such that $a\neq 0$ and $\alpha^3\neq a^2$, where $e^3=a$, $e\xi=\rho(\xi)e$ for all $\xi\in L$,

$$\xi_0=e,\quad \eta_0=(\alpha+a^{-1}\alpha^2e+e^2)t,\quad \zeta_0=(\alpha+a^{-1}\alpha^2e+e^2)\rho(t),$$

[2] A cubic pair is non-singular if its associated cubic form is non-singular.

and $t \in L$ is such that $1, t, \rho(t)$ span L and

$$(x - t)\big(x - \rho(t)\big)\big(x - \rho^2(t)\big) = x^3 - 3x + \lambda$$

for some $\lambda \in F$. Such a cubic pair is of the second kind if and only if $\alpha = 0$. The associated cubic form is semi-trace, and we may choose the cubic étale algebra over F to be $F \times F(\omega)$. However the cubic form is not semi-diagonal.

For the classification of singular cubic pairs, we use the fact that there are nine different kinds of singular cubic curves: the zero curve, a triple line, a double line plus simple line, three concurrent lines, a triangle, a conic plus tangent, a conic plus chord, a cuspidal curve and a nodal curve. Thus we split the classification of singular cubic pairs into nine parts. We prove that the cubic curve associated to a singular pair (A, V) where A is division, is necessarily a triangle: to classify the singular cubic pairs over F such that the associated cubic curve is not a triangle, we can therefore make computations in the matrix algebra $\mathsf{M}_3(F)$. We find in particular the following results:

Theorem III (cf. p. 134) *There is no cubic pair over F such that the associated cubic curve is three concurrent lines or a conic plus tangent. There exists at least one cubic pair over F such that the associated cubic curve is cuspidal if and only if F contains a primitive cube root of unity. There always exists at least one cubic pair over F such that the associated cubic curve is the zero curve, a triple line, a double line plus simple line, a conic plus chord or a nodal curve.*

For the classification of cubic pairs with a triangle as associated cubic curve, we prove:

Theorem IV (cf. p. 134) *Suppose that F contains a primitive cube root of unity. Then, up to F-isomorphism, the F-cubic pairs with a triangle as associated cubic curve are the pairs*

$$\big((a,b)_{\omega,F}, \mathsf{span}_F\langle \xi_0, \eta_0, \xi_0^2\eta_0^2\rangle\big)$$

for all $a, b \in F^\times$, where ξ_0 and η_0 are generators of the symbol algebra such that $\xi_0^3 = a$, $\eta_0^3 = b$ and $\xi_0\eta_0 = \omega\eta_0\xi_0$. The associated cubic form is semi-diagonal:

$$(x\xi_0 + y\eta_0 + z\xi_0^2\eta_0^2)^3 = ax^3 + by^3 + cz^3 - 3\lambda xyz$$

where $c = a^2b^2$, $\lambda = \omega^2 ab$ and $abc = \lambda^3$.

Theorem V (cf. p. 136) *Suppose that F is infinite and does not contain a primitive cube root of unity. Up to F-isomorphism, the F-cubic pairs with a triangle as associated cubic curve, are the pairs*

$$\Big(\bigoplus_{i=0}^{2} Le^i, \mathsf{span}_F\langle e, et, e\rho(t)\rangle\Big)$$

for all Galois $\mathbb{Z}/3$-algebras (L,ρ) and $a \in F^\times$, where $e^3 = a$, $e\xi = \rho(\xi)e$ for all $\xi \in L$, and $t \in L$ is such that $1, t, \rho(t)$ span L and

$$(x-t)\big(x-\rho(t)\big)\big(x-\rho^2(t)\big) = x^3 - 3x + \lambda$$

for some $\lambda \in F$. The associated cubic form is semi-trace, and we may choose the cubic étale F-algebra to be $F \times F(\omega)$.

Our classification implies in particular the following result on division algebras:

Theorem VI (cf. p. 136) *Let (A,V) be a cubic pair over F such that A is a division algebra. If F contains a primitive cube root of unity then the associated cubic form $f_{A,V}$ is semi-diagonal. If F is infinite and does not contain a primitive cube root of unity then $f_{A,V}$ is semi-trace, and we may choose the cubic étale algebra over F to be $F \times F(\omega)$.*

This allows us to sharpen the result of Haile and Tignol mentioned earlier:

Theorem VII (cf. p. 136) *Suppose that (A,V) is a cubic pair over F where F contains a primitive cube root of unity and A is a division algebra, and let $(\varphi_1,\varphi_2,\varphi_3)$ be a basis of $V^\star$, $a_1,a_2,a_3,\lambda \in F$ such that*

$$f = a_1\varphi_1^3 + a_2\varphi_2^3 + a_3\varphi_3^3 - 3\lambda\varphi_1\varphi_2\varphi_3,$$

Then either $a_1a_2a_3 = \lambda^3$ or there exists one and only one $i \in \{1,2,3\}$ such that $(a_1a_2a_3 - \lambda^3)a_i^{-2}$ is a non-zero cube in F; in the first case necessarily

$$A \cong (a_1,a_2)_{\omega^{\pm1},F} \cong (a_1,a_3)_{\omega^{\pm1},F} \cong (a_2,a_3)_{\omega^{\pm1},F};$$

in the second case necessarily $A \cong (a_i,a_j)$ for all $j \in \{1,2,3\}$, $i \neq j$.

Moreover our classification of cubic pairs and the result of Haile and Tignol imply the following result:

Theorem VIII (cf. p. 137) *Let A, A' be division algebras of degree 3 over F. If V, V' are such that (A, V) and (A', V') are F-cubic pairs and $f_{A,V}$ is equivalent to $f_{A',V'}$, then the algebras A and A' are either isomorphic or anti-isomorphic.*

Overview of contents

In Chapter 1 we state well-known results on ternary cubic forms and projective cubic curves. We present them by means of the symmetric trilinear form associated to a cubic form. These results are used in the other chapters.

We study the geometry of particular points and lines of a non-singular cubic curve in Chapter 2. We define the Hessian point, the harmonic points and the harmonic polar of a flex of a non-singular cubic curve, and prove some remarkable properties of these points and lines. Further on we shall use the Hessian point and the harmonic points to classify the non-singular cubic pairs.

In the third chapter we give criterions for a non-singular cubic form to be semi-diagonal or semi-trace. These criterions involve the flexes of the associated cubic curve, and shall be used to describe the cubic form associated to a non-singular cubic pair further on.

In Chapter 4 we classify the non-singular cubic pairs over a field F. First we classify up to isomorphism the non-singular cubic pairs over the separable closure of F; next we compute the automorphism group of an arbitrary cubic pair over F_{sep}; finally, we use Galois cohomology to obtain the non-singular cubic pairs over F.

In the fifth chapter we classify the singular cubic pairs over F. Since we have nine different kinds of singular cubic curves we split the classification into nine parts. In all cases except the triangle, the algebra of a singular cubic pair is split. Since we classify the cubic pairs up to isomorphism, we may assume in those cases that the algebra is the matrix algebra $\mathsf{M}_3(F)$. In the remaining case we use the same method as for the classification of non-singular cubic pairs.

In the Conclusion we summarize our results on the classification of cubic pairs and their associated cubic forms.

Some heavy computations in Chapter 4 were done with the aid of Mathematica: in a short Appendix we explain this on a couple of examples. Finally we have of course included a bibliography, an index of notation and an index of terms.

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1

Ternary cubic forms and cubic curves

We define cubic forms and cubic curves and we give their properties that we need in the following chapters. We present these notions and the results by means of the symmetric trilinear form associated to a cubic form. The main purpose of this chapter is to fix notation and not to present original results. Whenever the stated results are well-known we omit a detailed proof but we give an appropriate reference.

1.1 Forms and curves

In this chapter, as well as in all following chapters, F is a field of characteristic neither 2 nor 3. We denote by $\overline{F}$ an algebraic closure of F and by F_{sep} the separable closure of F in $\overline{F}$. For a commutative F-algebra R and an F-vector space V, we put $V_R := V \otimes_F R$. We have an injection $V \hookrightarrow V_R$ defined by $u \mapsto u \otimes 1$ and we identify an element of V with its image by this injection. For brevity we write $\overline{V}$ instead of $V_{\overline{F}}$, and V_{sep} instead of $V_{F_{\mathsf{sep}}}$. As usual, $V^\star$ is the dual space of V. Henceforth V is a 3-dimensional F-vector space.

If d is a strictly positive integer, we denote by $\mathsf{S}^d(V^\star)$ the d-th symmetric power of the vector space $V^\star$ and we put $\mathsf{S}^0(V^\star) := F$. For all strictly positive integers d and d', we have a map

$$\mathsf{S}^d(V^\star) \times \mathsf{S}^{d'}(V^\star) \to \mathsf{S}^{d+d'}(V^\star)$$

sending $(\varphi_1 \dots \varphi_d, \psi_1 \dots \psi_{d'})$ to $\varphi_1 \dots \varphi_d \psi_1 \dots \psi_{d'}$. This endows

$$\mathsf{S}(V^\star) := \oplus_{d \geq 0} \mathsf{S}^d(V^\star)$$

with a structure of graded F-algebra.

In general, an element of $\mathsf{S}^d(V^\star)$ is called a *degree d form* over V.

Definition 1.1.1 *We say that an element of* $\mathsf{S}^3(V^\star)$ *is a* ternary cubic form *over V, or more briefly a cubic form.*

If $f \in \mathsf{S}^3(V^\star)$, thus $f = \sum_{i=0}^r \varphi_i\psi_i\theta_i$ for some $\varphi_i, \psi_i\, \theta_i \in V^\star$, we can define for every F-algebra R, a map $V_R \to R$ by

$$u \otimes \lambda \mapsto \sum_{i=0}^r \lambda\varphi_i(u)\psi_i(u)\theta_i(u)$$

which we also denote by f.

Let (e_1, e_2, e_3) be a basis of V. We consider the element $xe_1+ye_2+ze_3$ of $V \otimes_F F[x, y, z]$. One can check that $f(xe_1 + ye_2 + ze_3)$ is a degree 3 homogeneous polynomial in the variables x, y, z over F.

If $f \in \mathsf{S}^3(V^\star)$ can be written as $l \cdot q$ for some $l \in \overline{V}^\star$ and $q \in \mathsf{S}^2(\overline{V}^\star)$, we call f *reducible* and f is *irreducible* otherwise.

Definition 1.1.2 *Given $f \in \mathsf{S}^3(V^\star)$ there exists a unique symmetric trilinear form t over V such that $t(\xi, \xi, \xi) = f(\xi)$ for all $\xi \in V$. Namely,*

$$t(\xi, \eta, \zeta) = \frac{1}{6}\big(f(\xi+\eta+\zeta)-f(\xi+\eta)-f(\xi+\zeta)-f(\eta+\zeta)+f(\xi)+f(\eta)+f(\zeta)\big).$$

We call t the symmetric trilinear form *associated to f.*

For $f \in \mathsf{S}^3(V^\star)$, w denote by t_f the symmetric trilinear form associated to f. For any commutative algebra R, we can also define a trilinear form over V_R using the map $f\colon V_R \to R$, and we also denote it by t_f.

We denote by $\mathbb{P}(V)$ the projective space associated to V, i.e. the set of non-zero elements of V quotiented by the equivalence relation $u \sim \lambda u$ for $u \in V \setminus \{0\}$ and $\lambda \in F^\times$. For $u \in V \setminus \{0\}$, we write uF for the equivalence class of u in $\mathbb{P}(V)$. Let Ext_F denote the category of field extensions of F and Set the category of sets.

Definition 1.1.3 *Let $f \in \mathsf{S}^3(V^\star)$. The* projective cubic curve *associated to f over F is the functor $\mathcal{F}\colon \mathsf{Ext}_F \to \mathsf{Set}$ defined by*

$$\mathcal{F}(L) = \{uL \in \mathbb{P}(V_L) \mid f(u) = 0\}$$

for an object L of Ext_F and $\mathcal{F}\sigma\colon \mathcal{F}(L) \to \mathcal{F}(M)\colon uL \mapsto \sigma(u)M$ for a morphism $\sigma\colon L \to M$ in Ext_F.

If $f = 0$, the set

$$\{uL \in \mathbb{P}(V_L) \mid f(u) = 0\}$$

is equal to $\mathbb{P}(V_L)$ for all field extensions L/F: in that case we call the associated projective curve the *zero projective curve.*

To have a more suggestive notation we write $\{f(\xi) = 0\}_L$ for

$$\{uL \in \mathbb{P}(V_L) \mid f(u) = 0\}$$

and $\{f(\xi) = 0\}$ for the projective cubic curve associated to f. We call an element of $\{f(\xi) = 0\}_L$ an *L-point* of $\{f(\xi) = 0\}$.

Note that the projective cubic curve associated to f is the same as the one associated to λf for $\lambda \in F^\times$. In fact, by Theorem 9.7, page 26, in [Walker, 1950], if two irreducible cubic forms have the same associated projective cubic curve then they are equal up to a non-zero scalar.

For $L \subset M$ field extensions of F and $p \in \mathbb{P}(V_M)$, we say that p is *defined over* L if there exists $u \in V_L$ such that $p = uM$. We have a natural injection $\mathbb{P}(V_L) \hookrightarrow \mathbb{P}(V_M)$ which we consider as an inclusion. Thus, if $p \in \mathbb{P}(V_M)$ is defined over L, we may consider p as an element of $\mathbb{P}(V_L)$.

Since $f \in \mathsf{S}^3(V^\star)$ can also be viewed as an element of $\mathsf{S}^3\big((V_L)^\star\big)$ for any algebraic field extension L over F, we may also associate to f a projective cubic curve over L, i.e. a functor $\mathsf{Ext}_L \to \mathsf{Set}$. By abuse of notation, we also write $\{f(\xi) = 0\}$ for this functor. For $L \subset M$ algebraic field extensions of F and $f \in \mathsf{S}^3\big((V_M)^\star\big)$, we say that the cubic curve $\{f(\xi) = 0\}$ is *defined over* L if there exists $g \in \mathsf{S}^3\big((V_L)^\star\big)$ and $\lambda \in M^\times$ such that $f = \lambda g$.

In the same way, we may define the projective curve associated to a form of any degree. In the case of degree one, say $l \in \mathsf{S}^1(V^\star) = V^\star$, we call the projective curve $\{l(\xi) = 0\}$ a *projective line*; and in the case of degree two, say $q \in \mathsf{S}^2(V^\star)$, $\{q(\xi) = 0\}$ is a *projective conic.*

1.2 Tangents

We want to define an intersection multiplicity between a projective cubic curve and a projective line at a point of the projective plane. Thereto, let $f \in \mathsf{S}^3(V^\star)$, $l \in \overline{V}^\star$ non-zero and $p = u\overline{F} \in \mathbb{P}(\overline{V})$. We write $m_p(f, l)$ for the multiplicity of the root $\lambda = 0$ of the polynomial

$$f(u + \lambda v) = f(u) + 3\lambda t_f(u, u, v) + 3\lambda^2 t_f(u, v, v) + \lambda^3 f(v)$$

if $l(u) = 0$ and u and v are linearly independent vectors of $\mathsf{ker}(l)$; and $m_p(f, l) = 0$ if $l(u) \neq 0$.

Definition 1.2.1 *The number* $m_p(f,l) \in \{0,1,2,3\} \cup \{\infty\}$ *is the* intersection multiplicity *of the cubic curve* $\{f(\xi)=0\}$ *with the line* $\{l(\xi)=0\}$ *at the point* p.

Thus we have that $m_p(f,l)=0$ if and only if

$$p \notin \{f(\xi)=0\}_{\overline{F}} \ \cap \ \{l(\xi)=0\}_{\overline{F}}$$

and by Theorem 9.7, page 26, in [Walker, 1950], we have $m_p(f,l)=\infty$ if and only if l divides f (in particular f is reducible) and $p \in \{l(\xi)=0\}_{\overline{F}}$. Note that $f=0$ if and only if $m_p(f,l)=\infty$ for all non-zero $l \in \overline{V}^\star$ such that $l(u)=0$.

Now we define the multiplicity of a cubic curve at a point.

Definition 1.2.2 *Let* $f \in \mathsf{S}^3(V^\star)$ *and* $p = u\overline{F} \in \mathbb{P}(\overline{V})$. *If* $f \neq 0$ *then the* multiplicity *of the cubic curve* $\{f(\xi)=0\}$ *at the point* p *is the least integer* $m_p(f)$ *such that there exists a non-zero* $l \in \overline{V}^\star$ *with* $l(u)=0$ *and* $m_p(f)=m_p(f,l)$. *If* $f=0$, *then the multiplicity of the cubic curve* $\{f(\xi)=0\}$ *at the point* p *is* $m_p(f)=\infty$.

If $f \in \mathsf{S}^3(V^\star)$ is non-zero then $m_p(f) \in \{0,1,2,3\}$. We observe that, if $t_f(u,\xi,\xi)=0$ for all $\xi \in V$, then $t_f(u,u,\xi)=0$ for all $\xi \in V$. Indeed, we have $t_f(u,u,u)=0$ and for all $\xi \in V$,

$$t_f(u,\xi,\xi) = t_f(u,u+\xi,u+\xi) = 0,$$

hence $t_f(u,u,\xi)=0$. Since

$$f(u+\lambda\xi) = f(u) + 3\lambda t_f(u,u,\xi) + 3\lambda^2 t_f(u,\xi,\xi) + \lambda^3 f(\xi)$$

for all $\xi \in \overline{V}$,

- $m_p(f)=0$ if and only if $f(u)\neq 0$,
- $m_p(f)=1$ if and only if $f(u)=0$ and there exists $\xi_0 \in \overline{V}$ such that $t_f(u,u,\xi_0)\neq 0$,
- $m_p(f)=2$ if and only if $t_f(u,u,\xi)=0$ for all $\xi \in \overline{V}$ and there exists $\xi_0 \in \overline{V}$ such that $t_f(u,\xi_0,\xi_0)\neq 0$,
- $m_p(f)=3$ if and only if $t_f(u,\xi,\xi)=0$ for all $\xi \in \overline{V}$ and $f \neq 0$.

There exist at most $m_p(f)$ lines $\{l(\xi)=0\}$ such that $m_p(f,l) > m_p(f)$.

Definition 1.2.3 *The lines* $\{l(\xi)=0\}$ *such that* $m_p(f,l) > m_p(f)$ *are called the* tangents *to the cubic curve* $\{f(\xi)=0\}$ *at the point* p.

Definition 1.2.4 *We say that a point $u\overline{F} \in \{f(\xi) = 0\}_{\overline{F}}$ is* singular *if $t_f(u, u, \xi) = 0$ for all $\xi \in \overline{V}$ and is* non-singular *otherwise. We call f (or the cubic curve $\{f(\xi) = 0\}$)* singular *if there exists a singular point and we call f (or $\{f(\xi) = 0\}$)* non-singular *otherwise.*

Let (e_1, e_2, e_3) be a basis of V, put $\xi := x_1e_1 + x_2e_2 + x_3e_3$ and

$$c(x_1, x_2, x_3) := f(\xi).$$

It is clear that $m_p(f) = 1$ if and only if $p = u\overline{F}$ is a non-singular point of $\{f(\xi) = 0\}$. In that case the unique tangent to $\{f(\xi) = 0\}$ at p is the line $\{t_f(u, u, \xi) = 0\}$; moreover we have

$$t_f(u, u, \xi) = \frac{1}{3}\sum_{i=1}^{3} \frac{\partial c}{\partial x_i}(a_1, a_2, a_3)x_i$$

where a_1, a_2, a_3 are the coordinates of u in the basis (e_1, e_2, e_3). If $m_p(f) = 2$ then the tangents at p are contained in $\{t_f(u, \xi, \xi) = 0\}$ (we say that $\mathcal{F}\colon \mathsf{Ext}_F \to \mathsf{Set}$ *is contained in* $\mathcal{G}\colon \mathsf{Ext}_F \to \mathsf{Set}$ if $\mathcal{F}(L) \subset \mathcal{G}(L)$ for all field extensions L/F). Also $6t_f(u, \xi, \xi)$ is equal to

$$\sum_{i,j=1}^{3} \frac{\partial^2 c}{\partial x_i \partial x_j}(a_1, a_2, a_3)x_ix_j$$

where a_1, a_2, a_3 are the coordinates of u in the basis (e_1, e_2, e_3). If $m_p(f) = 3$, then f is reducible and the tangents at p are contained in $\{f(\xi) = 0\}$. If $f = 0$ then all the points of $\mathbb{P}(\overline{V})$ are singular points of $\{f(\xi) = 0\}$: this case is not interesting and we will not consider it.

The following theorem is a weaker version of Bézout's Theorem.

Theorem 1.2.5 *Let $f \in \mathsf{S}^3(V^\star)$ and $l \in \mathsf{S}^3(\overline{V}^\star)$ be non-zero such that l does not divide f. Then there are 3 intersection $\overline{F}$-points between the cubic curve $\{f(\xi) = 0\}$ and the line $\{l(\xi) = 0\}$, counting multiplicities.*

Proof : See Proposition 1, page 208, in [Brieskorn and Knörrer, 1986]. □

The notions of intersection multiplicity, multiplicity and tangents can also be defined for an arbitrary projective curve; we leave that to the reader. In what follows we shall mainly use these notions in the case of cubic curves (but Section 1.6 is an exception).

1.3 Hessian curve, flexes and normal form

Definition 1.3.1 *Let* $f \in \mathsf{S}^3(V^\star)$. *The functor* $\mathsf{H}_f\colon \mathsf{Ext}_F \to \mathsf{Set}$ *defined by*

$$\mathsf{H}_f(L) = \{uL \in \mathbb{P}(V_L) \mid \textit{the form } (\xi,\eta) \mapsto t_f(u,\xi,\eta) \textit{ is singular}\}$$

is called the Hessian curve[1] *of* f.

The Hessian curve H_f of a cubic form f is itself a cubic curve[2]. Indeed, let (e_1, e_2, e_3) be a basis of V, put

$$c(x_1, x_2, x_3) := f(x_1e_2 + x_2e_2 + x_3e_3)$$

and define $h\colon V \to F$ by

$$h(a_1e_1 + a_2e_2 + a_3e_3) = \det \begin{pmatrix} \frac{\partial^2 c}{\partial x_1^2}(a) & \frac{\partial^2 c}{\partial x_1 \partial x_2}(a) & \frac{\partial^2 c}{\partial x_1 \partial x_3}(a) \\ \frac{\partial^2 c}{\partial x_1 \partial x_2}(a) & \frac{\partial^2 c}{\partial x_2^2}(a) & \frac{\partial^2 c}{\partial x_2 \partial x_3}(a) \\ \frac{\partial^2 c}{\partial x_1 \partial x_3}(a) & \frac{\partial^2 c}{\partial x_2 \partial x_3}(a) & \frac{\partial^2 c}{\partial x_3^2}(a) \end{pmatrix},$$

where $a = (a_1, a_2, a_3)$. Then $h \in \mathsf{S}^3(V^\star)$ and $\mathsf{H}_f = \{h(\xi) = 0\}$. Alternatively in terms of the associated trilinear form, we may also define H_f as $\{h(\xi) = 0\}$ with

$$h(\xi) = \det \begin{pmatrix} t_f(\xi, e_1, e_1) & t_f(\xi, e_1, e_2) & t_f(\xi, e_1, e_3) \\ t_f(\xi, e_1, e_2) & t_f(\xi, e_2, e_2) & t_f(\xi, e_2, e_3) \\ t_f(\xi, e_1, e_3) & t_f(\xi, e_2, e_3) & t_f(\xi, e_3, e_3) \end{pmatrix},$$

where (e_1, e_2, e_3) is a basis of V.

By Remark (i), page 289, in [Brieskorn and Knörrer, 1986], the Hessian curve H_f is the zero curve only if f decomposes into a product of linear forms over the algebraic closure. Remark (v) in *loc. cit.* says that H_f goes through each singular point of $\{f(\xi) = 0\}$.

Definition 1.3.2 *Let* $p = u\overline{F}$ *be an* $\overline{F}$*-point of* $\{f(\xi) = 0\}$. *We say that* p *is a* flex *of* $\{f(\xi) = 0\}$ *if* p *is non-singular and the intersection multiplicity of the curve* $\{f(\xi) = 0\}$ *with the tangent* $\{t_f(u, u, \xi) = 0\}$ *at* p *is greater than or equal to 3.*

[1] The Hessian curve is named after the German mathematician Otto Hesse (1811–1874) who defined this curve in 1844... as a polynomial, of course!

[2] We can define a Hessian curve for a form of any degree but in general it is not necessarily a projective curve of the same degree.

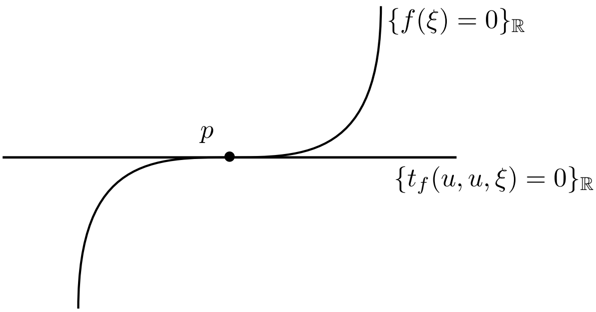

Figure 1.1: Illustration of a flex

In the following, we give an example of a flex of a cubic curve .

Example 1.3.3 Let $f\colon \mathbb{R}^3 \to \mathbb{R}$ be defined by $f(x, y, z) = yz^2 - x^3$. Then f is a cubic form over $\mathbb{R}^3$ and $p = (0, 0, 1)\mathbb{C}$ is a flex of the curve $\{f(\xi) = 0\}$. In Figure 1.1 we draw the $\mathbb{R}$-points of $\{f(\xi) = 0\}$ in the affine plane obtained by choosing the line of equation $z = 1$ as line at infinity.

Let $f \in \mathsf{S}^3(V^\star)$ and $p = u\overline{F}$ a non-singular $\overline{F}$-point of $\{f(\xi) = 0\}$. If $t_f(u, u, v) = 0$ then $f(u + \lambda v) = 3\lambda^2 t_f(u, v, v) + \lambda^3 f(v)$, so p is a flex if and only if the tangent to $\{f(\xi) = 0\}$ at p is contained in the conic $\{t_f(u, \xi, \xi) = 0\}$. We have another way to characterize the flexes of a cubic curve.

Proposition 1.3.4 *Let $f \in \mathsf{S}^3(V^\star)$ and p a non-singular point of the curve $\{f(\xi) = 0\}$. Then p is a flex of $\{f(\xi) = 0\}$ if and only if $p \in \mathsf{H}_f(\overline{F})$.*

Proof : See [Knapp, 1992], Proposition 2.12. □

If f is non-singular, one can prove that $\{f(\xi) = 0\}$ has at least one and at most nine flexes using the resultant of polynomials. We will see that $\{f(\xi) = 0\}$ has in fact nine distinct flexes. But first we need other results.

Lemma 1.3.5 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular. Then $\{f(\xi) = 0\}$ has at least two flexes.*

Proof : Lemma 15.3 in [Gibson, 1998] says that a non-singular cubic curve has nine distinct flexes if $F = \mathbb{C}$. But the proof can be adapted to show that a non-singular cubic curve has at least two flexes over an algebraically closed field. □

Definition 1.3.6 *We say that $f \in \mathsf{S}^3(V^\star)$ is a* normal form *if there exist linearly independent $\varphi_1, \varphi_2, \varphi_3 \in V^\star$ and $\lambda, \mu \in F$ not both zero such that*

$$f = \mu(\varphi_1^3 + \varphi_2^3 + \varphi_3^3) + \lambda\varphi_1\varphi_2\varphi_3.$$

Suppose that $f = \mu(\varphi_1^3 + \varphi_2^3 + \varphi_3^3) + \lambda\varphi_1\varphi_2\varphi_3$ for some linearly independent $\varphi_1, \varphi_2, \varphi_3 \in \overline{V}^\star$ and $\lambda, \mu \in \overline{F}$ not both zero. Then f is singular if and only if $\mu = 0$ or $\lambda^3 = \mu^3$.

Theorem 1.3.7 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular. Then f is a normal form as an element of $\mathsf{S}^3(\overline{V}^\star)$. In particular, $\{f(\xi) = 0\}$ has exactly nine flexes in $\mathbb{P}(\overline{V})$.*

Proof : See Theorem 4, page 293, in [Brieskorn and Knörrer, 1986]. □

A priori the nine flexes are in $\mathbb{P}(\overline{V})$. The next theorem says that they are defined over F_{sep}.

Theorem 1.3.8 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular. The nine flexes of $\{f(\xi) = 0\}$ are defined over F_{sep} and f is a normal form as an element of $\mathsf{S}^3\big((V_{\mathsf{sep}})^\star\big)$.*

Proof : Let (e_1, e_2, e_3) be a basis of $\overline{V}$ and put

$$c(x, y, z) := f(xe_1 + ye_2 + ze_3).$$

We may assume that none of the flexes are in $\{\alpha e_1 + \beta e_2 \mid \alpha, \beta \in \overline{F}\}$. Let $a_i, b_i \in \overline{F}$, $i = 1, \ldots, 9$, be such that the $(a_i e_1 + b_i e_2 + e_3)\overline{F}$ are the nine flexes. Changing the basis if necessary, we may assume that $a_i \neq a_j$ for $i \neq j$. Let

$$h(x, y, z) = \det \begin{pmatrix} \frac{\partial^2 c}{\partial x^2}(x, y, z) & \frac{\partial^2 c}{\partial x \partial y}(x, y, z) & \frac{\partial^2 c}{\partial x \partial z}(x, y, z) \\ \frac{\partial^2 c}{\partial x \partial y}(x, y, z) & \frac{\partial^2 c}{\partial y^2}(x, y, z) & \frac{\partial^2 c}{\partial y \partial z}(x, y, z) \\ \frac{\partial^2 c}{\partial x \partial z}(x, y, z) & \frac{\partial^2 c}{\partial y \partial z}(x, y, z) & \frac{\partial^2 c}{\partial z^2}(x, y, z) \end{pmatrix}$$

and $r(x, z)$ (respectively $s(x)$) the resultant of the polynomials $c(x, y, z)$ and $h(x, y, z)$ (respectively $c(x, y, 1)$ and $h(x, y, 1)$) with respect to y. One can check that $s(a_i) = 0$ for all i and $s(x) = r(x, 1) \neq 0$. As the degree of $s(x)$ is less than or equal to 9 and $s(a_i) = 0$ for all i, it follows that s has degree 9 and all its roots are simple. So $a_i \in F_{\mathsf{sep}}$ for all i.

The polynomial $c(a_i, y, 1)$ is not constant since otherwise $c(a_i, y, 1)$ would be constant equal to zero and f would be singular. If $h(a_i, y, 1)$ is not constant, the polynomials $c(a_i, y, 1)$ and $h(a_i, y, 1)$ have a non-constant common factor in $F_{\mathsf{sep}}[y]$ which is a multiple of $(y - b_i)^n$ for some $n = 1, 2, 3$ (the degree of $f(a_i, y, 1)$ is less or equal to 3); so $b_i \in F_{\mathsf{sep}}$. If $h(a_i, y, 1)$ is constant, then $h(a_i, y, 1) = 0$ and so $c(a_i, y, 1)$ is a multiple of $(y - b_i)^n$ for some $n = 1, 2, 3$. As $c(a_i, y, 1) \in F_{\mathsf{sep}}[y]$ we have that $b_i \in F_{\mathsf{sep}}$. Hence $\{f(\xi) = 0\}$ has its nine flexes defined over F_{sep}.

In particular $\{f(\xi) = 0\}$ has at least two flexes in $\mathbb{P}(V_{\mathsf{sep}})$ and we can adapt the proof of Theorem 4, page 293, in [Brieskorn and Knörrer, 1986] to see that f is a normal form as an element of $\mathsf{S}^3\big((V_{\mathsf{sep}})^\star\big)$. $\square$

Suppose that f is a non-singular cubic form. The nine flexes of the curve $\{f(\xi) = 0\}$ have the following property: a line passing through two flexes passes through a third one. So we have 9 flexes and 12 lines which pass through two of the flexes; through each flex pass four lines among the 12 lines; there are four triples of lines such that each flex lies on one and only one line of the triple. Figure 1.2 shows the incidences just described, where p_{ij} are the flexes and C_0, C_1, C_2 and C_∞ are the triples of lines.

We summarize the properties of the flexes.

Proposition 1.3.9 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular. The nine flexes of the cubic curve $\{f(\xi) = 0\}$ and the twelve lines which go through two of the flexes have the configuration of the points and the lines of the affine plane $\mathbb{F}_3^2$.*

Proof: Pages 295-296 in [Brieskorn and Knörrer, 1986] treat the configuration of the flexes of the projective curve associated to a non-singular normal form. Since a non-singular cubic form is a normal form as an element of $\mathsf{S}^3\big((V_{\mathsf{sep}})^\star\big)$, the flexes of its associated cubic curve have the same configuration. $\square$

1.4 j-Invariant

Let $f \in \mathsf{S}^3(V^\star)$ be a non-singular cubic form. By Theorem 1.3.8, there exist linearly independent $\varphi_1, \varphi_2, \varphi_3 \in (V_{\mathsf{sep}})^\star$ and $\lambda \in F_{\mathsf{sep}}$ such that

$$f = \varphi_1^3 + \varphi_2^3 + \varphi_3^3 - 3\lambda\varphi_1\varphi_2\varphi_2.$$

and $\lambda^3 \neq 1$.

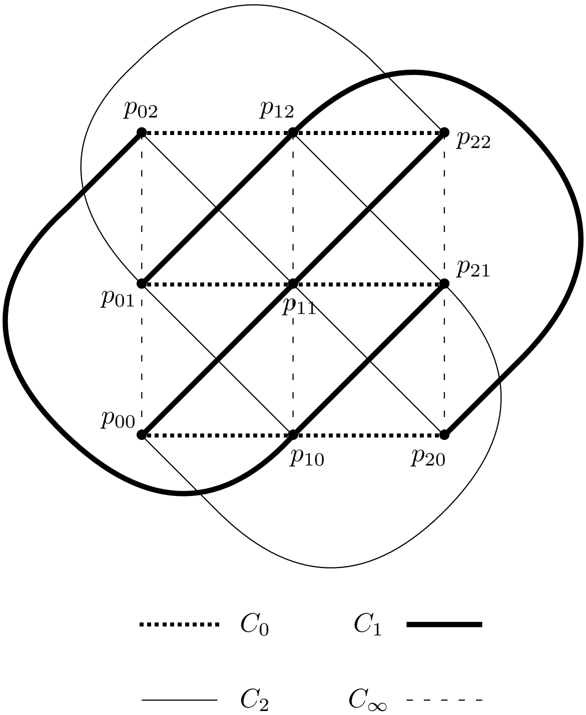

Figure 1.2: Flexes of a non-singular cubic curve

Lemma 1.4.1 *There exist other linearly independent ψ_1, ψ_2, ψ_3 in $(V_{\mathsf{sep}})^\star$ and $\mu \in F_{\mathsf{sep}}$ such that $f = \psi_1^3 + \psi_2^3 + \psi_3^3 - 3\mu\psi_1\psi_2\psi_3$ if and only if*

$$\frac{\lambda^3(\lambda^3+8)^3}{(\lambda^3-1)^3} = \frac{\mu^3(\mu^3+8)^3}{(\mu^3-1)^3}.$$

Proof : See Theorem 10, page 302, in [Brieskorn and Knörrer, 1986]. □

We shall prove that

$$\frac{\lambda^3(\lambda^3+8)^3}{(\lambda^3-1)^3} \in F.$$

To do this we need some lemmas.

We denote by Γ the absolute Galois group $\mathsf{Gal}(F_{\mathsf{sep}}/F)$. We consider several continuous actions of Γ. First Γ acts naturally on V_{sep}: for $\sigma \in \Gamma$, $v \otimes \lambda \in V_{\mathsf{sep}}$,

$$\sigma(v \otimes \lambda) = v \otimes \sigma(\lambda).$$

Next we have an action of Γ on $(V_{\mathsf{sep}})^\star$: for $\sigma \in \Gamma$, $\varphi \in (V_{\mathsf{sep}})^\star$ and $\xi \in V_{\mathsf{sep}}$,

$${}^\sigma\varphi(\xi) = \sigma\left(\varphi\left(\sigma^{-1}(\xi)\right)\right).$$

Finally Γ acts naturally on $V^\star \otimes_F F_{\mathsf{sep}}$: for $\sigma \in \Gamma$ and $\varphi \otimes \lambda \in V^\star \otimes_F F_{\mathsf{sep}}$,

$${}^\sigma(\varphi \otimes \lambda) = \varphi \otimes \sigma(\lambda).$$

Lemma 1.4.2 *There exists an F_{sep}-vector space isomorphism between $V^\star \otimes_F F_{\mathsf{sep}}$ and $(V_{\mathsf{sep}})^\star$, which is compatible with the action of Γ.*

Proof : We may choose the linear map $\Theta\colon V^\star \otimes_F F_{\mathsf{sep}} \to (V_{\mathsf{sep}})^\star$ which sends $\varphi \otimes \lambda$ to the linear form mapping $v \otimes \mu$ to $\varphi(v)\lambda\mu$. □

Thus we may identify $V^\star \otimes_F F_{\mathsf{sep}}$ and $(V_{\mathsf{sep}})^\star$ and we denote them by $V_{\mathsf{sep}}^\star$.

We define two other actions of Γ. We have an action of Γ on $\mathsf{S}^d(V_{\mathsf{sep}}^\star)$ induced by the action on $V_{\mathsf{sep}}^\star$:

$${}^\sigma(\varphi_1 \ldots \varphi_d) = {}^\sigma\varphi_1 \ldots {}^\sigma\varphi_d;$$

and Γ acts naturally on $\mathsf{S}^d(V^\star) \otimes_F F_{\mathsf{sep}}$:

$${}^\sigma(\varphi_1 \ldots \varphi_d \otimes \lambda) = \varphi_1 \ldots \varphi_d \otimes \sigma(\lambda).$$

Lemma 1.4.3 *There exists an F_{sep}-vector space isomorphism between $\mathsf{S}^d(V^\star) \otimes_F F_{\mathsf{sep}}$ and $\mathsf{S}^d(V_{\mathsf{sep}}^\star)$ which is compatible with the action of Γ.*

Proof : The linear map sending $\varphi_1 \ldots \varphi_d \otimes \lambda$ onto $(\varphi_1 \otimes 1 \ldots \varphi_d \otimes 1)\lambda$ defines an F_{sep}-vector space isomorphism between $\mathsf{S}^d(V^\star) \otimes_F F_{\mathsf{sep}}$ and $\mathsf{S}^d(V^\star_{\mathsf{sep}})$ and is compatible with the action of Γ. □

Let $f \in \mathsf{S}^3(V^\star)$ be non singular, then ${}^\sigma f = f$. Let $\varphi_1, \varphi_2, \varphi_3 \in V^\star_{\mathsf{sep}}$ be linearly independent and $\lambda \in F_{\mathsf{sep}}$ such that

$$f = \varphi_1^3 + \varphi_2^3 + \varphi_3^3 - 3\lambda\varphi_1\varphi_2\varphi_3.$$

For all $\sigma \in \Gamma$, we have

$$f = {}^\sigma\varphi_1^3 + {}^\sigma\varphi_2^3 + {}^\sigma\varphi_3^3 - 3\sigma(\lambda)\ {}^\sigma\varphi_1 {}^\sigma\varphi_2 {}^\sigma\varphi_3$$

where ${}^\sigma\varphi_1, {}^\sigma\varphi_2, {}^\sigma\varphi_3 \in V^\star_{\mathsf{sep}}$ are linearly independent. By the lemma above,

$$\sigma\left(\frac{\lambda^3(\lambda^3+8)^3}{(\lambda^3-1)^3}\right) = \frac{\sigma(\lambda)^3(\sigma(\lambda)^3+8)^3}{(\sigma(\lambda)^3-1)^3} = \frac{\lambda^3(\lambda^3+8)^3}{(\lambda^3-1)^3},$$

and hence

$$\frac{\lambda^3(\lambda^3+8)^3}{(\lambda^3-1)^3} \in F.$$

Definition 1.4.4 *In the situation above, we define the* j-invariant *of* f *as*

$$j(f) = \frac{\lambda^3(\lambda^3+8)^3}{(\lambda^3-1)^3} \in F.$$

We say that two ternary cubic forms $f \in \mathsf{S}^3(V^\star)$ and $f' \in \mathsf{S}^3(V'^\star)$ are *equivalent* if there exists an F-vector space isomorphism $\Theta \colon V \to V'$ such that $f = f' \circ \Theta$. By Lemma 1.4.1, the j-invariant has the following property: suppose $f \in \mathsf{S}^3(V^\star)$ and $f' \in \mathsf{S}^3(V'^\star)$ are non-singular, then $j(f) = j(f')$ if and only if f and f' are equivalent as elements of $\mathsf{S}^3(V^\star_{\mathsf{sep}})$ and $\mathsf{S}^3(V'^\star_{\mathsf{sep}})$ respectively.

1.5 Canonical pencil

Let $f \in \mathsf{S}^3(V^\star)$ and $h \in \mathsf{S}^3(V^\star)$ be such that $\{h(\xi) = 0\}$ is the Hessian curve H_f of f.

Definition 1.5.1 *The* canonical pencil *associated to* f *is the collection of the cubic curves* $\{(\alpha f + \beta h)(\xi) = 0\}$ *for all* $\alpha, \beta \in F_{\mathsf{sep}}$ *not both zero.*

If f is non-singular, then the flexes of $\{f(\xi) = 0\}$ are common points of all the cubic curves in the pencil.

Let $g \in \mathsf{S}^3(V^\star)$: the cubic curve $\{g(\xi) = 0\}$ is a *triangle* if $g = l_1 l_2 l_3$ for some $l_i \in \overline{V}^\star$ such that the lines $\{l_i(\xi) = 0\}$, for $i = 1, 2, 3$, are distinct and non-concurrent.

Proposition 1.5.2 *Suppose that $f \in \mathsf{S}^3(V^\star)$ is non-singular. The cubic curves in the canonical pencil of f are exactly the cubic curves over F_{sep} passing through the nine flexes of $\{f(\xi) = 0\}$. These nine points are also flexes for any cubic curve of the pencil. If $\alpha f + \beta h$ is singular, then $\{(\alpha f + \beta h)(\xi) = 0\}$ is a triangle. There are exactly four triangles in the canonical pencil.*

Proof : Let $\varphi_1, \varphi_2, \varphi_3 \in V^\star_{\mathsf{sep}}$ be linearly independent and $\lambda \in F_{\mathsf{sep}}$ such that

$$f = \varphi_1^3 + \varphi_2^3 + \varphi_3^3 - 3\lambda\varphi_1\varphi_2\varphi_3.$$

Let (e_1, e_2, e_3) is a basis of V_{sep} such that $\varphi_i(e_j) = \delta_{ij}$ where δ_{ij} denotes the Kronecker symbol (such a basis exists since the linear forms $\varphi_1, \varphi_2, \varphi_3$ are linearly independent) and put

$$c(x_1, x_2, x_3) := f(x_1 e_1 + x_2 e_2 + x_3 e_3).$$

Then

$$\det\Big(\frac{\partial^2 c}{\partial x_i \partial x_j}\Big) = -54\big(\lambda^2(x_1^3 + x_2^3 + x_3^3) + (\lambda^3 - 4)x_1 x_2 x_3\big).$$

Thus we may replace h by a multiple so that

$$h = \lambda^2(\varphi_1^3 + \varphi_2^3 + \varphi_3^3) + (\lambda^3 - 4)\varphi_1\varphi_2\varphi_3.$$

Since $\lambda^3 \neq 1$ the canonical pencil of f is equal to

$$\{\nu(\varphi_1^3 + \varphi_2^3 + \varphi_3^3) + \mu\varphi_1\varphi_2\varphi_3 \mid \nu, \mu \in F_{\mathsf{sep}} \text{ not both zero}\}.$$

Then Proposition 5, page 295, in [Brieskorn and Knörrer, 1986] completes the proof. □

The triangles in the canonical pencil of f are called the *inflexional triangles* of f.

Proposition 1.5.3 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular and $h \in \mathsf{S}^3(V^\star)$ such that $\{h(\xi) = 0\}$ is the Hessian curve. Then h is singular if and only if $j(f) = 0$. Moreover, if h is singular then H_f is an inflexional triangle.*

Proof: Let $\varphi_1, \varphi_2, \varphi_3 \in V^\star_{\mathsf{sep}}$ be linearly independent and $\lambda \in F_{\mathsf{sep}}$ such that

$$f = \varphi_1^3 + \varphi_2^3 + \varphi_3^3 - 3\lambda\varphi_1\varphi_2\varphi_3.$$

Then we may assume that

$$h = \lambda^2(\varphi_1^3 + \varphi_2^3 + \varphi_3^3) + (\lambda^3 - 4)\varphi_1\varphi_2\varphi_3,$$

so h is singular if and only if $\lambda = 0$ or $(\lambda^3 - 4)^3 = (-3\lambda^2)^3$. Since $\lambda^3 \neq 1$,

$$(\lambda^3 - 4)^3 = (-3\lambda^2)^3 \iff \lambda^3 + 8 = 0.$$

Thus h is singular if and only if $j(f) = 0$. Since H_f is in the canonical pencil of f, if h is singular then H_f is an inflexional triangle of f. □

1.6 Singular cubic forms

In this section we classify the singular cubic forms over V by giving a representative of each F_{sep}-equivalence class of F-cubic pairs. The classification splits into two parts: the reducible cubic forms and the irreducible ones.

First we consider a non-zero reducible cubic form f (note that a reducible cubic form is singular): thus $f = l \cdot q$ for some $l \in \overline{V}^\star$ and $q \in \mathsf{S}^2(\overline{V}^\star)$. It is easy to see that we may assume $l \in V^\star_{\mathsf{sep}}$ and $q \in \mathsf{S}^2(V^\star_{\mathsf{sep}})$ since $f \in \mathsf{S}^3(V^\star)$.

If q itself is reducible, then $f = l_1 l_2 l_3$ for some $l_i \in V^\star_{\mathsf{sep}}$. Depending on the number of distinct lines $\{l_i(\xi) = 0\}$ and on their intersection points, one can show by straightforward computations that there exists a basis (e_1, e_2, e_3) of V_{sep} such that $f(xe_1 + ye_2 + ze_3)$ is one of the following polynomials:

(1) x^3,

(2) $x^2 y$,

(3) $xy(x + y)$,

(4) xyz.

In those cases we call the projective cubic curve respectively *triple line*, *double line plus simple line*, *three concurrent lines* and *triangle*. The singular points of these curves are the intersection points between the lines. Thus a triple line and a double line plus simple line have infinitely many singular points, but three concurrent lines have one singular point and a triangle has three singular points. If one among these singular curves has finitely many singular points then all its singular points are defined over F_{sep}.

If on the contrary q is irreducible, then we may find a basis (e_1, e_2, e_3) of V_{sep} such that $f(xe_1 + ye_2 + ze_3)$ is equal to one of the following:

(5) $(y^2 - xz)z$,

(6) $(z^2 - xy)z$.

In case (5) the line $\{l(\xi) = 0\}$ is tangent to $\{q(\xi) = 0\}$ and in case (6) the line $\{l(\xi) = 0\}$ has two distinct intersection points with $\{q(\xi) = 0\}$. The associated cubic curves are called *conic plus tangent* and *conic plus chord* respectively. The singular points of these curves are the intersection points between the conic and the line. A conic plus tangent has one singular point and a conic plus chord has two singular points.

Having dealt with the singular reducible forms, we now classify the singular irreducible cubic forms with a series of lemmas.

Lemma 1.6.1 *Suppose that f is a non-zero singular irreducible cubic form. Then $\{f(\xi) = 0\}$ has a unique singular point and the multiplicity of $\{f(\xi) = 0\}$ at this point is equal to 2.*

Proof: See Lemma 15.1 in [Gibson, 1998]. □

Let $p = u\overline{F}$ be the unique singular point of $\{f(\xi) = 0\}$. Since the multiplicity $m_p(f) = 2$ there exists $\xi_0 \in \overline{V}$ such that $t_f(u, \xi_0, \xi_0) \neq 0$ and the tangents to $\{f(\xi) = 0\}$ at p are contained in $\{t_f(u, \xi, \xi) = 0\}$. If $t_f(u, \xi, \xi) = l(\xi)^2$ for some $l \in \overline{V}^\star$ then we have a unique tangent $\{l(\xi) = 0\}$ and we say that the tangent is *double*. If $t_f(u, \xi, \xi) = l_1(\xi)l_2(\xi)$ for some $l_1, l_2 \in \overline{V}^\star$ linearly independent then the tangents to $\{f(\xi) = 0\}$ are $\{l_1(\xi) = 0\}$ and $\{l_2(\xi) = 0\}$ and we say that the tangents are *simple*.

Lemma 1.6.2 *Suppose that f is a non-zero singular irreducible cubic form with two simple tangents at the singular point. Then there exists a basis (e_1, e_2, e_3) of V_{sep} such that*

$$f(xe_1 + ye_2 + ze_3) = z^3 + xz^2 - xy^2.$$

Moreover the cubic curve $\{f(\xi) = 0\}$ has exactly three collinear flexes; the flexes and the singular points are defined over F_{sep}.

Proof : See Proposition 12, page 304, in [Brieskorn and Knörrer, 1986]. Those authors assume that $F = \mathbb{C}$, but their argument can straightforwardly be generalized for an arbitrary algebraically closed field. Then one can check that the singular point, its tangents and the flexes are defined over F_{sep}, so the result is in fact also true over a separably closed field. □

If f is a non-zero singular irreducible cubic form with two simple tangents at the singular point, we say that the cubic curve $\{f(\xi) = 0\}$ is *nodal.*

Lemma 1.6.3 *Suppose that f is a non-zero singular irreducible cubic form with a double tangent at the singular point. Then there exists a basis (e_1, e_2, e_3) of V_{sep} such that*

$$f(xe_1 + ye_2 + ze_3) = z^3 - xy^2.$$

Moreover the cubic curve $\{f(\xi) = 0\}$ has a unique flex; the flex and the singular point are defined over F_{sep}.

Proof : See Proposition 13, page 304, in [Brieskorn and Knörrer, 1986]: the result is stated in the case where $F = \mathbb{C}$ but one can prove that it is also true for an arbitrary separably closed field. □

If f is a non-zero singular irreducible cubic form with a double tangent at the singular point then we say that $\{f(\xi) = 0\}$ is *cuspidal.*

Thus, to repeat: if f is a non-zero irreducible singular cubic form, there exists a basis (e_1, e_2, e_3) of V_{sep} such that $f(xe_1 + ye_2 + ze_3)$ is one of the following:

(7) $z^3 + xz^2 - xy^2$,

(8) $z^3 - xy^2$.

In conclusion, we have eight different kinds of non-zero singular cubic forms, six of them are reducible and two of them are irreducible.

2

Particular points and lines

For a flex of a non-singular cubic curve, we define its Hessian point, its harmonic points and the associated harmonic polar. We study the geometry of these points and lines which – much like the flexes – have interesting properties. The Hessian points and the harmonic points will play a crucial role in the classification of non-singular cubic pairs in Chapter 4. We believe that most of the results in this chapter are original; we did not find any references in the literature.

2.1 Hessian points

Let $f \in \mathsf{S}^3(V^\star)$ be non-singular and $p = u\overline{F}$ a flex of $\{f(\xi) = 0\}$. Then the bilinear form

$$\overline{V} \times \overline{V} \to \overline{F} \colon (\xi, \eta) \mapsto t_f(u, \xi, \eta)$$

is singular and hence there exists a non-zero vector $u' \in \overline{V}$ such that $t_f(u, u', \xi) = 0$ for all $\xi \in \overline{V}$. We note that $u'\overline{F}$ is an $\overline{F}$-point of the Hessian curve H_f and u and u' are linearly independent since p is non-singular.

Proposition 2.1.1 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular and $p = u\overline{F}$ a flex of $\{f(\xi) = 0\}$. Then there exists a unique $u'\overline{F} \in \mathbb{P}(\overline{V})$ such that $t_f(u, u', \xi) = 0$ for all $\xi \in \overline{V}$.*

Proof : Suppose there exist linearly independent $u', u'' \in \overline{V}$ such that $t_f(u, u', \xi) = 0$ and $t_f(u, u'', \xi) = 0$ for all $\xi \in \overline{V}$. It is clear that $u\overline{F}, u'\overline{F}$ and $u''\overline{F}$ are $\overline{F}$-points of the tangent $\{t_f(u, u, \xi) = 0\}$. Hence there exist $\alpha, \beta \in \overline{F}$ with $\alpha \neq 0$ such that $u'' = \alpha u + \beta u'$. Because

$$t_f(u, u'', \xi) = \alpha t_f(u, u, \xi) + \beta t_f(u, u', \xi)$$

we have $t_f(u,u,\xi) = 0$ for all $\xi \in \overline{V}$. This contradicts the assumption that f is non-singular. $\square$

Definition 2.1.2 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular and $p = u\overline{F}$ a flex of $\{f(\xi) = 0\}$. We denote by $p' = u'\overline{F}$ the (unique) point satisfying $t_f(u,u',\xi) = 0$ for all $\xi \in \overline{V}$, and (in view of Proposition 2.1.3) we call p' the* Hessian point *of p.*

We observe that p' is not an $\overline{F}$-point of $\{f(\xi) = 0\}$. Indeed, the intersection multiplicity of $\{f(\xi) = 0\}$ with the tangent $\{t_f(u,u,\xi) = 0\}$ at p is three. By Theorem 1.2.5, since f is irreducible, the point p is the only intersection point between the cubic curve $\{f(\xi) = 0\}$ and the tangent $\{t_f(u,u,\xi) = 0\}$; thus p' is not a point of $\{f(\xi) = 0\}$.

It is clear that p and the Hessian point p' are intersection points of the Hessian curve H_f and the tangent $\{t_f(u,u,\xi) = 0\}$. But we can prove more.

Proposition 2.1.3 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular and p a flex of $\{f(\xi) = 0\}$. The points p and p' are the only intersection points of H_f and the tangent to $\{f(\xi) = 0\}$ at p. Moreover, the intersection multiplicity at p' is equal to two. In particular , if p' is a non-singular point of H_f then the tangent to $\{f(\xi) = 0\}$ at p is the tangent to H_f at p'.*

Proof : Let $u, u' \in \overline{V}$ be such that $p = u\overline{F}$ and $p' = u'\overline{V}$ and let $v \in \overline{V}$ be such that u, u' and v are linearly independent. Then (u, u', v) is a basis of $\overline{V}$, so $\xi\overline{F}$ is an $\overline{F}$-point of the Hessian curve if and only if

$$\det \begin{pmatrix} t_f(\xi,u,u) & t_f(\xi,u,u') & t_f(\xi,u,v) \\ t_f(\xi,u,u') & t_f(\xi,u',u') & t_f(\xi,u',v) \\ t_f(\xi,u,v) & t_f(\xi,u',v) & t_f(\xi,v,v) \end{pmatrix} = 0.$$

The points $(\alpha u + \beta u')\overline{F}$, for $\alpha, \beta \in \overline{F}$ not both zero, are the $\overline{F}$-points of the tangent $\{t_f(u,u,\xi) = 0\}$. Hence the intersection $\overline{F}$-points of the Hessian curve and the tangent $\{t_f(u,u,\xi) = 0\}$ at p are the points $(\alpha u + \beta u')\overline{F}$ with $\alpha, \beta \in \overline{F}$ not both zero such that

$$\alpha^2 \beta f(u') t_f(u,u,v) = 0.$$

The Hessian point p' is not on $\{f(\xi) = 0\}$, so $f(u') \neq 0$. Since u, u', v are linearly independent, $t_f(u,u,v) \neq 0$. Thus, p and p' are the only intersection points of the Hessian curve and the tangent $\{t_f(u,u,\xi) = 0\}$ and the intersection multiplicity at p' is equal to two. $\square$

The next lemma gives a criterion for a Hessian point to be singular.

Lemma 2.1.4 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular, $p = u\overline{F}$ a flex of the cubic curve $\{f(\xi) = 0\}$ and $v\overline{F}$ not on the tangent $\{t_f(u,u,\xi) = 0\}$. Then $p' = u'\overline{F}$ is a singular point of H_f if and only if*

$$f(u')t_f(u',v,v) = t_f(u',u',v)^2.$$

Proof : Put

$$h(\xi) := \det \begin{pmatrix} t_f(u,u,\xi) & t_f(u,u',\xi) & t_f(u,v,\xi) \\ t_f(u,u',\xi) & t_f(u',u',\xi) & t_f(u',v,\xi) \\ t_f(u,v,\xi) & t_f(u',v,\xi) & t_f(v,v,\xi) \end{pmatrix}.$$

Then the coefficient of λ in $h(u' + \lambda\xi)$ is equal to

$$t_f(u,u,\xi)\left(f(u')t_f(u',v,v) - t_f(u',u',v)^2\right).$$

So p' is a singular point of H_f if and only if

$$t_f(u,u,\xi)\left(f(u')t_f(u',v,v) - t_f(u',u',v)^2\right) = 0$$

for all $\xi \in \overline{V}$. Since f is non-singular, there exists a $\xi_0 \in \overline{V}$ such that $t_f(u,u,\xi_0) \neq 0$. Hence p' is a singular point of H_f if and only if

$$f(u')t_f(u',v,v) = t_f(u',u',v)^2.$$

□

Next we give a condition, related to the Hessian points of the flexes, for the Hessian curve of a non-singular form to be singular, and describe the configuration of the Hessian points in that case.

Proposition 2.1.5 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular and p a flex of the cubic curve $\{f(\xi) = 0\}$. The following conditions are equivalent:*

1. *p' is a singular point of H_f;*
2. *there exists a flex q of $\{f(\xi) = 0\}$ such that $q \neq p$ and $q' = p'$;*
3. *H_f is singular.*

In this situation, the flexes of $\{f(\xi) = 0\}$ may be named $p_1, \ldots, p_9$ in such a way that $p'_i = p'_{i+1} = p'_{i+2}$ and p_i, p_{i+1}, p_{i+2} lie on a line contained in the Hessian curve, for $i \in \{1,4,7\}$. For all i, the Hessian point p'_i is a singular point of H_f and it is the intersection point of the lines contained in the Hessian curve which do not pass through p_i, as illustrated in Figure 2.1.

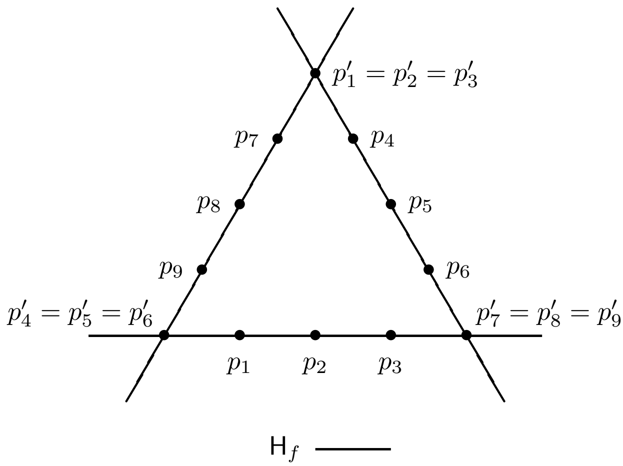

Figure 2.1: Hessian points on a singular Hessian curve

Proof: Suppose p' is a singular point of H_f. Then H_f is singular and by Proposition 1.5.3 it is a triangle. Suppose $q = v\overline{F}$ is a flex of $\{f(\xi) = 0\}$. Proposition 2.1.3 says that the intersection multiplicity of the Hessian curve with the tangent $\{t_f(v, v, \xi) = 0\}$ at q' is equal to two. Hence q' is singular (it is clear that the intersection multiplicity of a triangle with a line at a non-singular point is $0, 1$ or ∞). Assume $q' \neq p'$ for all flexes $q \neq p$ of $\{f(\xi) = 0\}$. As there are only two singular points of H_f distinct from p', there exist four distinct flexes such that their Hessian points are equal. Thus, there exist three non-collinear flexes q_1, q_2, q_3 of $\{f(\xi) = 0\}$ such that $q_1' = q_2' = q_3'$. Let $v_i, v_i' \in \overline{V}$ be such that $q_i = v_i\overline{F}$ and $q_i' = v_i'\overline{F}$. Then $t_f(v_i, v_1', \xi) = 0$ for all $\xi \in \overline{V}$ and for all $i \in \{1, 2, 3\}$. Since q_1, q_2, q_3 are non-collinear, the vectors v_1, v_2, v_3 are linearly independent. So, $f(v_1') = t_f(v_1', v_1', v_1') = 0$ which is impossible. Thus, (1) $\Rightarrow$ (2).

Suppose that there exists a flex $q \neq p$ of $\{f(\xi) = 0\}$ such that $q' = p'$. Let $u, u', v \in \overline{F}$ be such that $p = u\overline{F}$, $p' = u'\overline{F}$ and $q = v\overline{F}$. Then $v\overline{F}$ does not lie on the tangent $\{t_f(u, u, \xi) = 0\}$ and

$$f(u')t_f(u', v, v) = 0 = t_f(u', u', v)^2.$$

So by Lemma 2.1.4, the point p' is a singular point of H_f and we have (2) $\Rightarrow$ (1).

It is clear that (1) $\Rightarrow$ (3). By Proposition 2.1.3, if H_f is singular, then p is singular; so also (3) $\Rightarrow$ (1).

Now assume that these equivalent conditions hold. Since the line through a flex and its Hessian point intersects the Hessian curve at the Hessian point with multiplicity two, it is not contained in the Hessian curve (the intersection multiplicity of a triangle with a line of the triangle at a point is either 0 or ∞). So the Hessian points of three flexes on a line contained in the Hessian curve are equal; this single point is the intersection point of the lines contained in the Hessian curve which do not pass through the flexes. $\square$

For $p \neq q \in \mathbb{P}(V)$ we shall denote by $\langle p, q\rangle$ the line passing through p and q, i.e. the functor $\mathcal{F}\colon \mathsf{Ext}_F \to \mathsf{Set}$ defined by

$$\mathcal{F}(L) = \{(\alpha u + \beta v)L \mid \alpha, \beta \in L \text{ not both zero}\}$$

for L/F a field extension, where $p = uF$ and $q = vF$.

Proposition 2.1.6 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular such that H_f is non-singular. Suppose that p and q are distinct flexes of $\{f(\xi) = 0\}$. Then the lines $\langle p, q\rangle$ and $\langle p', q'\rangle$ are distinct and intersect at the third flex on the line $\langle p, q\rangle$.*

Proof: If the lines $\langle p, q\rangle$ and $\langle p', q'\rangle$ coincide then p, p' and q are intersection points of the Hessian curve with the tangent at p, which contradicts Proposition 2.1.3. So the lines are distinct.

The third flex on the line $\langle p, q\rangle$ is the third intersection point of the cubic curve $\{f(\xi) = 0\}$ with the line $\langle p, q\rangle$. Let u, u', v be such that $p = u\overline{F}$, $p' = u'\overline{F}$ and $q = v\overline{F}$. Because

$$f(\alpha u + \beta v) = 3\alpha\beta\big(\alpha t_f(u, u, v) + \beta t_f(u, v, v)\big),$$

the third flex on the line $\langle p, q\rangle$ is the point

$$\big(t_f(u, v, v)u - t_f(u, u, v)v\big)\overline{F}.$$

Put

$$h(\xi) := \det\begin{pmatrix} t_f(u,u,\xi) & t_f(u,u',\xi) & t_f(u,v,\xi) \\ t_f(u,u',\xi) & t_f(u',u',\xi) & t_f(u',v,\xi) \\ t_f(u,v,\xi) & t_f(u',v,\xi) & t_f(v,v,\xi) \end{pmatrix}$$

so that $\{h(\xi) = 0\}$ is the Hessian curve. Since

$$h(v) = -t_f(u, v, v)^2 t_f(u', u', v) - t_f(u, u, v)t_f(u', v, v)^2 = 0,$$

the vector

$$v' = -t_f(u, v, v)t_f(u', v, v)u + t_f(u, v, v)^2 u' + t_f(u, u, v)t_f(u', v, v)v$$

satisfies $t_f(v, v', \xi) = 0$ for all $\xi \in \overline{V}$, so $q' = v'\overline{F}$. Since $h(u' + \lambda v')$ is equal to

$$\lambda t_f(u, u, v') \left(f(u')t_f(u', v, v) - t_f(u', u', v)^2 + \lambda t_f(u', u', v')t_f(u', v, v) \right),$$

the third intersection $\overline{F}$-point of the Hessian curve and the line $\langle p', q' \rangle$ is the point

$$r = \left(t_f(u', u', v')t_f(u', v, v)u' + \left(t_f(u', u', v)^2 - f(u')t_f(u', v, v) \right) v' \right) \overline{F}.$$

Replacing v' in r we get that $r = (t_f(u, v, v)u - t_f(u, u, v)v)\overline{F}$. So the lines $\langle p, q \rangle$ and $\langle p', q' \rangle$ intersect at the third flex on the line $\langle p, q \rangle$. □

The previous proposition is also true if H_f is singular and $p' \neq q'$. Indeed, by Proposition 2.1.5, the flexes p and q lie on distinct lines contained in the Hessian curve. So the third flex on $\langle p, q \rangle$ lies on the line contained in the Hessian curve which does not pass through p and q; this is the line $\langle p', q' \rangle$.

2.2 Harmonic polars

Let $f \in \mathsf{S}^3(V^\star)$ be non-singular and $p = u\overline{F}$ a flex of $\{f(\xi) = 0\}$. We know by the remark preceding Proposition 1.3.4 that the tangent at p is contained in the conic $\{t_f(u, \xi, \xi) = 0\}$. Hence the quadratic form

$$\overline{V} \to \overline{F} \colon \xi \mapsto t_f(u, \xi, \xi)$$

is reducible and the conic $\{t_f(u, \xi, \xi) = 0\}$ is composed of two lines. These two lines intersect only at one point, namely the Hessian point p', thus the conic $\{t_f(u, \xi, \xi) = 0\}$ consists of two distinct lines, one of them being the tangent at p.

Definition 2.2.1 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular and $p = u\overline{F}$ a flex of $\{f(\xi) = 0\}$. The line different from the tangent at p contained in the conic $\{t_f(u, \xi, \xi) = 0\}$ is called the* harmonic polar *of the cubic curve $\{f(\xi) = 0\}$ at the flex p.*

We write $p^\star$ for the harmonic polar of $\{f(\xi) = 0\}$ at a flex p.

Lemma 2.2.2 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular and p and q distinct flexes of $\{f(\xi) = 0\}$. Then the harmonic polar at p is distinct from the harmonic polar at q.*

Proof: Let $u, v \in \overline{V}$ be such that $p = u\overline{F}$ and $q = v\overline{F}$. Suppose that the harmonic polars $p^\star$ and $q^\star$ coincide. Then there exists a point $r = w\overline{F}$ of $\{f(\xi) = 0\}$ which lies on both the harmonic polars. In particular, we have $t_f(u, w, w) = 0$ and $t_f(v, w, w) = 0$ (the harmonic polar at p is contained in the conic $\{t_f(u, \xi, \xi) = 0\}$). Since $\{t_f(w, w, \xi) = 0\}$ is the tangent to $\{f(\xi) = 0\}$ at r, the intersection multiplicity of $\{f(\xi) = 0\}$ with the tangent $\{t_f(w, w, \xi) = 0\}$ at r is greater than or equal to two. Thus the number of intersection $\overline{F}$-points of $\{f(\xi) = 0\}$ with the line $\{t_f(w, w, \xi) = 0\}$ is greater than or equal to four. By Theorem 1.2.5, the line $\{t_f(w, w, \xi) = 0\}$ is contained in the cubic curve $\{f(\xi) = 0\}$. This contradicts the fact that f is non-singular. $\square$

The following proposition exhibits the configuration of the nine harmonic polars of a non-singular cubic curve.

Proposition 2.2.3 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular and p, q, r distinct flexes of $\{f(\xi) = 0\}$. Then the harmonic polars $p^\star, q^\star$ and $r^\star$ are concurrent if and only if the flexes are collinear.*

Proof: Suppose that p, q, r are collinear. If $p' = q' = r'$ then the lines $p^\star, q^\star$ and $r^\star$ are concurrent at p'. Assume we do not have $p' = q' = r'$, then by Proposition 2.1.5 the points p', q' and r' are distinct pairwise because p, q, r are collinear. Let $u, u', v \in \overline{V}$ be such that $p = u\overline{F}$, $p' = u'\overline{V}$ and $q = v\overline{F}$. Then $r = w\overline{F}$ with

$$w = t_f(u, v, v)u - t_f(u, u, v)v$$

and the proof of Proposition 2.1.6 shows in particular that $q' = v'\overline{F}$ with

$$v' = -t_f(u, v, v)t_f(u', v, v)v + t_f(u, v, v)^2 u' + t_f(u, u, v)t_f(u', v, v)v.$$

Let $\xi_0\overline{F}$ be the intersection $\overline{F}$-point of $p^\star$ with $q^\star$. Then in particular $t_f(u, \xi_0, \xi_0) = 0$ and $t_f(v, \xi_0, \xi_0) = 0$. Thus

$$t_f(w, \xi_0, \xi_0) = t_f(u, v, v)t_f(u, \xi_0, \xi_0) - t_f(u, u, v)t_f(v, \xi_0, \xi_0) = 0$$

and $\xi_0\overline{F}$ is either on the tangent at r or on the harmonic polar at r. Assume $\xi_0\overline{F}$ is on the tangent at r. Let $\alpha, \beta, \gamma \in \overline{F}$ be scalars such that $\xi_0 = \alpha u + \beta u' + \gamma v$. The point $\xi_0\overline{F}$ is not on the tangent at p. Indeed, if $\xi_0\overline{F}$ lies on the tangent at p then it is the intersection point of the tangent at p with the harmonic polar at p, namely p'. But p' does not lie on the tangent at r since the only intersection points of the Hessian curve with the tangent at r are r and r', and $p' \neq r, r'$. Hence $\gamma \neq 0$. In the same way, it can be seen that the tangent at q does not pass through $\xi_0\overline{F}$. Because $t_f(w,w,\xi_0)=0$ implies

$$t_f(u,u,v)t_f(u',v,v)\beta = t_f(u,v,v)t_f(u,u,v)\alpha + t_f(u,v,v)^2\gamma$$

and $t_f(u,\xi_0,\xi_0)=0$ implies

$$2\alpha\gamma t_f(u,u,v)+\gamma^2 t_f(u,v,v)=0,$$

the point $\xi_0\overline{F}$ is equal to

$$\big(-t_f(u',v,v)t_f(u,v,v)u + t_f(u,v,v)^2u' + 2t_f(u,u,v)t_f(u',v,v)v\big)\,\overline{F}.$$

Hence $\xi_0\overline{F} = (v' + t_f(u,u,v)t_f(u',v,v)v)\,\overline{F}$ is an $\overline{F}$-point of the tangent $\{t_f(v,v,\xi)=0\}$ and we have a contradiction. Thus, the point $\xi_0\overline{F}$ is on the harmonic polar $r^\star$ and the harmonic polars $p^\star$, $q^\star$ and $r^\star$ are concurrent.

Now assume that the harmonic polars $p^\star, q^\star$ and $r^\star$ are concurrent at $\xi_0\overline{F}$. Suppose p,q,r are non-collinear. Let $u,v,w\in\overline{V}$ be such that $p=u\overline{F}$, $q=v\overline{V}$ and $r=w\overline{V}$. Then $t_f(u,\xi_0,\xi_0)=0$, $t_f(v,\xi_0,\xi_0)=0$ and $t_f(w,\xi_0,\xi_0)=0$. Since p,q,r are non-collinear, the vectors u,v,w are linearly independent. Thus, $t_f(\xi_0,\xi_0,\xi)=0$ for all $\xi\in\overline{V}$ and $\xi_0\overline{F}$ is a singular point of $\{f(\xi)=0\}$; this is impossible because f is non-singular. $\square$

Let us summarize the properties which we obtained on the harmonic polars. For a non-singular $f\in\mathsf{S}^3(V^\star)$, there are exactly nine harmonic polars of the cubic curve $\{f(\xi)=0\}$. Through the intersection point of two harmonic polars passes a third harmonic polar; through any given point pass at most three harmonic polars. There are four triples of points which satisfy the following property: a harmonic polar passes through one and only one point of the triple. Hence the configuration of the nine harmonic polars is dual to the configuration of the nine flexes in the following sense: to obtain the properties of the harmonic polars we replace "point" by "line", "lie on" by "pass through", "collinear" by "concurrent", etc. in the properties of the flexes. Moreover the two configurations are connected by Proposition 2.2.3: the harmonic polars at three flexes are concurrent if and only if the flexes are collinear.

2.3 Harmonic points

In this section, we define another class of particular points of a non-singular cubic curve.

Proposition 2.3.1 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular and p a flex of $\{f(\xi) = 0\}$. There are exactly three distinct intersection $\overline{F}$-points between the cubic curve $\{f(\xi) = 0\}$ and the harmonic polar $p^\star$.*

Proof : Let $u \in \overline{F}$ be such that $p = u\overline{F}$. Suppose that there is an intersection $\overline{F}$-point $q = v\overline{F}$ between $\{f(\xi) = 0\}$ and $p^\star$ with multiplicity greater than or equal to two. Then the tangent $\{t_f(v, v, \xi) = 0\}$ at q is the harmonic polar $p^\star$. Since q is on the harmonic polar $p^\star$, we have in particular $t_f(u, v, v) = 0$. Thus p is also on the harmonic polar $p^\star$. This is impossible since p' is the only intersection point between the tangent at p and the harmonic polar at p. Thus, there are exactly three intersection points between the cubic curve $\{f(\xi) = 0\}$ and the harmonic polar $p^\star$. □

Definition 2.3.2 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular and p a flex of the curve $\{f(\xi) = 0\}$. The three intersection points of the cubic curve $\{f(\xi) = 0\}$ with the harmonic polar at p are called the* harmonic points *of the flex p.*

In [1950], page 124, Walker defines a sextatic point of an irreducible cubic curve. An harmonic point is in particular a sextatic point.

Proposition 2.3.3 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular, p a flex of the curve $\{f(\xi) = 0\}$ and $q \neq p$ an $\overline{F}$-point of $\{f(\xi) = 0\}$. Then the tangent to the curve $\{f(\xi) = 0\}$ at q passes through p if and only if q is a harmonic point of p.*

Proof : Let $u, v \in \overline{V}$ be such that $p = u\overline{F}$ and $q = v\overline{F}$. Suppose that q is a harmonic point of p. Then in particular $t_f(u, v, v) = 0$ and so p is on the tangent to $\{f(\xi) = 0\}$ at q. Conversely, if p is on the tangent to $\{f(\xi) = 0\}$ at q then $t_f(u, v, v) = 0$. So q is either on the tangent at p or on the harmonic polar at p. Suppose that q is on the tangent at p. Because q is on the cubic curve $\{f(\xi) = 0\}$ we have $q = p$; it contradicts the hypothesis. Thus q is on the harmonic polar $p^\star$ and then q is a harmonic point of p. □

As in [Walker, 1950] we shall define a group law on the $\overline{F}$-points of the cubic curve $\{f(\xi) = 0\}$. To that end, it is useful to introduce some more notation. Recall that we write $\langle p, q\rangle$ for the line through given points $p \neq q \in \mathbb{P}(V)$. For a $p \in \{f(\xi) = 0\}_{\overline{F}}$ we shall now write $\langle p, p\rangle$ for the tangent of $\{f(\xi) = 0\}$ at p.

Let o be a flex of $\{f(\xi) = 0\}$. We define a group law on the $\overline{F}$-points of $\{f(\xi) = 0\}$ which depends on the flex o. Let a, b be two $\overline{F}$-points of $\{f(\xi) = 0\}$. By Theorem 1.2.5, there are exactly three intersection $\overline{F}$-points of $\{f(\xi) = 0\}$ with the line $\langle a, b\rangle$, counting multiplicity. Let c_1 be the third intersection $\overline{F}$-point (for instance, if the intersection multiplicity at a is two and $a \neq b$ then $c_1 = a$). Now we set $a +_o b$ to be the third intersection point of $\{f(\xi) = 0\}$ with $\langle o, c_1\rangle$. By Theorem 9.1, page 191, in [Walker, 1950] this addition on the $\overline{F}$-points of $\{f(\xi) = 0\}$ is a commutative group law with o as the zero element. Theorem 9.2, page 192, in *op. cit.*, says in particular the following:

Theorem 2.3.4 *Let o be a flex of $\{f(\xi) = 0\}$ and a_1, a_2, a_3 $\overline{F}$-points of $\{f(\xi) = 0\}$. Then a_1, a_2, a_3 are the intersection $\overline{F}$-points of $\{f(\xi) = 0\}$ with $\{l(\xi) = 0\}$ for some $l \in \overline{V}^\star$, counted with multiplicity, if and only if $a_1 +_o a_2 +_o a_3 = o$.*

Now we can state a property of the harmonic points.

Proposition 2.3.5 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular, p_1, p_2, p_3 distinct collinear flexes of $\{f(\xi) = 0\}$ and q_1 a harmonic point of p_1. Then the line $\langle q_1, p_3\rangle$ intersects the cubic curve $\{f(\xi) = 0\}$ at a third point, which is a harmonic point of p_2.*

Proof: We put $o := p_1$ and $q_2 := p_2 +_o q_1$. Then $q_2 \neq p_2$ since otherwise $q_1 = p_1$. By Theorem 2.3.4, since p_1, p_2, p_3 are distinct collinear $\overline{F}$-points of $\{f(\xi) = 0\}$, we have $p_2 +_o p_3 = o$. Also $2q_1 = o$ and $3p_2 = o$ because p_1 is on the tangent at q_1 and p_2 is a flex. Then

$$\begin{cases} p_2 +_o 2q_2 = p_2 +_o 2p_2 +_o 2q_1 = o, \\ q_1 +_o p_3 +_o q_2 = q_1 +_o p_3 +_o p_2 +_o q_1 = o. \end{cases}$$

Thus, p_2 is on the tangent to $\{f(\xi) = 0\}$ at q_2 and q_1, p_3, q_2 are the intersection $\overline{F}$-points of $\{f(\xi) = 0\}$ with the line $\langle q_1, p_3\rangle$. So the third intersection point of the cubic curve $\{f(\xi) = 0\}$ with the line $\langle q_1, p_3\rangle$ is the points q_2 which is a harmonic point of p_2. $\square$

3

Particular ternary cubic forms

We introduce and study particular ternary cubic forms which we call semi-diagonal form and semi-trace form, the latter generalizing the former. We give a criterion for a non-singular cubic form to be semi-diagonal or semi-trace. In Chapter 4 we shall use this criterion to show that the cubic form associated to a non-singular cubic pair is always a semi-trace form.

3.1 Semi-diagonal forms

Definition 3.1.1 *We say that* $f \in \mathsf{S}^3(V^\star)$ *is a* semi-diagonal form *if there exist linearly independent forms* $\varphi_1, \varphi_2, \varphi_3 \in V^\star$ *and scalars* $\alpha_1, \alpha_2, \alpha_3, \lambda \in F$ *such that*

$$f = \alpha_1 \varphi_1^3 + \alpha_2 \varphi_2^3 + \alpha_3 \varphi_3^3 - 3\lambda \varphi_1 \varphi_2 \varphi_3.$$

If moreover $\lambda = 0$ *we say that* f *is a* diagonal form.

Note that $\varphi_1, \varphi_2, \varphi_3$ are linearly independent if and only if the cubic curve $\{(\varphi_1\varphi_2\varphi_3)(\xi) = 0\}$ is a triangle. Also note that a non-singular cubic form is a diagonal form only if its j-invariant is equal to zero.

The following lemma gives a condition for a semi-diagonal form to be non-singular.

Lemma 3.1.2 *Let* $f \in \mathsf{S}^3(V^\star)$ *be a semi-diagonal form:*

$$f = \alpha_1 \varphi_1^3 + \alpha_2 \varphi_2^3 + \alpha_3 \varphi_3^3 - 3\lambda \varphi_1 \varphi_2 \varphi_3.$$

Then f *is non-singular if and only if* $\alpha_1, \alpha_2, \alpha_3 \neq 0$ *and* $\lambda^3 \neq \alpha_1\alpha_2\alpha_3$*.*

Proof : The symmetric trilinear form t_f associated to f is defined by

$$t_f(\xi,\eta,\zeta) = \alpha_1\varphi_1(\xi)\varphi_1(\eta)\varphi_1(\zeta) + \alpha_2\varphi_2(\xi)\varphi_2(\eta)\varphi_2(\zeta)$$
$$+ \alpha_3\varphi_3(\xi)\varphi_3(\eta)\varphi_3(\zeta) - \frac{\lambda}{18}\sum_{\sigma\in\mathsf{S}_3}\varphi_{\sigma(1)}(\xi)\varphi_{\sigma(2)}(\eta)\varphi_{\sigma(3)}(\zeta).$$

Let (e_1,e_2,e_3) be a basis of V such that $\varphi_i(e_j)=\delta_{ij}$ for all i,j. Then $(x_0e_1+y_0e_2+z_0e_3)\overline{F}$ is singular if and only if $(x_0,y_0,z_0)\neq 0$ and

$$\begin{cases} \alpha_1x_0^2 = \lambda y_0z_0, \\ \alpha_2y_0^2 = \lambda x_0z_0, \\ \alpha_3z_0^2 = \lambda x_0y_0. \end{cases}$$

If $\alpha_1=0$ then $e_1\overline{F}$ is a singular point of $\{f(\xi)=0\}$; hence f is singular. In the same way, if $\alpha_2=0$ or $\alpha_3=0$ then f is singular. Assume that $\alpha_1,\alpha_2,\alpha_3\neq 0$ and $\lambda^3=\alpha_1\alpha_2\alpha_3$. Let $\theta\in\overline{F}$ be a cube root of $\alpha_1\alpha_2^{-1}$, then $(\theta\lambda e_1+\theta^2\lambda e_2+\alpha_1e_3)\overline{F}$ is a singular point, so f is singular. Conversely, suppose that $\alpha_1,\alpha_2,\alpha_3\neq 0$ and f is singular. Let p be a singular point. There exist $x_0,y_0,z_0\in\overline{F}$ such that $(x_0e_1+y_0e_2+z_0e_3)\overline{F}$. Then $x_0\neq 0$ because otherwise $x_0,y_0,z_0=0$ and

$$\alpha_2\alpha_3\alpha_1^2x_0^4 = \alpha_2\alpha_3\lambda^2y_0^2z_0^2 = \lambda^4x_0^2y_0z_0 = \lambda^3\alpha_1x_0^4;$$

thus, $\lambda^3=\alpha_1\alpha_2\alpha_3$. □

Next we give a criterion for a non-singular cubic form to be a semi-diagonal form.

Theorem 3.1.3 *Let $f\in\mathsf{S}^3(V^\star)$ be non-singular. Then f is a semi-diagonal form if and only if there exists an inflexional triangle of f whose lines are defined over F. Moreover, the cubic curve associated to $\varphi_1\varphi_2\varphi_3$, with $\varphi_i\in V^\star$, is an inflexional triangle of f if and only if*

$$f = \alpha_1\varphi_1^3+\alpha_2\varphi_2^3+\alpha_3\varphi_3^3-3\lambda\varphi_1\varphi_2\varphi_3$$

for some $\alpha_1,\alpha_2,\alpha_3,\lambda\in F$.

Proof : Suppose that f is a semi-diagonal form. Let $\varphi_1,\varphi_2,\varphi_3\in V^\star$ be linearly independent and $\alpha_1,\alpha_2,\alpha_3,\lambda\in F$ such that

$$f = \alpha_1\varphi_1^3+\alpha_2\varphi_2^3+\alpha_3\varphi_3^3-3\lambda\varphi_1\varphi_2\varphi_3.$$

Put

$$h := -2\lambda^2(\alpha_1\varphi_1^3+\alpha_2\varphi_2^3+\alpha_3\varphi_3^3)+(8\alpha_1\alpha_2\alpha_3-2\lambda^3)\varphi_1\varphi_2\varphi_3,$$

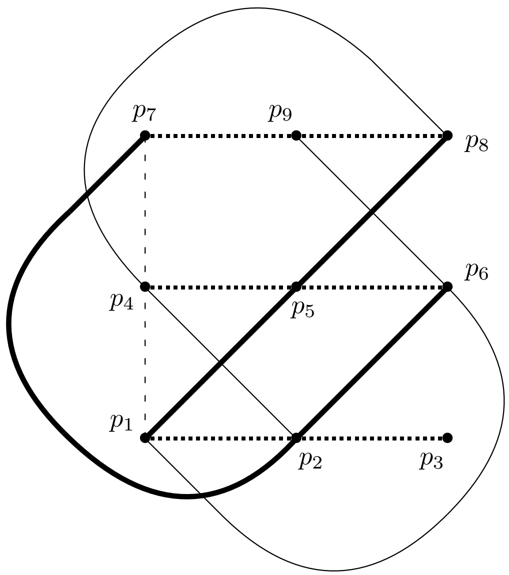

Figure 3.1: Flexes of $\{f(\xi) = 0\}$

then $\{h(\xi) = 0\}$ is the Hessian curve of $\{f(\xi) = 0\}$. Let $\xi_0\overline{F}$ be a flex of $\{f(\xi) = 0\}$. Since $\lambda^3 \neq \alpha_1\alpha_2\alpha_3$ and $\xi_0\overline{F}$ is an intersection point of $\{f(\xi) = 0\}$ and the Hessian curve H_f, we deduce that

$$\begin{cases} \alpha_1\varphi_1(\xi_0)^3 + \alpha_2\varphi_2(\xi_0)^3 + \alpha_3\varphi_3(\xi_0)^3 = 0, \\ \varphi_1(\xi_0)\varphi_2(\xi_0)\varphi_3(\xi_0) = 0. \end{cases}$$

Thus the cubic curve associated to $g = \varphi_1\varphi_2\varphi_3$ is an inflexional triangle of f whose lines are defined over F.

Conversely, assume there exists an inflexional triangle $g = \varphi_1\varphi_2\varphi_3$ of f with $\varphi_1, \varphi_2, \varphi_3 \in V^\star$. Let (e_1, e_2, e_3) be a basis of V such that $\varphi_i(e_j) = \delta_{ij}$. Let p_1, p_2, p_3 be the flexes of $\{f(\xi) = 0\}$ on the line $\{\varphi_1(\xi) = 0\}$ and let p_4 be a flex on the line $\{\varphi_2(\xi) = 0\}$. Then there exist $b \in F^\times$ and distinct $a_1, a_2, a_3 \in F^\times$ such that $p_4 = (e_1 + be_3)\overline{F}$ and $p_i = (a_ie_2 + e_3)\overline{F}$ for all $i = 1, 2, 3$ (the scalars a_1, a_2, a_3, b are non-zero because the flexes of a non-singular cubic curve are not intersection points between lines of an inflexional triangle). Let p_{6+i} be the third flex on the line $\langle p_i, p_4\rangle$ for $i = 1, 2, 3$, p_5 the third flex on the line $\langle p_1, p_8\rangle$ and p_6 the third flex on the line $\langle p_1, p_9\rangle$ (the incidences of the points are showed in Figure 3.1). Then we have $p_{6+i} = (e_1 - a_ibe_2)\overline{F}$ for all $i = 1, 2, 3$, $p_5 = (a_1e_1 + a_2be_3)\overline{F}$ and $p_6 = (a_1e_1 + a_3be_3)\overline{F}$. Using the configuration of the flexes of a non-singular cubic curve, we shall deduce which flexes are collinear. For distinct collinear points $p, q, r \in \mathbb{P}(\overline{V})$, we write $\langle p, q, r\rangle$ for the line passing through this points. Since the lines $\langle p_2, p_4, p_8\rangle$ and $\langle p_1, p_6, p_9\rangle$ do not pass through a common flex, they are contained in the same inflexional triangle and $\langle p_3, p_5, p_7\rangle$ is the last line of the triangle; so p_3, p_5, p_7 are collinear. We deduce similarly that p_2, p_6, p_7 are collinear. The lines $\langle p_1, p_2, p_3\rangle$, $\langle p_2, p_4, p_8\rangle$ and $\langle p_2, p_6, p_7\rangle$ pass all through p_2. Thus the last line passing through p_2 is $\langle p_2, p_5, p_9\rangle$ and p_2, p_5, p_9 are collinear. In the same way, we can prove that p_3, p_6, p_8 are collinear. Since p_5 lies on the line $\langle p_3, p_7\rangle$, there exist $\alpha, \beta, \lambda \in \overline{F}$ such that $\lambda \neq 0$ and

$$(a_1e_1 + a_2be_3)\lambda = \alpha(a_3e_2 + e_3) + \beta(e_1 - a_1be_2).$$

So $\alpha = \lambda a_2 b$, $\beta = \lambda a_1$ and $a_1^2 = a_2a_3$. Similarly, we have $a_2^2 = a_1a_3$ because p_2, p_5, p_9 are collinear and $a_3^2 = a_1a_2$ because p_3, p_6, p_8 are collinear. In particular

$$a_1^3 = a_2^3 = a_3^3 = a_1a_2a_3.$$

Since the a_i's are distinct, we have $a_2 = \omega a_1$ and $a_3 = \omega^2 a_1$ for some primitive cube root $\omega \in \overline{F}$ of unity. We write

$$f = \sum \lambda_{i_1,i_2,i_3}\varphi_1^{i_1}\varphi_2^{i_2}\varphi_3^{i_3}$$

where the sum runs over all the positive integers i_1, i_2 and i_3 such that $i_1 + i_2 + i_3 = 3$. Since $f(a_ie_2 + e_3) = 0$ for all $i = 1, 2, 3$, the a_i's are roots of the polynomial

$$\lambda_{0,3,0}t^3 + \lambda_{0,2,1}t^2 + \lambda_{0,1,2}t + \lambda_{0,0,3}.$$

Thus $\lambda_{0,2,1} = \lambda_{0,1,2} = 0$. Exchanging the role of the e_i's we also get that $\lambda_{2,1,0} = \lambda_{1,2,0} = 0$ and $\lambda_{2,0,1} = \lambda_{1,0,2} = 0$. Hence

$$f = \lambda_{3,0,0}\varphi_1^3 + \lambda_{0,3,0}\varphi_2^3 + \lambda_{0,0,3}\varphi_3^3 + \lambda_{1,1,1}\varphi_1\varphi_2\varphi_3,$$

and f is a semi-diagonal form. □

3.2 Semi-trace forms

For an F-algebra K, we denote by $\mathsf{Tr}_{K/F}$ and $\mathsf{N}_{K/F}$ the trace form and the norm form of the F-algebra K, i.e. the maps

$$\mathsf{Tr}_{K/F} \colon K \to F \colon \xi \mapsto \mathsf{tr}(l_\xi),$$

$$\mathsf{N}_{K/F}\colon K \to F\colon \xi \mapsto \mathsf{det}(l_\xi),$$

where l_ξ denotes the endomorphism of K of left multiplication by ξ. Suppose that K is d-dimensional over F and fix an $\alpha \in K$; then the forms $\mathsf{N}_{K/F}$ and $K \to F\colon \xi \mapsto \mathsf{Tr}_{K/F}(\alpha\xi^d)$ may be considered as elements of $\mathsf{S}^d(K^\star)$.

Definition 3.2.1 *Let $f \in \mathsf{S}^3(V^\star)$ be a ternary cubic form. We say that f is a* semi-trace form *if there exist a cubic étale F-algebra K, elements $\alpha \in K$ and $\lambda \in F$, and an F-vector space isomorphism $\Theta\colon V \to K$ such that*

$$f(\xi) = \mathsf{Tr}_{K/F}(\alpha\Theta(\xi)^3) - 3\lambda\mathsf{N}_{K/F}(\Theta(\xi))$$

for all $\xi \in V$.

A semi-diagonal form is in particular a semi-trace form. Indeed, suppose $f \in \mathsf{S}^3(V^\star)$ is a semi-diagonal form; so let $\varphi_1, \varphi_2, \varphi_3 \in V^\star$ be linearly independent and $\alpha_1, \alpha_2, \alpha_3, \lambda \in F$ such that

$$f = \alpha_1\varphi_1^3 + \alpha_2\varphi_2^3 + \alpha_3\varphi_3^3 - 3\lambda\varphi_1\varphi_2\varphi_3.$$

Let (e_1, e_2, e_3) be a basis of V such that $\varphi_i(e_j) = \delta_{ij}$ for all $i, j = 1, 2, 3$. We put $K := F \times F \times F$, $\alpha := (\alpha_1, \alpha_2, \alpha_3)$ and we define $\Theta\colon V \to K$ as the F-vector space isomorphism for which $\Theta(e_1) = (1, 0, 0)$, $\Theta(e_2) = (0, 1, 0)$ and $\Theta(e_3) = (0, 0, 1)$. Then

$$f(\xi) = \mathsf{Tr}_{K/F}(\alpha^3\Theta(\xi)) - 3\lambda\mathsf{N}_{K/F}(\Theta(\xi)).$$

We will give a criterion for a non-singular ternary cubic form to be a semi-trace form but first we need preliminaries.

As in [Knus *et al.*, 1998], we say that G is a Γ-*group* if G is a group equipped with a continuous action of Γ, denoted $(\sigma, a) \mapsto \sigma \star a$, such that

$$\sigma \star (ab) = (\sigma \star a)(\sigma \star b)$$

for all $\sigma \in \Gamma$ and $a, b \in G$. We denote by $\mathsf{Map}(\{1,2,3\}, F_{\mathsf{sep}}^\times)$ the group of the set maps between $\{1,2,3\}$ and $F_{\mathsf{sep}}^\times$. For $a \in \mathsf{Map}(\{1,2,3\}, F_{\mathsf{sep}}^\times)$ and $i \in \{1,2,3\}$, we write $\langle a, i\rangle$ for the image of i by the map a.

Suppose that Γ acts continuously on $\{1,2,3\}$. Then it induces a continuous action of Γ on $\mathsf{Map}(\{1,2,3\}, F_{\mathsf{sep}}^\times)$: for $\sigma \in \Gamma$, $i \in \{1,2,3\}$ and $a \in \mathsf{Map}(\{1,2,3\}, F_{\mathsf{sep}}^\times)$,

$$\langle {}^\sigma a, i\rangle = \sigma\left(\langle a, \sigma^{-1} \star i\rangle\right);$$

this endows $\mathsf{Map}(\{1,2,3\}, F_{\mathsf{sep}}^\times)$ with a Γ-group structure.

Lemma 3.2.2 *Suppose that* Γ *acts continuously on* $\{1,2,3\}$*. Then the first cohomology group* $\mathsf{H}^1(\Gamma, \mathsf{Map}(\{1,2,3\}, F_{\mathsf{sep}}^{\times}))$ *is trivial.*

Proof : Assume that Γ acts transitively on $\{1,2,3\}$. Let Γ_0 denote the stabilizer of 1 under the action of Γ on $\{1,2,3\}$. Then we have a bijection

$$\Gamma/\Gamma_0 \to \{1,2,3\} \colon \gamma\Gamma_0 \mapsto \gamma \star 1.$$

We put $A_0 := F_{\mathsf{sep}}^{\times}$, then A_0 is a Γ-group. Then Γ_0 is an open-closed subgroup of Γ (for the Krull topology). Let A be the group of continuous maps $a \colon \Gamma \to A_0$ such that

$$a(\gamma_0\gamma) = \gamma_0\,(a(\gamma))$$

for all $\gamma_0 \in \Gamma_0$ and $\gamma \in \Gamma$. We define an action of Γ on A as follows:

$${}^{\sigma}a(\gamma) = a(\gamma\sigma)$$

for all $\sigma, \gamma \in \Gamma$. Then A equipped with this action is a Γ-group. Remark (28.19) in [Knus *et al.*, 1998] says that we may identify A with the Γ-group of continuous maps from Γ/Γ_0 to A_0. Thus we may identify the Γ-groups A and $\mathsf{Map}\,(\{1,2,3\}, A_0)$. By Corollary (28.18) in *op. cit.*,

$$\mathsf{H}^1(\Gamma_0, A_0) = \mathsf{H}^1(\Gamma, A).$$

But Hilbert's Theorem 90 says that $\mathsf{H}^1(\Gamma_0, A_0) = 1$. Thus

$$\mathsf{H}^1(\Gamma, \mathsf{Map}(\{1,2,3\}, F_{\mathsf{sep}}^{\times})) = 1,$$

as wanted.

Now assume that the action of Γ on $\{1,2,3\}$ is not transitive. Suppose that X_1 and X_2 are disjoint non-empty subsets of $\{1,2,3\}$ such that $\{1,2,3\} = X_1 \cup X_2$ and the X_i's are stable under the action of Γ. For $i = 1,2$, we put $A_i := \mathsf{Map}(X_i, F_{\mathsf{sep}}^{\times})$, then the action of Γ restricts to A_i. We have a split exact sequence of abelian Γ-groups:

$$\begin{tikzcd} A_1 \arrow[r, "f"'] & A_1 \times A_2 \arrow[l, bend right, "r"'] \arrow[r, "g"'] & A_2 \arrow[l, bend right, "s"'] \end{tikzcd}$$

with $f(a_1) = (a_1, 1)$, $g(a_1, a_2) = a_2$, $r(a_1, a_2) = a_1$ and $s(a_2) = (1, a_2)$. It induces a split exact sequence of abelian groups:

$$\begin{tikzcd} 1 \arrow[r] & \mathsf{H}^1(\Gamma, A_1) \arrow[r] & \mathsf{H}^1(\Gamma, A_1 \times A_2) \arrow[r] & \mathsf{H}^1(\Gamma, A_2). \arrow[l, bend right] \end{tikzcd}$$

Hence, $\mathsf{H}^1(\Gamma, A_1 \times A_2) \cong \mathsf{H}^1(\Gamma, A_1) \times \mathsf{H}^1(\Gamma, A_2)$. Let $Y_1, \ldots, Y_r$ be the orbits of $\{1,2,3\}$ under the action of Γ and put $B_i := \mathsf{Map}(Y_i, F_{\mathsf{sep}}^{\times})$ for all i. Then $\mathsf{H}^1(\Gamma, B_i) = 1$ because Γ acts transitively on Y_i. Using the preceding

$$\mathsf{H}^1(\Gamma, B_1 \times \ldots \times B_r) \cong \mathsf{H}^1(\Gamma, B_1) \times \ldots \times \mathsf{H}^1(\Gamma, B_r) \cong 1.$$

Since we have a Γ-group isomorphism

$$\mathsf{Map}(\{1,2,3\}, F_{\mathsf{sep}}^{\times}) \to B_1 \times \ldots \times B_r \colon f \mapsto (f|_{Y_1}, \ldots, f|_{Y_r}),$$

we obtain that $\mathsf{H}^1(\Gamma, \mathsf{Map}(\{1,2,3\}, F_{\mathsf{sep}}^{\times})) = 1$. □

The next lemma gives a relation between a cubic form whose cubic curve is a triangle and the norm form of a cubic étale F-algebra.

Lemma 3.2.3 *Let $f \in \mathsf{S}^3(V^\star)$ be such that the curve $\{f(\xi) = 0\}$ is a triangle. Then there exist a cubic étale F-algebra K, a unit $\lambda \in F^\times$ and an F-vector space isomorphism $\Theta\colon V \to K$ such that $f(\xi) = \lambda \mathsf{N}_{K/F}(\Theta(\xi))$.*

Proof : Let $\varphi_1, \varphi_2, \varphi_3 \in V_{\mathsf{sep}}^\star$ be linearly independent forms such that $f = \varphi_1\varphi_2\varphi_3$. Because $f \in \mathsf{S}^3(V^\star)$, we have ${}^\sigma\varphi_1{}^\sigma\varphi_2{}^\sigma\varphi_3 = \varphi_1\varphi_2\varphi_3$ for all $\sigma \in \Gamma$. By uniqueness of factorization in $\mathsf{S}(V^\star)$, there exist a permutation π_σ of $\{1,2,3\}$ and scalars $\lambda_{\pi_\sigma(i),\sigma} \in F_{\mathsf{sep}}^{\times}$ such that

$${}^\sigma\varphi_i = \lambda_{\pi_\sigma(i),\sigma}\ \varphi_{\pi_\sigma(i)},$$

for all $i \in \{1,2,3\}$. Since ${}^{\sigma\tau}\varphi_i = {}^\sigma({}^\tau\varphi_i)$, we have

$$\lambda_{\pi_{\sigma\tau}(i),\sigma\tau}\ \varphi_{\pi_{\sigma\tau}(i)} = \lambda_{\pi_\sigma\pi_\tau(i),\sigma}\ \sigma(\lambda_{\pi_\tau(i),\tau})\ \varphi_{\pi_\sigma\pi_\tau(i)}.$$

Thus, $\pi_{\sigma\tau} = \pi_\sigma\pi_\tau$ and

$$\lambda_{\pi_{\sigma\tau}(i),\sigma\tau} = \lambda_{\pi_{\sigma\tau}(i),\sigma}\ \sigma(\lambda_{\pi_\tau(i),\tau}). \tag{3.1}$$

We define an action of Γ on the F_{sep}-algebra $\mathsf{Map}(\{1,2,3\}, F_{\mathsf{sep}})$ as follows: for $\sigma \in \Gamma$, $a \in \mathsf{Map}\,(\{1,2,3\}, F_{\mathsf{sep}})$ and $i \in \{1,2,3\}$,

$$\langle {}^\sigma a, i\rangle = \sigma(\langle a, \pi_\sigma^{-1}(i)\rangle).$$

The group Γ acts continuously by semilinear algebra automorphisms. By Galois descent, the F-algebra $K := \mathsf{Map}\,(\{1,2,3\}, F_{\mathsf{sep}})^\Gamma$ is such that the F_{sep}-linear map

$$K \otimes_F F_{\mathsf{sep}} \to \mathsf{Map}(\{1,2,3\}, F_{\mathsf{sep}})$$

mapping $a \otimes \lambda$ to $a\lambda$ is an F_{sep}-algebra isomorphism. Since the map

$$\mathsf{Map}(\{1,2,3\}, F_{\mathsf{sep}}) \to F_{\mathsf{sep}} \times F_{\mathsf{sep}} \times F_{\mathsf{sep}} \colon a \mapsto (\langle a,1\rangle, \langle a,2\rangle, \langle a,3\rangle)$$

defines an F_{sep}-algebra isomorphism, the F-algebra K is cubic étale. For $\sigma \in \Gamma$, we consider the map $a_\sigma \colon \{1,2,3\} \to F_{\mathsf{sep}}$ defined by

$$\langle a_\sigma, i\rangle = \lambda_{i,\sigma}.$$

Then $a_\sigma \in \mathsf{Map}\left(\{1,2,3\}, F_{\mathsf{sep}}^\times\right)$ because $\lambda_{i,\sigma} \neq 0$ for all i. In fact $(a_\sigma)_{\sigma\in\Gamma}$ is a 1-cocycle with values in $\mathsf{Map}(\{1,2,3\}, F_{\mathsf{sep}}^\times)$. Indeed,

$$\begin{aligned}\langle a_\sigma {}^\sigma a_\tau, i\rangle &= \langle a_\sigma, i\rangle\langle {}^\sigma a_\tau, i\rangle \\ &= \lambda_{i,\sigma}\ \sigma(\langle a_\tau, \pi_\sigma^{-1}(i)\rangle) \\ &= \lambda_{i,\sigma}\ \sigma(\lambda_{\pi_\sigma^{-1}(i),\tau}).\end{aligned}$$

for all $i \in \{1,2,3\}$. But relation (3.1) implies $\lambda_{i,\sigma\tau} = \lambda_{i,\sigma}\ \sigma(\lambda_{\pi_\sigma^{-1}(i),\tau})$, thus $a_\sigma {}^\sigma a_\tau = a_{\sigma\tau}$. We write u for the map $\{1,2,3\} \to F_{\mathsf{sep}}$ defined by $\langle u, i\rangle = 1$ for all i. Let E be a finite field extension of F such that $\varphi_i \in V_E^\star$ for all i. The continuity of the map

$$\Gamma \to \mathsf{Map}(\{1,2,3\}, F_{\mathsf{sep}}^\times) \colon \sigma \mapsto a_\sigma$$

follows from the fact that $\{\sigma \in \Gamma \mid a_\sigma = u\}$ contains the Galois group $\mathsf{Gal}(F_{\mathsf{sep}}/E)$ of F_{sep} over E. Thus, by Lemma 3.2.2, there exists a map $b \in \mathsf{Map}(\{1,2,3\}, F_{\mathsf{sep}}^\times)$ such that $a_\sigma = b\ {}^\sigma b^{-1}$ for all $\sigma \in \Gamma$. We put $\psi_i := \langle b, i\rangle\varphi_i$. Then

$$\begin{aligned}{}^\sigma\psi_i &= \sigma(\langle b,i\rangle)^\sigma\varphi_i \\ &= \sigma(\langle b,i\rangle)\lambda_{\pi_\sigma(i),\sigma}\ \varphi_{\pi_\sigma(i)} \\ &= \sigma(\langle b,i\rangle)\langle a_\sigma, \pi_\sigma(i)\rangle\ \varphi_{\pi_\sigma(i)} \\ &= \sigma(\langle b,i\rangle)\langle b, \pi_\sigma(i)\rangle\ \sigma(\langle b,i\rangle)^{-1}\varphi_{\pi_\sigma(i)} \\ &= \psi_{\pi_\sigma(i)}.\end{aligned}$$

Put $\lambda = \langle b,1\rangle^{-1}\langle b,2\rangle^{-1}\langle b,3\rangle^{-1}$, then $f = \lambda\psi_1\psi_2\psi_3$. Since f and $\psi_1\psi_2\psi_3$ are invariant under the action of Γ, we have $\lambda \in F^\times$. Let

$$\Theta \colon V_{\mathsf{sep}} \to \mathsf{Map}(\{1,2,3\}, F_{\mathsf{sep}})$$

be defined by $\langle\Theta(\xi), i\rangle = \psi_i(\xi)$ for all $\xi \in V_{\mathsf{sep}}$ and $i \in \{1,2,3\}$. Since the cubic curve $\{f(\xi) = 0\}$ is a triangle, the linear forms ψ_i are linearly independent and we deduce that Θ is an F_{sep}-vector space isomorphism. The map Θ is also compatible with the actions of Γ:

$$\begin{aligned}\langle {}^{\sigma}\Theta(\xi), i\rangle &= \sigma(\langle \Theta(\xi), \pi_{\sigma}^{-1}(i)\rangle) \\ &= \sigma(\psi_{\pi_{\sigma}^{-1}(i)}(\xi)) \\ &= {}^{\sigma}\psi_{\pi_{\sigma}^{-1}(i)}(\sigma(\xi)) \\ &= \psi_i(\sigma(\xi)) \\ &= \langle \Theta(\sigma(\xi)), i\rangle.\end{aligned}$$

Thus $\Theta|_V$ is an F-vector space isomorphism between V and K. Now, for $\xi \in V$, we have

$$\begin{aligned}\mathsf{N}_{K/F}(\Theta(\xi)) &= \mathsf{N}_{K\otimes_F F_{\mathsf{sep}}}(\Theta(\xi)\otimes 1) \\ &= \mathsf{N}_{F^3_{\mathsf{sep}}/F_{\mathsf{sep}}}(\langle\Theta(\xi),1\rangle, \langle\Theta(\xi),2\rangle, \langle\Theta(\xi),3\rangle) \\ &= \langle\Theta(\xi),1\rangle\langle\Theta(\xi),2\rangle\langle\Theta(\xi),3\rangle \\ &= \psi_1(\xi)\psi_2(\xi)\psi_3(\xi).\end{aligned}$$

Thus, $f(\xi) = \lambda\mathsf{N}_{K/F}(\Theta(\xi))$. □

Now we can give a criterion for a non-singular cubic form to be a semi-trace form.

Theorem 3.2.4 *Let $f \in \mathsf{S}^3(V^\star)$ be non-singular. Then f is a semi-trace form if and only if there exists an inflexional triangle of f defined over F.*

Proof: Suppose $g = \varphi_1\varphi_2\varphi_3 \in \mathsf{S}^3(V^\star)$ is such that the associated cubic curve $\{g(\xi) = 0\}$ is an inflexional triangle of f. Let K, Θ and π_σ, for $\sigma \in \Gamma$, be as in the proof of Lemma 3.2.3. Then the linear forms $\psi_1, \psi_2, \psi_3 \in \mathsf{S}^3(V^\star_{\mathsf{sep}})$ with $\langle\Theta(\xi), i\rangle = \psi_i(\xi)$ are such that ${}^{\sigma}\psi_i = \psi_{\pi_\sigma(i)}$ and $g(\xi) = \mu\mathsf{N}_{K/F}(\Theta(\xi)) = \mu\psi_1(\xi)\psi_2(\xi)\psi_3(\xi)$ for some $\mu \in F^\times$. By Theorem 3.1.3 and since the cubic curve associated to $\psi_1\psi_2\psi_3$ is also an inflexional triangle of f, there exist scalars $\alpha_1, \alpha_2, \alpha_3, \lambda \in F_{\mathsf{sep}}$ such that

$$f = \alpha_1\psi_1^3 + \alpha_2\psi_2^3 + \alpha_3\psi_3^3 - 3\lambda\psi_1\psi_2\psi_3.$$

Because ${}^{\sigma}f = f$ and ${}^{\sigma}\psi_i = \psi_{\pi_\sigma(i)}$, we have $\sigma(\alpha_i) = \alpha_{\pi_\sigma(i)}$ and $\sigma(\lambda) = \lambda$ for all $\sigma \in \Gamma$; thus in particular $\lambda \in F$. Let $\alpha\colon \{1,2,3\} \to F_{\mathsf{sep}}$ be the map defined by $\langle \alpha, i\rangle = \alpha_i$, then $\alpha \in K$. Indeed,

$$\begin{aligned} \langle {}^{\sigma}\alpha, i\rangle &= \sigma\left(\langle \alpha, \pi_\sigma^{-1}(i)\rangle\right) \\ &= \sigma(\alpha_{\pi_\sigma^{-1}(i)}) \\ &= \alpha_i \\ &= \langle \alpha, i\rangle. \end{aligned}$$

Therefore, for $\xi \in V$,

$$\begin{aligned} \mathsf{Tr}_{K/F}\left(\alpha\Theta(\xi)^3\right) &= \mathsf{Tr}_{F_{\mathsf{sep}}^3/F_{\mathsf{sep}}}\left(\alpha_1\psi_1(\xi)^3, \alpha_2\psi_2(\xi)^3, \alpha_3\psi_3(\xi)^3\right) \\ &= \alpha_1\psi_1(\xi)^3 + \alpha_2\psi_2(\xi)^3 + \alpha_3\psi_3(\xi)^3, \end{aligned}$$

from which

$$f(\xi) = \mathsf{Tr}_{K/F}\left(\alpha\Theta(\xi)^3\right) - 3\lambda\mathsf{N}_{K/F}\left(\Theta(\xi)\right)$$

and f is a semi-trace form.

Conversely, assume that f is a semi-trace form: let K be a cubic étale F-algebra, $\alpha \in K$, $\lambda \in F$ and $\Theta\colon V \to K$ an F-vector space isomorphism such that

$$f(\xi) = \mathsf{Tr}_{K/F}\left(\alpha\Theta(\xi)^3\right) - 3\lambda\mathsf{N}_{K/F}\left(\Theta(\xi)\right).$$

By Theorem (18.4) in [Knus *et al*, 1998], there exist an action of Γ on $\{1,2,3\}$ and an F-algebra isomorphism $\Phi\colon K \to \mathsf{Map}\left(\{1,2,3\}, F_{\mathsf{sep}}\right)^\Gamma$, where the action on $\mathsf{Map}\left(\{1,2,3\}, F_{\mathsf{sep}}\right)$ is induced by the one on $\{1,2,3\}$:

$$\langle {}^{\sigma}a, i\rangle = \sigma\left(\langle a, \sigma^{-1}\star i\rangle\right).$$

Replacing K by $\mathsf{Map}\left(\{1,2,3\}, F_{\mathsf{sep}}\right)^\Gamma$ and Θ by $\Phi\circ\Theta$, we may assume that $K = \mathsf{Map}\left(\{1,2,3\}, F_{\mathsf{sep}}\right)^\Gamma$. Let $\varphi_i \in V_{\mathsf{sep}}^\star$ be defined by $\varphi_i(\xi) = \langle\Theta(\xi), i\rangle$ for $\xi \in V$ and put $\alpha_i := \langle\Theta(\alpha), i\rangle$ for $i = 1,2,3$, then

$$f = \alpha_1\varphi_1^3 + \alpha_2\varphi_2^3 + \alpha_3\varphi_3^3 - 3\lambda\varphi_1\varphi_2\varphi_3.$$

By Theorem 3.1.3, the cubic curve associated to $g = \varphi_1\varphi_2\varphi_3$ is an inflexional triangle of f which is a priori defined over F_{sep}. Since ${}^{\sigma}\Theta(\xi) = \Theta(\xi)$ for all $\xi \in V$ and $\sigma \in \Gamma$, we have

$${}^{\sigma}\varphi_i(\xi) = \sigma\big(\varphi_i(\xi)\big) = \sigma\big(\langle\Theta(\xi), i\rangle\big) = \langle\Theta(\xi), \sigma\star i\rangle = \varphi_{\sigma\star i}(\xi)$$

for all $\xi \in V$ and $i \in \{1,2,3\}$. Thus ${}^{\sigma}\varphi_i = \varphi_{\sigma\star i}$ and ${}^{\sigma}g = g$. So $\{g(\xi) = 0\}$ is an inflexional triangle of f which is defined over F. □

4

Classification of non-singular cubic pairs

We introduce the notion of cubic pair over a field and we use Galois cohomology to classify the isomorphism classes of non-singular cubic pairs: we explicitly give a representative of each such isomorphism class. To each cubic pair is associated a ternary cubic form, and our classification of the non-singular cubic pairs allows us to describe explicitly those cubic forms. It turns out that all of these cubic forms are semi-trace forms, and they are even semi-diagonal if the ground field contains a primitive cube root of unity.

4.1 Cubic pairs

For A a central simple F-algebra, we let $\mathsf{Trd}_A\colon A \to F$ denote the reduced trace of A and A° the subspace of A of reduced trace zero elements. The *trace quadratic form* of A is the quadratic form

$$q_A\colon A \to F\colon \xi \mapsto \mathsf{Trd}_A(\xi^2).$$

Let A be a central simple F-algebra of degree 3. There exists a field extension L of degree 3 over F such that $A \otimes_F L \cong \mathsf{M}_3(L)$. Let $\Theta\colon A \otimes_F L \to \mathsf{M}_3(L)$ be an L-algebra isomorphism, then

$$\mathsf{Trd}_A(\xi) = \mathsf{Tr}\big(\Theta(\xi)\big)$$

for all $\xi \in A$. For a matrix $m \in \mathsf{M}_3(L)$, let m_{ij} denote the element at the i-th row and the j-th column in m. For all $m \in \mathsf{M}_3(L)^\circ$, we have

$$\mathsf{tr}(m^2) = 2(m_{12}m_{21} + m_{13}m_{31} + m_{23}m_{32} + m_{11}^2 + m_{22}^2 + m_{11}m_{22}).$$

Thus the restriction of q_{A_L} to $(A_L)^\circ$ is isometric to the diagonal form

$$\langle 1, -1, 1, -1, 1, -1, 1, 3\rangle.$$

Since the degree of the field extension L over F is odd, by Springer's Theorem there exist 3-dimensional subspaces of A° which are totally isotropic for the quadratic form q_A.

Definition 4.1.1 *A pair (A, V) where A is a central simple algebra of degree 3 over F and V is a 3-dimensional subspace of A° which is totally isotropic for the trace quadratic form, is called a* cubic pair *over F.*

An *F-isomorphism* between two cubic pairs (A, V) and (A', V') is an isomorphism $\Theta\colon A \to A'$ of F-algebras such that $\Theta(V) = V'$.

To a cubic pair (A, V) we can naturally associate a ternary cubic form $f_{A,V}$ over V: define

$$f_{A,V}\colon V \to F\colon \xi \mapsto \xi^3.$$

We say that (A, V) is a *non-singular cubic pair* if $f_{A,V}$ is non-singular and (A, V) is a *singular cubic pair* otherwise.

We observe that the cubic forms $f_{A,V}$ and $f_{A',V'}$ are equivalent if the F-cubic pairs (A, V) and (A', V') are isomorphic (but the converse need not hold). Indeed let $\Theta\colon A \to A'$ be an F-algebra isomorphism such that $\Theta(V) = V'$. Then, for all $\xi \in V$,

$$f_{A,V}(\xi) = \xi^3 = \Theta(\xi^3) = \Theta(\xi)^3 = f_{A',V'}(\Theta(\xi)).$$

Furthermore, it is clear that, if $f \in \mathsf{S}^3(V^\star)$ and $f' \in \mathsf{S}^3(V'^\star)$ are equivalent, then f is singular if and only if f' is singular. Thus we can split the classification of cubic pairs over F, up to isomorphism, into two parts: the singular cubic pairs and the non-singular ones.

To classify cubic pairs, we shall use a method based on Galois cohomology justified by the following theorem. In this theorem, we use the following notation: $\mathsf{Aut}(A, V)$ is the group scheme of automorphisms of a cubic pair (A, V). Explicitly, $\mathsf{Aut}(A, V)$ is the functor from the category of F-algebras to the category of groups that sends an F-algebra R on the group of R-automorphisms of $(A, V)_R := (A_R, V_R)$.

Theorem 4.1.2 *Let (A, V) be an F-cubic pair. We have a bijection*

$$\mathsf{H}^1(F, \mathsf{Aut}(A, V)) \longleftrightarrow \left\{ \begin{array}{c} F\text{-isomorphism classes of} \\ \text{the } F\text{-cubic pairs which are} \\ \text{isomorphic to } (A, V)_{F_{\mathsf{sep}}} \text{ over } F_{\mathsf{sep}} \end{array} \right\}.$$

Proof : Let $G \subset \mathsf{GL}(A)$ be the subscheme of automorphisms of the flag of vector spaces $A \supset V$ over F. Let W be the F-vector space of the F-homomorphisms of vector spaces between $A \otimes_F A$ and A. We define $\rho\colon G \to \mathsf{GL}(W)$ by

$$\rho_R(g)(\varphi)(\xi \otimes \eta) = g \circ \varphi\big(g^{-1}(\xi) \otimes g^{-1}(\eta)\big),$$

for $g \in G(R)$, $\varphi \in W_R$ and $\xi, \eta \in A_R$. We denote by $m\colon A \otimes_F A \to A$ the multiplication in the algebra A, then $m \in W$. As in [Knus, *et al.*, 1998], page 392, let $\mathsf{Aut}_G(m)$ denote the stabilizer of m. It is a subgroup of the group scheme G and, for every F-algebra R,

$$\mathsf{Aut}_G(m)(R) = \{R\text{-algebra automorphisms of } (A,V)_R\},$$

so $\mathsf{Aut}_G(m) = \mathsf{Aut}(A,V)$. Let $\mathsf{A}(\rho,m)$ be the category whose objects are the $\varphi \in W$ such that $\varphi = \rho_{F_{\mathsf{sep}}}(g)(m)$ for some $g \in G(F_{\mathsf{sep}})$, and whose morphisms $\varphi \to \psi$ are the elements $g \in G(F)$ such that $\rho(g)(\varphi) = \psi$. By Corollary (29.5) in *op. cit.*, $\mathsf{H}^1(F,G) = 1$. So by Proposition (29.1) in *op. cit.*, there is a bijection

$$\mathsf{Isom}\big(\mathsf{A}(\rho,m)\big) \longleftrightarrow \mathsf{H}^1(F, \mathsf{Aut}_G(m))$$

where $\mathsf{Isom}\big(\mathsf{A}(\rho,m)\big)$ denotes the set of isomorphism classes of objects of the category $\mathsf{A}(\rho,m)$. To finish the proof, we show that

$$\mathsf{Isom}(\mathsf{A}(\rho,m)) \longleftrightarrow \left\{ \begin{array}{c} F\text{-isomorphism classes of} \\ \text{the } F\text{-cubic pairs which are} \\ \text{isomorphic to } (A,V)_{F_{\mathsf{sep}}} \text{ over } F_{\mathsf{sep}} \end{array} \right\}.$$

Let $\varphi \in \mathsf{A}(\rho,m)$. We define A' to be the F-algebra such that A' is equal to A as an F-vector space and the multiplication in A' is given by φ. Let $g \in G(F_{\mathsf{sep}})$ be such that $\varphi = \rho_{F_{\mathsf{sep}}}(g)(m)$, then g is an F_{sep}-isomorphism between $(A,V)_{F_{\mathsf{sep}}}$ and $(A',V')_{F_{\mathsf{sep}}}$, where $V' = V$. Hence we obtain an F-cubic pair (A',V') such that $(A,V)_{F_{\mathsf{sep}}}$ and $(A',V')_{F_{\mathsf{sep}}}$ are isomorphic. If $\psi \in \mathsf{A}(\rho,m)$ is isomorphic to φ, then there exists $h \in G(F)$ such that $\psi = \rho(h)(\varphi)$. If (A'',V'') is the F-cubic pair associated to ψ in the same way, then h is an F-isomorphism between (A',V') and (A'',V''). So the mapping $[\varphi] \mapsto [(A,V)]$ is well-defined, where $[\varphi]$ and $[(A,V)]$ are the isomorphism classes of φ and (A,V) respectively. We prove that this mapping is bijective. Let (A',V') be an F-cubic pair and $\Theta\colon (A,V)_{F_{\mathsf{sep}}} \to (A',V')_{F_{\mathsf{sep}}}$ an isomorphism of F_{sep}-cubic pairs. Let m' denote the multiplication in A'. There exists an isomorphism of F-vector spaces $\Phi\colon A \to A'$ such that $\Phi(V) = V'$. We define $\varphi\colon A \otimes_F A \to A$ by

$$\varphi(\xi \otimes \eta) = \Phi^{-1} \circ m'\big(\Phi(\xi) \otimes \Phi(\eta)\big),$$

for $\xi, \eta \in A$. Then $\varphi = \rho_{F_{\mathsf{sep}}}(g)(m)$, where $g = \Phi^{-1} \circ \Theta \in G(F_{\mathsf{sep}})$. If $\Psi \colon A \to A'$ is another F-vector space isomorphism such that $\Psi(V) = V'$, let $\psi \in \mathsf{A}(\rho, m)$ be defined by

$$\psi(\xi, \eta) = \Psi^{-1} \circ m'\big(\Psi(\xi) \otimes \Psi(\eta)\big).$$

Then $\Psi^{-1} \circ \Phi \in G(F)$ and $\rho(\Psi^{-1} \circ \Phi)(\varphi) = \psi$. Hence $[\varphi]$ does not depend on the choice of Φ. Now suppose that $\Psi \colon (A', V') \to (A'', V'')$ is an F-cubic pair isomorphism. Then $\Psi \circ \Theta$ is an F_{sep}-cubic pair isomorphism between $(A, V)_{F_{\mathsf{sep}}}$ and $(A'', V'')_{F_{\mathsf{sep}}}$ and $\Psi \circ \Phi$ is an F-vector space isomorphism from A to A'' such that $\Psi \circ \Phi(V) = V''$. Let m'' be the multiplication in A''. Then the element of $\mathsf{Isom}(A(\rho, m))$ associated to the pair (A'', V'') in the same way is $[\psi]$ where

$$\psi(\xi \otimes \eta) = \Phi^{-1} \circ \Psi^{-1} \circ m''\big(\Psi \circ \Phi(\xi) \otimes \Psi \circ \Phi(\eta)\big) = \varphi(\xi \otimes \eta).$$

Therefore $[\varphi]$ does not depend on the representative of $[(A', V')]$. We obtain a mapping $[(A', V')] \mapsto [\varphi]$ which is the inverse of the former. □

The bijection in Theorem 4.1.2 goes as follows. Let (A', V') be a cubic pair over F such that $\Theta \colon (A', V')_{F_{\mathsf{sep}}} \to (A, V)_{F_{\mathsf{sep}}}$ is an F_{sep}-isomorphism. Then the corresponding 1-cocycle is $(a_\sigma F_{\mathsf{sep}}^\times)_{\sigma \in \Gamma}$ with

$$\mathsf{int}(a_\sigma) = \Theta \circ (\mathsf{id}_{A'} \otimes \sigma) \circ \Theta^{-1} \circ (\mathsf{id}_A \otimes \sigma^{-1}),$$

where $\mathsf{int}(a_\sigma)$ is the inner automorphism $\xi \mapsto a_\sigma \xi a_\sigma^{-1}$ of $A_{F_{\mathsf{sep}}}$. Conversely, for $(a_\sigma F_{\mathsf{sep}}^\times)_{\sigma \in \Gamma} \in \mathsf{Z}^1(F, \mathsf{Aut}(A, V))$, we let

$$A' = \{\xi \in A_{F_{\mathsf{sep}}} \mid a_\sigma \sigma(\xi) a_\sigma^{-1} = \xi \ \text{ for all } \sigma \in \Gamma\}$$

and

$$V' = \{\xi \in V_{\mathsf{sep}} \mid a_\sigma \sigma(\xi) a_\sigma^{-1} = \xi \ \text{ for all } \sigma \in \Gamma\}.$$

Then (A', V') is the F-cubic pair corresponding to $(a_\sigma)_{\sigma \in \Gamma}$.

The previous theorem gives us a method to classify the cubic pairs over F: first classify the cubic pairs over F_{sep}; then compute the automorphism group of a representative of any F_{sep}-isomorphism class of cubic pairs; and finally for a representative of any F_{sep}-isomorphism class of cubic pairs which is defined over F, give all the F-isomorphism classes of F-cubic pairs which are F_{sep}-isomorphic to the former.

The rest of this chapter is devoted to the classification of the non-singular cubic pairs; the singular ones are classified in Chapter 5.

4.2 Non-singular cubic pairs over F_{sep}

Let (A, V) be a non-singular cubic pair over F_{sep}. We want to describe (A, V) up to F_{sep}-isomorphism. Since A is a degree 3 central simple algebra over F_{sep}, we may assume that $A = \mathsf{M}_3(F_{\mathsf{sep}})$ and V is a 3-dimensional subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$ of trace zero matrices which is totally isotropic for the trace quadratic form

$$q_A\colon \mathsf{M}_3(F_{\mathsf{sep}}) \to F_{\mathsf{sep}}\colon \xi \mapsto \mathsf{tr}(\xi^2).$$

By the Skolem-Noether Theorem, an isomorphism between the cubic pairs $(\mathsf{M}_3(F_{\mathsf{sep}}), V)$ and $(\mathsf{M}_3(F_{\mathsf{sep}}), V')$ over F_{sep} is an inner automorphism

$$\mathsf{int}(m)\colon \mathsf{M}_3(F_{\mathsf{sep}}) \to \mathsf{M}_3(F_{\mathsf{sep}})\colon \xi \mapsto m\xi m^{-1}$$

such that $mVm^{-1} = V'$ for some $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$. Thus, to classify the isomorphism classes of non-singular cubic pairs over F_{sep}, we may classify, up to conjugacy, the 3-dimensional subspaces of $\mathsf{M}_3(F_{\mathsf{sep}})^\circ$ which are totally isotropic for the trace quadratic form $q_{\mathsf{M}_3(F_{\mathsf{sep}})}$ and such that f_V is non-singular.

For brevity, if (A, V) is a cubic pair over F with $A = \mathsf{M}_3(F)$, we write f_V instead of $f_{A,V}$; we call V a *cubic subspace* of $\mathsf{M}_3(F)$; we say that V is *singular* if f_V is singular and V is *non-singular* otherwise. Note that the symmetric trilinear form associated with f_V is the map

$$t_V\colon V \times V \times V\colon (\xi, \eta, \zeta) \mapsto \frac{1}{6}\mathsf{tr}(\xi\eta\zeta + \xi\zeta\eta).$$

Lemma 4.2.1 *Let V be a cubic subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$ and $u \in V$ non-zero such that $u^2 = 0$. Then $u\overline{F}$ is a singular point of the cubic curve $\{f_V(\xi) = 0\}$.*

Proof: Since $u^2 = 0$, we have $t_V(u, u, \xi) = \frac{1}{3}\mathsf{tr}(u^2\xi) = 0$ for all $\xi \in \overline{V}$. Therefore $u\overline{F}$ is a singular point of $\{f_V(\xi) = 0\}$. □

In other words, there is no non-zero $u \in V$ such that $u^2 = 0$ in a non-singular cubic subspace V of $\mathsf{M}_3(F_{\mathsf{sep}})$. The next proposition gives equivalent conditions for a matrix of a cubic subspace with a non-zero square, to be of determinant zero.

Proposition 4.2.2 *Let V be a cubic subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$. Then, for all $u \in \overline{V}$ such that $u^2 \neq 0$, the following statements are equivalent:*

1. $f_V(u) = 0$,

2. $\mathsf{det}(u) = 0$,

3. the rank of u *is equal to 2,*

4. $\mathsf{im}(u) = \mathsf{ker}(u^2)$.

Proof : The first two statements are equivalent since $\xi^3 = \mathsf{det}(\xi)$ for all $\xi \in V$. By hypothesis $u^2 \neq 0$, so $u^3 = 0$ implies that the Jordan normal form of u is

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Therefore $u^3 = 0$ implies (3) and (4). It is easy to see that (3) and (4) both imply $\mathsf{det}(u) = 0$. □

For a non-singular cubic subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$, we shall give explicit vectors which span the given subspace. To do this we need preliminary results.

Lemma 4.2.3 *Let* V *be a cubic subspace of* $\mathsf{M}_3(F_{\mathsf{sep}})$ *and* $u_1, u_2 \in V$ *determinant zero matrices. If* $u_2^2 \neq 0$, $\mathsf{tr}(u_1 u_2^2) = 0$ *and* $\mathsf{tr}(u_1^2 u_2) \neq 0$, *then* $\mathsf{ker}(u_2) \not\subset \mathsf{ker}(u_1^2)$.

Proof : Since $\mathsf{tr}(u_1^2 u_2) \neq 0$ we have $u_1^2 \neq 0$. Replacing u_1 and u_2 by conjugates in $\mathsf{M}_3(F_{\mathsf{sep}})$ if necessary, we may assume that

$$u_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

If $\mathsf{ker}(u_2) = \mathsf{ker}(u_1)$ then

$$u_2 = \begin{pmatrix} 0 & x_{12} & x_{13} \\ 0 & x_{22} & x_{23} \\ 0 & x_{32} & x_{33} \end{pmatrix}$$

for some $x_{ij} \in F_{\mathsf{sep}}$ and so we have $\mathsf{tr}(u_1^2 u_2) = 0$ which contradicts the hypothesis; hence $\mathsf{ker}(u_2) \neq \mathsf{ker}(u_1)$. Suppose that $\mathsf{ker}(u_2) \subset \mathsf{ker}(u_1^2)$. Let $a \in F_{\mathsf{sep}}^3$ be such that

$$\mathsf{ker}(u_2) = u_1 a \cdot F_{\mathsf{sep}}$$

(there exists such a vector because $\mathsf{ker}(u_1^2) = \mathsf{im}(u_1)$ by Proposition 4.2.2). Since $\mathsf{ker}(u_2) \neq \mathsf{ker}(u_1)$ we have $u_1^2 a \neq 0$, so $a \notin \mathsf{ker}(u_1^2) = \mathsf{im}(u_1)$

and $u_1 a \notin \mathsf{ker}(u_1)$. Thus we have $a \notin \mathsf{im}(u_1)$, $u_1 a \in \mathsf{im}(u_1) \setminus \mathsf{ker}(u_1)$ and $u_1^2 a \in \mathsf{ker}(u_1) \setminus \{0\}$, which means that $a, u_1 a, u_1^2 a$ are linearly independent in F_{sep}^3. Let m be the matrix with columns $u_1^2 a, u_1 a$ and a, then

$$m^{-1} u_1 m = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad m^{-1} u_2 m = \begin{pmatrix} x_{11} & 0 & x_{13} \\ x_{21} & 0 & x_{23} \\ x_{31} & 0 & x_{33} \end{pmatrix}$$

for some $x_{ij} \in F_{\mathsf{sep}}$, since $\mathsf{ker}(u_2) = u_1 a \cdot F_{\mathsf{sep}}$. Because $\mathsf{tr}(u_2) = 0$ and $\mathsf{tr}(u_1 u_2) = 0$, we have $x_{11} + x_{33} = x_{21} = 0$. So

$$m^{-1} u_2^2 m = \begin{pmatrix} x_{11}^2 + x_{13}x_{31} & 0 & 0 \\ x_{23}x_{31} & 0 & -x_{23}x_{11} \\ 0 & 0 & x_{11}^2 + x_{13}x_{31} \end{pmatrix}$$

and $\mathsf{tr}(u_1 u_2^2) = \mathsf{tr}(u_2^2) = 0$ implies

$$x_{23}x_{31} = x_{11}^2 + x_{13}x_{31} = 0.$$

On the other hand $\mathsf{tr}(u_1^2 u_2) = x_{31}$, so $x_{31} \neq 0$. Therefore we have $x_{23} = 0$ and $u_2^2 = 0$ which is impossible. Hence $\mathsf{ker}(u_2) \not\subset \mathsf{ker}(u_1^2)$. □

Lemma 4.2.4 *Let V be a cubic subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$ and $u_1, u_2 \in V$ determinant zero matrices. Suppose that $u_2^2 \neq 0$, $\mathsf{tr}(u_1 u_2^2) = 0$ and $\mathsf{tr}(u_1^2 u_2) \neq 0$, then there exist $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$ and $\lambda, \mu \in F_{\mathsf{sep}}^\times$ such that*

$$m u_1 m^{-1} = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad m u_2 m^{-1} = \mu \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

Proof: Let $a \in F_{\mathsf{sep}}^3$ be such that $\mathsf{ker}(u_2) = a \cdot F_{\mathsf{sep}}$. By Lemma 4.2.3, we have $a \notin \mathsf{ker}(u_1^2)$ and so $a, u_1 a, u_1^2 a$ are linearly independent. Let m_0 be the matrix with columns $u_1^2 a, u_1 a, a$, then

$$m_0^{-1} u_1 m_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad m_0^{-1} u_2 m_0 = \begin{pmatrix} x_{11} & x_{12} & 0 \\ x_{21} & x_{22} & 0 \\ x_{31} & x_{32} & 0 \end{pmatrix}$$

for some $x_{ij} \in F_{\mathsf{sep}}$. As $\mathsf{tr}(u_2) = \mathsf{tr}(u_1 u_2) = 0$, we have $x_{22} = -x_{11}$ and $x_{32} = -x_{21}$, thus

$$m_0^{-1} u_2^2 m_0 = \begin{pmatrix} x_{11}^2 + x_{12}x_{21} & 0 & 0 \\ 0 & x_{11}^2 + x_{12}x_{21} & 0 \\ x_{31}x_{11} - x_{21}^2 & x_{31}x_{12} + x_{11}x_{21} & 0 \end{pmatrix}.$$

From $\mathsf{tr}(u_1u_2^2) = \mathsf{tr}(u_2^2) = 0$ we deduce

$$x_{31}x_{12} + x_{11}x_{21} = 0 \tag{4.1}$$
$$x_{11}^2 + x_{12}x_{21} = 0 \tag{4.2}$$

whence

$$m_0^{-1}u_2^2m_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x_{31}x_{11} - x_{21}^2 & 0 & 0 \end{pmatrix}.$$

As $u_2^2 \neq 0$, we need $x_{31}x_{11} - x_{21}^2 \neq 0$. But

$$x_{11}(x_{21}^2 - x_{11}x_{31}) = (x_{31}x_{12} + x_{11}x_{21})x_{21} - (x_{11}^2 + x_{12}x_{21})x_{31} = 0$$

so $x_{11} = 0$ and $x_{21} \neq 0$. Using equation (4.2) we get $x_{12} = 0$ so

$$m_0^{-1}u_2m_0 = \begin{pmatrix} 0 & 0 & 0 \\ x_{21} & 0 & 0 \\ x_{31} & -x_{21} & 0 \end{pmatrix}$$

and $x_{31} \neq 0$ as $\mathsf{tr}(u_1^2u_2) \neq 0$. Now choosing

$$m = \begin{pmatrix} x_{21}^{-2}x_{31}^2 & 0 & 0 \\ 0 & x_{21}^{-1}x_{31} & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot m_0^{-1}$$

we get

$$mu_1m^{-1} = \frac{x_{31}}{x_{21}} \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad mu_2m^{-1} = \frac{x_{21}^2}{x_{31}} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

□

The matrix m in the previous lemma is unique in the following sense:

Lemma 4.2.5 *If $m_1, m_2 \in \mathsf{GL}_3(F_{\mathsf{sep}})$ and $\lambda_1, \lambda_2, \mu_1, \mu_2 \in F_{\mathsf{sep}}^\times$ are such that both m_1, λ_1, μ_1 and m_2, λ_2, μ_2 satisfy the conditions in Proposition 4.2.4, then $m_1F_{\mathsf{sep}}^\times = m_2F_{\mathsf{sep}}^\times$, $\lambda_1 = \lambda_2$ and $\mu_1 = \mu_2$.*

Proof : This follows easily from the relations

$$m_2^{-1}m_1 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = \lambda_1^{-1}\lambda_2 \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} m_2^{-1}m_1$$

and

$$m_2^{-1} m_1 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} = \mu_1^{-1} \mu_2 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} m_2^{-1} m_1.$$

□

Let V be a non-singular cubic subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$. By Theorem 1.3.8, we know that $\{f_V(\xi) = 0\}$ has exactly nine flexes which are defined over F_{sep}. Let $u \in V$ be such that $u\overline{F}$ is a flex of $\{f_V(\xi) = 0\}$. Then the conic $\{t_V(u, \xi, \xi) = 0\}$ and the tangent $\{t_V(u, u, \xi) = 0\}$ are defined over F_{sep}. Thus the harmonic polar at $u\overline{F}$, the Hessian point and the harmonic points of $u\overline{F}$ are also defined over F_{sep}.

To describe a non-singular cubic subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$ up to conjugacy, we shall use particular points related to the associated cubic curve. More precisely, let V be a non-singular cubic subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$, uF_{sep} a flex of $\{f_V(\xi) = 0\}$, vF_{sep} a harmonic point of uF_{sep} and wF_{sep} the Hessian point of uF_{sep}. Since vF_{sep} does not lie on the tangent to $\{f(\xi) = 0\}$ at uF_{sep} which is the line $\langle uF_{\mathsf{sep}}, wF_{\mathsf{sep}} \rangle$, the matrices u, v, w are linearly independent and V is spanned by u, v, w. Therefore, to describe V it is sufficient to describe u, v, w.

To state our next result, we introduce some notation that we shall use in the rest of this chapter. We write $\omega \in F_{\mathsf{sep}}$ for a primitive cube root of unity[1]. We also put

$$u := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad v := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix},$$

and for $\alpha \in F_{\mathsf{sep}}$,

$$w_1(\alpha) := \begin{pmatrix} \alpha & -\frac{1}{2} & 1 \\ 3\alpha^2 & -2\alpha & \frac{1}{2} \\ 0 & -3\alpha^2 & \alpha \end{pmatrix},$$

$$w_2(\alpha) := \begin{pmatrix} \alpha & \frac{1}{2}\left((\omega^2 - 1)\alpha - 1\right) & 1 \\ 0 & \omega\alpha & \frac{1}{2}\left((\omega^2 - 1)\alpha + 1\right) \\ 0 & 0 & \omega^2\alpha \end{pmatrix},$$

$$w_3(\alpha) := \begin{pmatrix} \alpha & \frac{1}{2}\left((\omega - 1)\alpha - 1\right) & 1 \\ 0 & \omega^2\alpha & \frac{1}{2}\left((\omega - 1)\alpha + 1\right) \\ 0 & 0 & \omega\alpha \end{pmatrix}.$$

[1] The element $\omega \in F_{\mathsf{sep}}$ shall denote a primitive cube root of unity in the rest of the chapters.

For $\xi_1, \dots, \xi_r \in \mathsf{M}_3(F)$, we write $\mathsf{span}_F\langle \xi_1, \dots, \xi_r \rangle$ for the F-vector subspace of $\mathsf{M}_3(F)$ spanned by $\xi_1, \dots, \xi_r$. We put

$$V_\alpha := \mathsf{span}_{F_{\mathsf{sep}}}\langle u, v, w_1(\alpha) \rangle$$

for $\alpha \in F_{\mathsf{sep}}$. We call a non-singular cubic subspace of $\mathsf{M}_3(F)$ which is spanned by $u, v, w_i(\alpha)$ for some $\alpha \in F$ and some $i = 1, 2, 3$, a *special subspace* of $\mathsf{M}_3(F)$.

Theorem 4.2.6 *Let V be a non-singular cubic subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$. Then V is conjugate to the $\mathsf{span}_{F_{\mathsf{sep}}}\langle u, v, w_i(\alpha) \rangle$ for some $\alpha \in F_{\mathsf{sep}}$ and some $i \in \{1, 2, 3\}$.*

Proof : Let $\tilde{u}, \tilde{v}, \tilde{w} \in V$ be such that $\tilde{u}\overline{F}$ is a flex of the cubic curve $\{f_V(\xi) = 0\}$, $\tilde{v}\overline{F}$ is a harmonic point of $\tilde{u}\overline{F}$ and $\tilde{w}\overline{F}$ is the Hessian point of $\tilde{u}\overline{F}$. Since $\tilde{u}\overline{F}$ is a flex of the cubic curve $\{f_V(\xi) = 0\}$ and $\tilde{v}\overline{F}$ is a harmonic point of $\tilde{u}\overline{F}$, the determinants of $\tilde{u}$ and $\tilde{v}$ are zero, $\mathsf{tr}(\tilde{u}\tilde{v}^2) = 0$ and $\mathsf{tr}(\tilde{u}^2\tilde{v}) \neq 0$. Moreover $\tilde{v}^2 \neq 0$ because V is non-singular. Thus, by Proposition 4.2.4, there exist a matrix $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$ and scalars $\lambda, \mu \in F_{\mathsf{sep}}^\times$ such that

$$m\tilde{u}m^{-1} = \lambda u \quad \text{and} \quad m\tilde{v}m^{-1} = \mu v.$$

Put $w := m\tilde{w}m^{-1}$. Let $w_{ij} \in F_{\mathsf{sep}}$ denote the element on row i and column j in w. Since $\mathsf{tr}(\tilde{w}) = 0$, $\mathsf{tr}(\tilde{u}\tilde{w}) = 0$ and $\mathsf{tr}(\tilde{v}\tilde{w}) = 0$, we have $\mathsf{tr}(w) = 0$, $\mathsf{tr}(uw) = 0$ and $\mathsf{tr}(vw) = 0$. Thus $w_{33} = -w_{11} - w_{22}$, $w_{32} = -w_{21}$ and $w_{23} = w_{12} + w_{13}$ and

$$w = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{12} + w_{13} \\ w_{31} & -w_{21} & -w_{11} - w_{22} \end{pmatrix}.$$

We have $w_{13} \neq 0$ for otherwise $\mathsf{tr}(\tilde{v}^2\xi) = 0$ for all $\xi \in \overline{V}$ and $\tilde{v}\overline{F}$ would be singular. Because $\mathsf{tr}(\tilde{w}^2) = 0$, we have

$$w_{31} = w_{21} - w_{13}^{-1}(w_{11}^2 + w_{22}^2 + w_{11}w_{22}).$$

Since $\tilde{w}\overline{F}$ is the Hessian point of $\tilde{u}\overline{F}$, we have $t_V(\tilde{u}, \tilde{w}, \xi) = 0$ for all $\xi \in \overline{F}$. But $t_V(\tilde{u}, \tilde{w}, \tilde{u}) = 0$ implies

$$w_{21} = w_{13}^{-1}(w_{11}^2 + w_{22}^2 + w_{11}w_{22});$$

next $t_V(\tilde{u}, \tilde{w}, \tilde{v}) = 0$ implies

$$w_{12} = -\frac{1}{2}(w_{13} + 2w_{11} + w_{22});$$

and finally $t_V(\tilde{u}, \tilde{w}, \tilde{w}) = 0$ implies

$$w_{22} = -2w_{11} \quad \text{or} \quad w_{11}^2 + w_{22}^2 + w_{11}w_{22} = 0.$$

If $w_{22} = -2w_{11}$, then $wF_{\mathsf{sep}} = w_1(\alpha)F_{\mathsf{sep}}$ with $\alpha = w_{11}w_{13}^{-1}$. On the other hand, if $w_{11}^2 + w_{22}^2 + w_{11}w_{22} = 0$, then $w_{22} = \omega w_{11}$ or $w_{22} = \omega^2 w_{11}$. Hence either $wF_{\mathsf{sep}} = w_2(\alpha)F_{\mathsf{sep}}$ or $wF_{\mathsf{sep}} = w_3(\alpha)F_{\mathsf{sep}}$ with $\alpha = w_{11}w_{13}^{-1}$. □

Therefore, up to conjugacy, the non-singular subspaces of $\mathsf{M}_3(F_{\mathsf{sep}})$ are special subspaces. We can prove more:

Theorem 4.2.7 *The pairs $(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha)$ for $\alpha \in F_{\mathsf{sep}} \setminus \{0, \frac{1}{8}, \frac{1}{9}\}$, are non-singular cubic pairs over F_{sep}, and any non-singular cubic pair over F_{sep} is isomorphic to $(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha)$ for some $\alpha \in F_{\mathsf{sep}} \setminus \{0, \frac{1}{8}, \frac{1}{9}\}$.*

Proof : It is easy to check that $(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha)$ is a cubic pair for all $\alpha \in F_{\mathsf{sep}}$, and is non-singular if and only if $\alpha \notin \{0, \frac{1}{8}, \frac{1}{9}\}$.

To prove that an arbitrary non-singular cubic pair over F_{sep} is isomorphic to $(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha)$ for some α, it is sufficient to prove that the non-singular cubic subspaces of $\mathsf{M}_3(F_{\mathsf{sep}})$ spanned by u, v and $w_i(\beta)$ are isomorphic to V_α for some $\alpha \in F_{\mathsf{sep}}$. Observe that $\mathsf{span}_{F_{\mathsf{sep}}}\langle u, v, w_2(\beta)\rangle$ is a cubic subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$ for all $\beta \in F_{\mathsf{sep}}$ and is non-singular if and only if $\beta \neq 0, \frac{-\omega^2}{3}, \frac{-\omega^2}{9}$. Let $\beta \in F_{\mathsf{sep}} \setminus \{0, \frac{-\omega^2}{3}, \frac{-\omega^2}{9}\}$, $\theta \in F_{\mathsf{sep}}$ a cube root of $9\omega\beta + 1$ and let m be the matrix

$$\begin{pmatrix} 1 & \frac{\omega\theta^2+\theta+\omega^2}{3\beta} & \frac{\theta-1}{(\omega-\omega^2)\beta} \\ \frac{\theta^2-9\omega\beta-1}{(\omega^2-1)(9\omega\beta+1)} & \frac{\omega^2\theta^2}{9\omega\beta+1} & \frac{-\omega^2\theta^2-\omega^2(9\beta+\omega^2)-9\omega\beta-1}{3\beta(9\omega\beta+1)} \\ \frac{-\theta^2-\omega^2\theta+3(\omega-\omega^2)\beta-\omega}{3(9\omega\beta+1)} & \frac{(\omega^2-\omega)(\theta-1)}{3(9\omega\beta+1)} & \frac{-\omega^2\theta-\omega}{9\omega\beta+1} \end{pmatrix}.$$

Then $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$ and

$$m \cdot \mathsf{span}_{F_{\mathsf{sep}}}\langle u, v, w_2(\beta)\rangle \cdot m^{-1} = V_\alpha$$

with $\alpha = \beta(9\beta + \omega^2)^{-1}$. In the same way, we can prove that any non-singular cubic subspaces of $\mathsf{M}_3(F_{\mathsf{sep}})$ spanned by u, v and $w_3(\beta)$ is conjugate to V_α for some $\alpha \in F$: the matrix obtained replacing ω by ω^2 and θ by a cube root of $9\omega^2\beta + 1$ in m, conjugates $\mathsf{span}_{F_{\mathsf{sep}}}\langle u, v, w_3(\beta)\rangle$ into V_α with $\alpha = \beta(9\beta + \omega)^{-1}$ (explanations on these computations are given in Section A.2 of the appendix). □

4.3 Automorphism group

In order to classify the non-singular cubic pairs over F, we compute the automorphism group of an arbitrary non-singular cubic pair over F_{sep}. Suppose that (A,V) and (A',V') are F-cubic pairs and there exists an F_{sep}-isomorphism $\Theta\colon (A,V)_{F_{\mathsf{sep}}} \to (A',V')_{F_{\mathsf{sep}}}$, then

$$\mathsf{Aut}(A',V')(F_{\mathsf{sep}}) = \Theta \circ \mathsf{Aut}(A,V)(F_{\mathsf{sep}}) \circ \Theta^{-1}.$$

By the previous section, we know that a non-singular cubic pair over F_{sep} is isomorphic to $(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha)$, for some $\alpha \in F_{\mathsf{sep}}$. Therefore we only need to compute the automorphism group of the pairs $(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha)$, for $\alpha \in F_{\mathsf{sep}}$. By the Skolem-Noether Theorem,

$$\mathsf{Aut}\big(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha\big)(F_{\mathsf{sep}}) = \{mF_{\mathsf{sep}}^\times \in \mathsf{PGL}_3(F_{\mathsf{sep}}) \mid mV_\alpha m^{-1} = V_\alpha\},$$

hence we want to find the invertible matrices, up to scalar, which conjugate V_α into itself.

First we give some results which hold for arbitrary cubic subspaces of $\mathsf{M}_3(F_{\mathsf{sep}})$.

Lemma 4.3.1 *Let V be a cubic subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$. If $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$ then mVm^{-1} is also a cubic subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$.*

Proof : Clearly, mVm^{-1} is a 3-dimensional subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$. The properties of the trace imply that the trace of any matrix in mVm^{-1} is zero and mVm^{-1} is totally isotropic for the trace quadratic form $q_{\mathsf{M}_3(F_{\mathsf{sep}})}$. Thus mVm^{-1} is a cubic subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$. □

The group $\mathsf{GL}_3(F_{\mathsf{sep}})$ acts on $\mathbb{P}\left(\mathsf{M}_3(F_{\mathsf{sep}})\right)$:

$$m \star uF_{\mathsf{sep}} = mum^{-1}F_{\mathsf{sep}}$$

for $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$ and $uF_{\mathsf{sep}} \in \mathbb{P}\left(\mathsf{M}_3(F_{\mathsf{sep}})\right)$. This action induces an action of $\mathsf{PGL}_3(F_{\mathsf{sep}})$ on $\mathbb{P}\left(\mathsf{M}_3(F_{\mathsf{sep}})\right)$:

$$mF_{\mathsf{sep}}^\times \star uF_{\mathsf{sep}} = m \star uF_{\mathsf{sep}}$$

for $mF_{\mathsf{sep}}^\times \in \mathsf{PGL}_3(F_{\mathsf{sep}})$ and $uF_{\mathsf{sep}} \in \mathbb{P}\left(\mathsf{M}_3(F_{\mathsf{sep}})\right)$.

The following lemma says that particular points of $\mathbb{P}(\mathsf{M}_3(F_{\mathsf{sep}}))$ are preserved under the action of $\mathsf{GL}_3(F_{\mathsf{sep}})$.

Lemma 4.3.2 *Let V be a cubic subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$, $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$ and put $V' := mVm^{-1}$. Suppose that $\tilde{u}, \tilde{v}, \tilde{w} \in V$ are such that $\tilde{u}F_{\mathsf{sep}}$*

is a flex of the cubic curve $\{f_V(\xi) = 0\}$, $\tilde{v}F_{\mathsf{sep}}$ *is a harmonic point of* $\tilde{u}F_{\mathsf{sep}}$ *and* $\tilde{w}F_{\mathsf{sep}}$ *is the Hessian point of* $\tilde{u}F_{\mathsf{sep}}$. *Then* $m \star \tilde{u}F_{\mathsf{sep}}$ *is a flex of* $\{f_{V'}(\xi) = 0\}$, $m \star \tilde{v}F_{\mathsf{sep}}$ *is a harmonic point of* $m \star \tilde{u}F_{\mathsf{sep}}$ *and* $m \star \tilde{w}F_{\mathsf{sep}}$ *is the Hessian point of* $m \star \tilde{u}F_{\mathsf{sep}}$.

Proof: This follows easily from the fact that $\mathsf{tr}(m\xi m^{-1}) = \mathsf{tr}(\xi)$ for all $\xi \in \mathsf{M}_3(F_{\mathsf{sep}})$. □

We use the notation introduced on page 53 to state the following theorem.

Theorem 4.3.3 *Let* V *be a non-singular cubic subspace of* $\mathsf{M}_3(F_{\mathsf{sep}})$ *and* $\tilde{u}, \tilde{v}, \tilde{w} \in V$ *such that* $\tilde{u}\overline{F}$ *is a flex of the cubic curve* $\{f_V(\xi) = 0\}$, $\tilde{v}\overline{F}$ *is a harmonic point of* $\tilde{u}\overline{F}$ *and* $\tilde{w}\overline{F}$ *is the Hessian point of* $\tilde{u}\overline{F}$. *Then there exists a unique* $mF_{\mathsf{sep}}^{\times} \in \mathsf{PGL}_3(F_{\mathsf{sep}})$ *such that*

$$mF_{\mathsf{sep}}^{\times} \star \tilde{u}F_{\mathsf{sep}} = uF_{\mathsf{sep}} \quad \textit{and} \quad mF_{\mathsf{sep}}^{\times} \star \tilde{v}F_{\mathsf{sep}} = vF_{\mathsf{sep}}.$$

Moreover, we have $mF_{\mathsf{sep}}^{\times} \star \tilde{w}F_{\mathsf{sep}} = w_i(\alpha)F_{\mathsf{sep}}$ *for some* $\alpha \in F_{\mathsf{sep}}$ *and* $i \in \{1, 2, 3\}$ *and in particular* mVm^{-1} *is a special subspace.*

Proof: The proof of Theorem 4.2.6 gives the existence of $mF_{\mathsf{sep}}^{\times}$. The unicity follows from Lemma 4.2.5. □

Thus, for a non-singular special subspace V of $\mathsf{M}_3(F_{\mathsf{sep}})$, the elements $mF_{\mathsf{sep}}^{\times} \in \mathsf{PGL}_3(F_{\mathsf{sep}})$ such that mVm^{-1} is special, are in correspondence with the pairs $(\tilde{u}F_{\mathsf{sep}}, \tilde{v}F_{\mathsf{sep}})$ where $\tilde{u}F_{\mathsf{sep}}$ is a flex of $\{f_V(\xi) = 0\}$ and $\tilde{v}F_{\mathsf{sep}}$ is a harmonic point of $\tilde{u}F_{\mathsf{sep}}$: the element $mF_{\mathsf{sep}}^{\times} \in \mathsf{PGL}_3(F_{\mathsf{sep}})$ such that mVm^{-1} is special, corresponds to $(m^{-1} \star uF_{\mathsf{sep}}, m^{-1} \star vF_{\mathsf{sep}})$ (the point $m^{-1} \star uF_{\mathsf{sep}}$ is a flex of $\{f_V(\xi) = 0\}$ and $m^{-1} \star vF_{\mathsf{sep}}$ is a harmonic point of $m^{-1} \star uF_{\mathsf{sep}}$ because uF_{sep} is a flex of $\{f_V(\xi) = 0\}$ and vF_{sep} is a harmonic point of uF_{sep}).

We can deduce an upper bound for the number of elements in the automorphism group of $\big(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha\big)$.

Lemma 4.3.4 *Let* $\alpha \in F_{\mathsf{sep}} \setminus \{0, \frac{1}{8}, \frac{1}{9}\}$. *There are exactly 27 elements* $mF_{\mathsf{sep}}^{\times} \in \mathsf{PGL}_3(F_{\mathsf{sep}})$ *such that* $mV_\alpha m^{-1}$ *is a special subspace. In particular, there are at most 27 elements in the automorphism group of* $(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha)$.

Proof: Since a non-singular cubic curve has exactly 9 flexes and given any flex, there are exactly 3 harmonic points of this flex, by Theorem 4.3.3, there are exactly 27 elements $mF_{\mathsf{sep}}^{\times} \in \mathsf{PGL}_3(F_{\mathsf{sep}})$ such that $mV_\alpha m^{-1}$ is a special subspace. □

The following is a corollary of Theorem 4.3.3.

Corollary 4.3.5 *Let $\alpha \in F_{\mathsf{sep}} \setminus \{0, \frac{1}{8}, \frac{1}{9}\}$. If $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$ is such that $m \star uF_{\mathsf{sep}} = uF_{\mathsf{sep}}$ and $m \star vF_{\mathsf{sep}} = vF_{\mathsf{sep}}$, then $m \in F_{\mathsf{sep}}^{\times}$.*

Since uF_{sep} is a flex of $\{f_{V_\alpha}(\xi) = 0\}$, there exist three distinct elements $mF_{\mathsf{sep}}^{\times} \in \mathsf{PGL}_3(F_{\mathsf{sep}})$ such that $mV_\alpha m^{-1}$ is a special subspace and $mF_{\mathsf{sep}}^{\times} \star uF_{\mathsf{sep}} = uF_{\mathsf{sep}}$, one element for each harmonic points of uF_{sep} and two of them being non-trivial.

Lemma 4.3.6 *Let $\alpha \in F_{\mathsf{sep}} \setminus \{0, \frac{1}{8}, \frac{1}{9}\}$ and $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$ such that $m \notin F_{\mathsf{sep}}^{\times}$, $mV_\alpha m^{-1}$ is a special subspace and $m \star uF_{\mathsf{sep}} = uF_{\mathsf{sep}}$. Then $mV_\alpha m^{-1} = V_\alpha$ if and only if $\alpha = \frac{1}{6}$.*

Proof : Since $mum^{-1} = \lambda u$, for some $\lambda \in F_{\mathsf{sep}}^{\times}$, by straightforward computations we deduce that

$$mF_{\mathsf{sep}}^{\times} = \begin{pmatrix} \lambda^2 & \lambda a & b \\ 0 & \lambda & a \\ 0 & 0 & 1 \end{pmatrix} F_{\mathsf{sep}}^{\times}$$

for some $a, b \in F_{\mathsf{sep}}$. Suppose that $mV_\alpha m^{-1} = V_\alpha$. Then by Lemma 4.3.2, the point $m \star w_1(\alpha)F_{\mathsf{sep}}$ is the Hessian point of $m \star uF_{\mathsf{sep}} = uF_{\mathsf{sep}}$ and by unicity of the Hessian point, we have $m \star w_1(\alpha)F_{\mathsf{sep}} = w_1(\alpha)F_{\mathsf{sep}}$. Hence $mw_1(\alpha) = \nu w_1(\alpha)m$ for some $\nu \in F_{\mathsf{sep}}$. For $\xi \in \mathsf{M}_3(F_{\mathsf{sep}})$, let ξ_{ij} denote the element on the i-th row and the the j-th column in ξ. Then

$$\begin{aligned} (mw_1(\alpha))_{21} = \nu(w_1(\alpha)m)_{21} \quad &\text{implies} \quad \lambda\nu = 1, \\ (mw_1(\alpha))_{33} = \nu(w_1(\alpha)m)_{33} \quad &\text{implies} \quad \nu = (1 - 3\alpha a)^{-1}. \end{aligned}$$

We deduce from $(mw_1(\alpha))_{12} = \nu(w_1(\alpha)m)_{12}$ that $b = a^2/2$. Then $(mw_1(\alpha))_{13} = \nu(w_1(\alpha)m)_{13}$ if and only if $(1-9\alpha)(3\alpha^2a^2 - 3\alpha a + 1) = 0$ or $a = 0$. If $a = 0$, then $m \in F_{\mathsf{sep}}^{\times}$ which contradicts the hypothesis. Thus $3\alpha^2a^2 - 3\alpha a + 1 = 0$ because $\alpha \neq \frac{1}{9}$. We obtain that

- either $\lambda = \omega^2, \nu = \omega$ and $mF_{\mathsf{sep}}^{\times} = \begin{pmatrix} \omega & \frac{\omega^2-\omega}{3\alpha} & \frac{-\omega^2}{6\alpha^2} \\ 0 & \omega^2 & \frac{1-\omega^2}{3\alpha} \\ 0 & 0 & 1 \end{pmatrix} F_{\mathsf{sep}}^{\times}$,

- or $\lambda = \omega, \nu = \omega^2$ and $mF_{\mathsf{sep}}^{\times} = \begin{pmatrix} \omega^2 & \frac{\omega-\omega^2}{3\alpha} & \frac{-\omega}{6\alpha^2} \\ 0 & \omega & \frac{1-\omega}{3\alpha} \\ 0 & 0 & 1 \end{pmatrix} F_{\mathsf{sep}}^{\times}$.

Since $m \star vF_{\mathsf{sep}}$ is a harmonic point of uF_{sep}, it lies on the line passing through uF_{sep} and wF_{sep}. Thus we have $mvm^{-1} = y_0 v + z_0 w_1(\alpha)$ for some $y_0, z_0 \in F_{\mathsf{sep}}$ and in particular $(mvm^{-1})_{12} = -\frac{1}{2}(mvm^{-1})_{13}$. In both cases, it implies that $\alpha = \frac{1}{6}$. Conversely, the distinct elements $m_1 F_{\mathsf{sep}}^{\times}, m_2 F_{\mathsf{sep}}^{\times} \in \mathsf{PGL}_3(F_{\mathsf{sep}})$ with

$$m_i = \begin{pmatrix} 1 & 2(\omega^i - 1) & -6\omega^i \\ 0 & \omega^i & 2(\omega^{2i} - \omega^i) \\ 0 & 0 & \omega^{2i} \end{pmatrix}$$

are such that $m_i \notin F_{\mathsf{sep}}^{\times}$, $m_i V_{\frac{1}{6}} m_i^{-1} = V_{\frac{1}{6}}$ and $m_i F_{\mathsf{sep}}^{\times} \star uF_{\mathsf{sep}} = uF_{\mathsf{sep}}$. Since there exist at most two such elements, we deduce that either $mF_{\mathsf{sep}}^{\times} = m_1 F_{\mathsf{sep}}^{\times}$ or $mF_{\mathsf{sep}}^{\times} = m_2 F_{\mathsf{sep}}^{\times}$ and thus $mV_{\frac{1}{6}} m^{-1} = V_{\frac{1}{6}}$. □

We can now give the automorphism group of $\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha)$ in the particular case $\alpha = \frac{1}{6}$. First we give a notation: we denote by μ_3 the set of cube roots of unity in F_{sep}.

Proposition 4.3.7 *For $\alpha = \frac{1}{6}$ we have a Γ-group isomorphism*

$$\mathsf{Aut}\big(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha)(F_{\mathsf{sep}}) \cong \mu_3 \times \mathbb{Z}/3.$$

Proof : Put

$$m := \begin{pmatrix} 1 & -3 & 6 \\ \frac{1}{2} & -2 & 3 \\ 0 & -\frac{1}{2} & 1 \end{pmatrix},\ m' = \begin{pmatrix} 1 & 2(\omega - 1) & -6\omega \\ 0 & \omega & 2(\omega^2 - \omega) \\ 0 & 0 & \omega^2 \end{pmatrix}$$

and $G := \{m^i m'^j F_{\mathsf{sep}}^{\times} \mid i, j \in \mathbb{Z}\}$. Then the group G contains exactly 9 elements, namely the $m^i m'^j$ for $i, j = 0, 1, 2$, and G is a subgroup of $\mathsf{Aut}(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha)(F_{\mathsf{sep}})$. Let $\theta \in F_{\mathsf{sep}}$ be a cube root of -2. Put

$$m_2 := \begin{pmatrix} -\theta^2 - \omega^2\theta + 2 & -6 & 6(\omega^2\theta + 2) \\ -\theta^2 + \theta & 2(\theta^2 + \omega\theta + \omega) & 2(\omega - 1)(\theta^2 + \omega\theta - 2) \\ \theta & 2(-\theta + \omega) & 2(\omega\theta^2 + \theta - 2\omega) \end{pmatrix}$$

then $m_2 \in \mathsf{GL}_2(F_{\mathsf{sep}})$ and $m_2 V_\alpha m_2^{-1}$ is the span of u, v and $w_2(\frac{-\omega^2}{3})$. Hence the set $m_2 F_{\mathsf{sep}}^{\times} G$ contains 9 elements and is a subset of

$$\Big\{nF_{\mathsf{sep}}^{\times} \in \mathsf{PGL}_3(F_{\mathsf{sep}}) \mid nV_\alpha n^{-1} \text{ is the span of } u, v, w_2(\frac{-\omega^2}{3})\Big\}.$$

Put

$$m_3 := \begin{pmatrix} -\theta^2 - \omega\theta + 2 & -6 & 6(\omega\theta + 2) \\ -\theta^2 + \theta & 2(\theta^2 + \omega^2\theta + \omega^2) & 2(\omega^2 - 1)(\theta^2 + \omega^2\theta - 2) \\ \theta & 2(-\theta + \omega^2) & 2(\omega^2\theta^2 + \theta - 2\omega^2) \end{pmatrix}$$

then $m_3 \in \mathsf{GL}_3(F_{\mathsf{sep}})$ and $m_3 V_\alpha m_3^{-1}$ is the span of u, v and $w_3(\frac{-\omega}{3})$. The set $m_3 F_{\mathsf{sep}}^\times G$ contains 9 elements and is a subset of

$$\Big\{nF_{\mathsf{sep}}^\times \in \mathsf{PGL}_3(F_{\mathsf{sep}}) \mid mV_\alpha m^{-1} \text{ is the span of } u, v, w_3(\frac{-\omega}{3})\Big\}.$$

Therefore, the set $G \cup m_2 F_{\mathsf{sep}}^\times G \cup m_3 F_{\mathsf{sep}}^\times G$ consists of 27 elements $nF_{\mathsf{sep}}^\times$ such that $nV_\alpha n^{-1}$ is a special subspace. So by Theorem 4.3.3, the set $G \cup m_2 F_{\mathsf{sep}}^\times G \cup m_3 F_{\mathsf{sep}}^\times G$ is equal to

$$\{nF_{\mathsf{sep}}^\times \in \mathsf{PGL}_3(F_{\mathsf{sep}}) \mid nV_\alpha n^{-1} \text{ is a special subspace}\}.$$

Hence $\mathsf{Aut}(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha)(F_{\mathsf{sep}}) = G$. Because

$$m'^2 F_{\mathsf{sep}}^\times = \begin{pmatrix} 1 & 2(\omega^2-1) & -6\omega^2 \\ 0 & \omega^2 & 2(\omega-\omega^2) \\ 0 & 0 & \omega \end{pmatrix} F_{\mathsf{sep}}^\times$$

the mappings $mF_{\mathsf{sep}}^\times \mapsto (1, 1+3\mathbb{Z})$ and $m'F_{\mathsf{sep}}^\times \mapsto (\omega, 3\mathbb{Z})$ define a group isomorphism $G \to \mu_3 \times \mathbb{Z}/3$ which is compatible with the action of Γ. Thus

$$\mathsf{Aut}(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha)(F_{\mathsf{sep}}) \cong \mu_3 \times \mathbb{Z}/3.$$

□

In fact, we can prove that, up to conjugacy, the only non-singular cubic subspace V of $\mathsf{M}_3(F_{\mathsf{sep}})$ such that $\mathsf{Aut}\big(\mathsf{M}_3(F_{\mathsf{sep}}), V\big)(F_{\mathsf{sep}})$ is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$ as an abstract group, is $V_{\frac{1}{6}}$. But first we need a lemma.

Lemma 4.3.8 *Let $G \subset \mathsf{PGL}_3(F_{\mathsf{sep}})$ be a subgroup which is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$ (as an abstract group). Then G is conjugate in $\mathsf{PGL}_3(F_{\mathsf{sep}})$ to the subgroup of $\mathsf{PGL}_3(F_{\mathsf{sep}})$ generated by*

- *either* $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} F_{\mathsf{sep}}^\times$ *and* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} F_{\mathsf{sep}}^\times$
- *or* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix} F_{\mathsf{sep}}^\times$ *and* $\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} F_{\mathsf{sep}}^\times$.

Proof: Let $\Theta\colon G \to \mathbb{Z}/3 \times \mathbb{Z}/3$ be a group isomorphism and $aF_{\mathsf{sep}}^\times$ and $bF_{\mathsf{sep}}^\times$ inverse images of $(1,0)$ and $(0,1)$. Since $a^3 F_{\mathsf{sep}}^\times = F_{\mathsf{sep}}^\times$, changing the representative of $aF_{\mathsf{sep}}^\times$ if necessary, we may assume that $a^3 = 1$. Similarly, we may assume that $b^3 = 1$. So the minimal polynomial of b

divides $x^3 - 1$ and thus b is diagonalizable and its eigenvalues are cube roots of unity. Hence there exists $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$ such that

$$mbm^{-1} = \begin{pmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{pmatrix},$$

where the θ_i's are cube roots of unity in F_{sep}. We want to describe G up to conjugacy, so we may assume that

$$b = \begin{pmatrix} \theta_1 & 0 & 0 \\ 0 & \theta_2 & 0 \\ 0 & 0 & \theta_3 \end{pmatrix}.$$

Because $abF_{\mathsf{sep}}^\times = baF_{\mathsf{sep}}^\times$, there exists $\rho \in F_{\mathsf{sep}}^\times$ such that $ba = \rho ab$. Let $e \in F_{\mathsf{sep}}^3$ be an eigenvector of b with eigenvalue θ. Then

$$ba(e) = \rho ab(e) = \rho\theta a(e).$$

Hence $a(e)$ is an eigenvector of b with eigenvalue $\rho\theta$. We deduce that ρ is a cube root of the unity. Let (e_1, e_2, e_3) be the canonical basis of F_{sep}^3, then, for all $i = 1, 2, 3$, the vector $a(e_i)$ is a multiple of some e_j. If $\rho = 1$, then $a(e_i)F_{\mathsf{sep}} = e_iF_{\mathsf{sep}}$ for all $i = 1, 2, 3$. Indeed, since $a^3 = 1$, either $a(e_i)F_{\mathsf{sep}} = e_iF_{\mathsf{sep}}$ for all i or $a(e_i)F_{\mathsf{sep}} \neq e_iF_{\mathsf{sep}}$ for all i. If $a(e_1)F_{\mathsf{sep}} \neq e_1F_{\mathsf{sep}}$ then we may assume that

$$a = \begin{pmatrix} 0 & \theta_2' & 0 \\ 0 & 0 & \theta_3' \\ \theta_1' & 0 & 0 \end{pmatrix}.$$

But $ba = ab$ implies $\theta_1 = \theta_2 = \theta_3$ which is impossible. Thus

$$a = \begin{pmatrix} \theta_1' & 0 & 0 \\ 0 & \theta_2' & 0 \\ 0 & 0 & \theta_3' \end{pmatrix}$$

for some $\theta_i' \in F_{\mathsf{sep}}$ with $\theta_i'^3 = 1$ since $a^3 = 1$. In this case, G is conjugate to the group

$$\left\{ \begin{pmatrix} \rho_1 & 0 & 0 \\ 0 & \rho_2 & 0 \\ 0 & 0 & \rho_3 \end{pmatrix} F_{\mathsf{sep}}^\times \mid \rho_i^3 = 1 \right\}$$

which is generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} F_{\mathsf{sep}}^\times \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix} F_{\mathsf{sep}}^\times.$$

Now assume that $\rho \neq 1$. If $a(e_1)F_{\mathsf{sep}}^{\times} = e_1 F_{\mathsf{sep}}^{\times}$ then $a(e_1) = \theta_1' e_1$ for some $\theta_1' \in F_{\mathsf{sep}}$ and

$$\theta_1 \theta_1' e_1 = ba(e_1) = \rho ab(e_1) = \rho \theta_1 \theta_1' e_1.$$

This implies $\rho = 1$, so $a(e_i)F_{\mathsf{sep}}^{\times} \neq e_i F_{\mathsf{sep}}^{\times}$ for all i and we may assume that

$$a = \begin{pmatrix} 0 & \theta_3' & 0 \\ 0 & 0 & \theta_2' \\ \theta_1' & 0 & 0 \end{pmatrix}$$

for some $\theta_1', \theta_2', \theta_3' \in F_{\mathsf{sep}}^{\times}$ with $\theta_1'\theta_2'\theta_3' = 1$ since $a^3 = 1$. Put

$$m := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \theta_1'\theta_2' & 0 \\ 0 & 0 & \theta_1' \end{pmatrix}$$

then $m^{-1}bm = b$ and

$$m^{-1}am = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

The scalars $\theta_1, \theta_2, \theta_3$ are distinct pairwise because $ba = \rho ab$ and $\rho \neq 1$. Therefore G is conjugate to the subgroup of $\mathsf{PGL}_3(F_{\mathsf{sep}})$ generated by

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} F_{\mathsf{sep}}^{\times} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} F_{\mathsf{sep}}^{\times}.$$

□

Theorem 4.3.9 *If V is a non-singular subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$ such that $\mathsf{Aut}\big(\mathsf{M}_3(F_{\mathsf{sep}}), V\big)(F_{\mathsf{sep}})$ is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$ as an abstract group then V is conjugate to $V_{\frac{1}{6}}$.*

Proof: By the previous lemma, the group $\mathsf{Aut}(\mathsf{M}_3(F_{\mathsf{sep}}), V)(F_{\mathsf{sep}})$ is conjugate to the subgroup G of $\mathsf{PGL}_3(F_{\mathsf{sep}})$ generated by $aF_{\mathsf{sep}}^{\times}$ and $bF_{\mathsf{sep}}^{\times}$ where

$$b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \text{ and}$$

$$\text{either } a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \text{ or } a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

Let $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$ be such that

$$mF_{\mathsf{sep}}^{\times} \cdot \mathsf{Aut}\big(\mathsf{M}_3(F_{\mathsf{sep}}), V\big)(F_{\mathsf{sep}}) \cdot m^{-1}F_{\mathsf{sep}}^{\times} = G.$$

Then $G = \mathsf{Aut}(\mathsf{M}_3(F_{\mathsf{sep}}), \tilde{V})$ with $\tilde{V} = mVm^{-1}$. We have a group homomorphism

$$G \to \mathsf{GL}(\tilde{V})\colon gF_{\mathsf{sep}}^{\times} \to (\hat{g}\colon \xi \mapsto g\xi g^{-1}).$$

So $\hat{a}\hat{b} = \hat{b}\hat{a}$ because $abF_{\mathsf{sep}}^{\times} = baF_{\mathsf{sep}}^{\times}$. We deduce from Corollary 4.3.5 that $\hat{g} \in F_{\mathsf{sep}}^{\times}$ implies $g \in F_{\mathsf{sep}}^{\times}$. Thus the subgroup of $\mathsf{PGL}_3(F_{\mathsf{sep}})$ generated by $\hat{a}F_{\mathsf{sep}}^{\times}$ and $\hat{b}F_{\mathsf{sep}}^{\times}$ is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$. Using the proof of the previous lemma, the eigenvalues of $\hat{a}$ and $\hat{b}$ are cube roots of unity and we may find a basis of $\tilde{V}$ which diagonalizes both $\hat{a}$ and $\hat{b}$.

Assume that

$$a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

Let D denote the subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$ of the diagonal matrices and $\{e_{ij} \mid i,j = 1,2,3\}$ the canonical basis of $\mathsf{M}_3(F_{\mathsf{sep}})$. Let $\tilde{v} \in \tilde{V}$ be a common eigenvector of $\hat{a}$ and $\hat{b}$, then $\hat{a}(\tilde{v}) = \lambda_1\tilde{v}$ and $\hat{b}(\tilde{v}) = \lambda_2\tilde{v}$ for some cube roots $\lambda_1, \lambda_2 \in F_{\mathsf{sep}}$ of unity. This implies that either $\tilde{v}F_{\mathsf{sep}} = e_{ij}$ for some i,j or $\tilde{v}$ is a diagonal matrix. Since $\tilde{V}$ is non-singular, it does not contain any e_{ij}. So $\tilde{V} = D$ which is impossible since $D \not\subset \mathsf{M}_3(F_{\mathsf{sep}})^{\circ}$. Therefore

$$a = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

We shall prove that $\tilde{V}$ is conjugate to $\mathsf{span}_{F_{\mathsf{sep}}}\langle a, b, ab\rangle$. Let $\tilde{v} \in \tilde{V}$ be a common eigenvector of $\hat{a}$ and $\hat{b}$. The set

$$\{a^ib^j \mid i,j = 0,1,2\}$$

is a basis of $\mathsf{M}_3(F_{\mathsf{sep}})$, thus there exist scalars $\alpha_{ij} \in F_{\mathsf{sep}}$ such that $\tilde{v} = \sum \alpha_{ij}a^ib^j$. Using the relations $a^3 = 1$, $b^3 = 1$ and $ba = \omega^2 ab$, we deduce that

$$\begin{cases} \hat{a}(\tilde{v}) = \omega^i\tilde{v} \\ \hat{b}(\tilde{v}) = \omega^j\tilde{v} \end{cases} \iff \tilde{v}F_{\mathsf{sep}} = a^{2j}b^iF_{\mathsf{sep}}.$$

Hence $\tilde{V}$ is spanned by $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$, where $\tilde{v}_1, \tilde{v}_2, \tilde{v}_3$ are distinct vectors among a^ib^j for $i,j \in \{0,1,2\}$. Since $\tilde{V}$ is a non-singular cubic subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$, we have $\tilde{V} \subset \mathsf{M}_3(F_{\mathsf{sep}})^{\circ}$; so $\tilde{v}_i \notin F_{\mathsf{sep}}$ for all i.

Also $\tilde{V}$ is totally isotropic for the trace quadratic form $q_{\mathsf{M}_3(F_{\mathsf{sep}})}$, so $\tilde{v}_i F_{\mathsf{sep}} \neq \tilde{v}_j^2 F_{\mathsf{sep}}$ for all $i \neq j$. The fact that $f_{\tilde{V}}$ is non-singular implies that $\tilde{v}_1\tilde{v}_2\tilde{v}_3 \notin F_{\mathsf{sep}}$. Clearly, the matrices $\tilde{v}_1$ and $\tilde{v}_2$ commute if and only if $\tilde{v}_2 F_{\mathsf{sep}} \in \{F_{\mathsf{sep}}, \tilde{v}_1 F_{\mathsf{sep}}, \tilde{v}_1^2 F_{\mathsf{sep}}\}$; hence $\tilde{v}_1$ and $\tilde{v}_2$ do not commute. Since the subgroup of $\mathsf{PGL}_3(F_{\mathsf{sep}})$ generated by $\tilde{v}_1 F_{\mathsf{sep}}^\times$ and $\tilde{v}_2 F_{\mathsf{sep}}^\times$ is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$ and v_1, v_2 do not commute, by the proof of the previous lemma, it is conjugate to the subgroup generated by $aF_{\mathsf{sep}}^\times$ and $bF_{\mathsf{sep}}^\times$. Therefore we may assume that $\tilde{v}_1 = a$ and $\tilde{v}_2 = b$ and so

$$\tilde{v}_3 \notin \{1, a, a^2, b, b^2, (ab)^2\}.$$

If $\tilde{v}_3 = a^2 b F_{\mathsf{sep}}$, using again the proof of the previous lemma, there exists an $m' \in \mathsf{GL}_3(F_{\mathsf{sep}})$ such that $m' \star aF_{\mathsf{sep}} = aF_{\mathsf{sep}}$ and $m' \star a^2 b F_{\mathsf{sep}} = bF_{\mathsf{sep}}$. Then $m' \star bF_{\mathsf{sep}} = abF_{\mathsf{sep}}$ and $\tilde{V}$ is conjugate to the span of a, b, ab. In the same way, if $\tilde{v}_3 F_{\mathsf{sep}} = ab^2 F_{\mathsf{sep}}$, then there exists an $m' \in \mathsf{GL}_3(F_{\mathsf{sep}})$ such that $m \star ab^2 F_{\mathsf{sep}} = aF_{\mathsf{sep}}$ and $m \star bF_{\mathsf{sep}} = bF_{\mathsf{sep}}$. Then $m \star aF_{\mathsf{sep}} = abF_{\mathsf{sep}}$ and $\tilde{V}$ is conjugate to the span of a, b, ab. Thus we may assume that $\tilde{v}_3 = ab$.

Finally, let $\theta \in F_{\mathsf{sep}}$ be a cube root of -2 and put

$$m' := \begin{pmatrix} 2(\omega^2\theta^2 + \omega^2\theta + 1) & 2(\theta^2 + \omega^2\theta + \omega^2) & 2(\theta^2 + \omega\theta + 1) \\ \omega^2\theta^2 + 2 & \theta^2 + 2\omega^2 & \theta^2 + 2 \\ 1 & \omega^2 & 1 \end{pmatrix}.$$

Then $m'\tilde{V}m'^{-1} = V_{\frac{1}{6}}$. Hence V is conjugate to $V_{\frac{1}{6}}$. □

We say that a special subspace is *exceptional* if it is conjugate to $V_{\frac{1}{6}}$ and it is *non-exceptional* otherwise. We shall now compute the automorphism group $\mathsf{Aut}\big(\mathsf{M}_3(F), V\big)(F_{\mathsf{sep}})$ for a non-exceptional subspace V. The group $\mathsf{PGL}_3(F_{\mathsf{sep}})$ acts on the pairs $(V, \tilde{u}F_{\mathsf{sep}})$, where V is a special subspace and $\tilde{u}F_{\mathsf{sep}}$ is a flex of $\{f_V(\xi) = 0\}$ as follows:

$$mF_{\mathsf{sep}}^\times \star (V, \tilde{u}F_{\mathsf{sep}}) = (mVm^{-1}, m \star \tilde{u}F_{\mathsf{sep}}).$$

Lemma 4.3.6 says that, if V is a non-exceptional special subspace, then the stabilizer $\mathsf{PGL}_3(F_{\mathsf{sep}})_{(V, uF_{\mathsf{sep}})}$ of (V, uF_{sep}) is trivial. We deduce the following:

Lemma 4.3.10 *Let V be a non-exceptional special subspace and $\tilde{u}F_{\mathsf{sep}}$ a flex of $\{f_V(\xi) = 0\}$. Then $\mathsf{PGL}_3(F_{\mathsf{sep}})_{(V, \tilde{u}F_{\mathsf{sep}})} = 1$.*

Proof: There exists an $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$ such that $m \star \tilde{u}F_{\mathsf{sep}} = uF_{\mathsf{sep}}$ and $\tilde{V} := mVm^{-1}$ is special. Hence

$$mF_{\mathsf{sep}}^\times \star (V, \tilde{u}F_{\mathsf{sep}}) = (\tilde{V}, uF_{\mathsf{sep}})$$

and also

$$\mathsf{PGL}_3(F_{\mathsf{sep}})_{(V,\tilde{u}F_{\mathsf{sep}})} \cong \mathsf{PGL}_3(F_{\mathsf{sep}})_{(\tilde{V},uF_{\mathsf{sep}})} = 1$$

as claimed. □

Thus, if V is non-exceptional, then the non-trivial elements of the automorphism group $\mathsf{Aut}(\mathsf{M}_3(F_{\mathsf{sep}}), V)$ do not fix any flex.

Lemma 4.3.11 *Let V be a non-exceptional subspace and $\tilde{u}F_{\mathsf{sep}}$ a flex of $\{f_V(\xi) = 0\}$. Then there exists at most one $mF_{\mathsf{sep}}^{\times} \in \mathsf{PGL}_3(F_{\mathsf{sep}})$ such that $mF_{\mathsf{sep}}^{\times} \star (V, \tilde{u}F_{\mathsf{sep}}) = (V, uF_{\mathsf{sep}})$.*

Proof: Suppose that $mF_{\mathsf{sep}}^{\times}, m'F_{\mathsf{sep}}^{\times} \in \mathsf{PGL}_3(F_{\mathsf{sep}})$ are such that

$$mF_{\mathsf{sep}}^{\times} \star (V, \tilde{u}F_{\mathsf{sep}}) = (V, uF_{\mathsf{sep}}) = m'F_{\mathsf{sep}}^{\times} \star (V, \tilde{u}F_{\mathsf{sep}}).$$

Then $m'm^{-1}F_{\mathsf{sep}}^{\times} \in \mathsf{PGL}_3(F_{\mathsf{sep}})_{(V,uF_{\mathsf{sep}})} = 1$, so $mF_{\mathsf{sep}}^{\times} = m'F_{\mathsf{sep}}^{\times}$. □

Recall that, given a special subspace V, the $mF_{\mathsf{sep}}^{\times} \in \mathsf{PGL}_3(F_{\mathsf{sep}})$ such that mVm^{-1} is a special subspace, are in correspondence with the pairs $(\tilde{u}F_{\mathsf{sep}}, \tilde{v}F_{\mathsf{sep}})$ where $\tilde{u}F_{\mathsf{sep}}$ is a flex of $\{f_V(\xi) = 0\}$ and $\tilde{v}F_{\mathsf{sep}}$ is a harmonic point of $\tilde{u}F_{\mathsf{sep}}$: the pair corresponding to an element $mF_{\mathsf{sep}}^{\times} \in \mathsf{PGL}_3(F_{\mathsf{sep}})$ such that mVm^{-1} is special, is $(m^{-1}\star uF_{\mathsf{sep}}, m^{-1}\star vF_{\mathsf{sep}})$. By the previous lemma, the map

$$\mathsf{Aut}\big(\mathsf{M}_3(F_{\mathsf{sep}}), V\big)(F_{\mathsf{sep}}) \to \big\{\text{flexes of } \{f_V(\xi) = 0\}\big\}$$

which maps $mF_{\mathsf{sep}}^{\times}$ to $m^{-1} \star uF_{\mathsf{sep}}$, is injective. Thus there exist at most 9 elements in $\mathsf{Aut}\big(\mathsf{M}_3(F_{\mathsf{sep}}), V\big)$.

Lemma 4.3.12 *Let V be a non-exceptional subspace and $mF_{\mathsf{sep}}^{\times}$ a non-trivial element of $\mathsf{PGL}_3(F_{\mathsf{sep}})$ such that $mVm^{-1} = V$. Then the order of $mF_{\mathsf{sep}}^{\times}$ is equal to 3.*

Proof: By Proposition 1.3.9, the flexes of a non-singular cubic curve and the lines through them have the configuration of the affine plane $\mathbb{F}_3^2$. We fix an isomorphism between the flexes of $\{f_V(\xi) = 0\}$ and $\mathbb{F}_3^2$. Since an element of $\mathsf{Aut}(\mathsf{M}_3(F_{\mathsf{sep}}), V)(F_{\mathsf{sep}})$ preserves the collinearity, the isomorphism induces a group homomorphism Θ from $\mathsf{Aut}(\mathsf{M}_3(F_{\mathsf{sep}}), V)(F_{\mathsf{sep}})$ to the group $\mathsf{A}_2(\mathbb{F}_3^2)$ of affine transformations of $\mathbb{F}_3^2$. By Lemma 4.3.10, if an element $mF_{\mathsf{sep}}^{\times}$ such that $mVm^{-1} = V$ preserves a flex of $\{f_V(\xi) = 0\}$ then it is trivial. In particular Θ is injective. Let G denotes the image of Θ, then it is sufficient to prove that a non-trivial element of G has order 3.

By Lemma 4.3.10, a non-trivial element of G does not fix any point of $\mathbb{F}_3^2$. Let $g \in G$ be non-trivial. We may change the affine coordinates so that $g(0,0) = (0,1)$. If $g^2 = 1$, then $g(0,1) = (0,0)$ and since g preserves the collinearity, we have $g(0,2) = (0,2)$; thus $g = 1$ which contradicts the hypothesis. Therefore the order of g is not 2. Suppose that g preserves the line passing through $(0,0), (0,1)$ and $(0,2)$, then $g(0,1) = (0,2)$ and $g(0,2) = (0,0)$. In particular, $g^3(0,0) = (0,0)$ and $g^3 = 1$; hence g has order 3. Now suppose that g does not preserve the line passing through $(0,0), (0,1)$ and $(0,2)$. Then we may assume that $g(0,1) = (1,2)$. We show that g is completely determined. There exist $a \neq b \in \mathbb{F}_3$ such that

$$g(x,y)^t = \begin{pmatrix} a & 1 \\ b & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Since g does not fix any points, the linear system

$$\begin{cases} (a-1)x + y = 0 \\ bx = -1 \end{cases}$$

has no solutions, and therefore $b = 0$. Would $a = 2$, then $g^3(0,2) = (0,2)$ and so $g^3 = 1$; but on the other hand $g^3(0,0) = (1,0)$ and thus we get a contradiction. Hence $a = 1$, so that $g^3(0,0) = (0,0)$ and g has order 3. □

Thus the group $\mathsf{Aut}(\mathsf{M}_3(F_{\mathsf{sep}}), V)$ is either trivial or isomorphic to $\mathbb{Z}/3$, because by Theorem 4.3.9 it is not isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$.

Lemma 4.3.13 *Let $\alpha \in F_{\mathsf{sep}} \setminus \{0, \frac{1}{8}, \frac{1}{9}\}$, then $\mathsf{Aut}(\mathsf{M}_3(F_{\mathsf{sep}}), V)$ is not trivial.*

Proof : Put

$$m := \begin{pmatrix} \alpha & 0 & -1 \\ \alpha & -2\alpha & 0 \\ 3\alpha^2 & -\alpha & \alpha \end{pmatrix}.$$

Then m is invertible and $mV_\alpha m^{-1} = V_\alpha$. Thus $mF_{\mathsf{sep}}^\times$ is a non-trivial element of $\mathsf{Aut}(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha)(F_{\mathsf{sep}})$. □

We proved the following theorem:

Theorem 4.3.14 *Let $\alpha \in F_{\mathsf{sep}} \setminus \{0, \frac{1}{8}, \frac{1}{9}\}$. Then*

$$\mathsf{Aut}(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha) \cong \begin{cases} \mathbb{Z}/3 \times \mathbb{Z}/3 & \text{if } \alpha = \frac{1}{6}, \\ \mathbb{Z}/3 & \text{otherwise} \end{cases}$$

as abstract groups.

We say that a non-singular cubic pair is *of the first kind* if its automorphism group is $\mathbb{Z}/3$ and *of the second kind* otherwise.

4.4 Classification of cubic pairs of the first kind

We shall classify the non-singular cubic pairs over F with an automorphism group isomorphic to $\mathbb{Z}/3$ (as an abstract group).

Fix (A, V) a non-singular F-cubic pair of the first kind. By Theorem 4.1.2, if $\mathsf{Aut}(A, V)(F_{\mathsf{sep}})$ is isomorphic to $\mathbb{Z}/3$ as Γ-groups then there is a bijection

$$\mathsf{H}^1(F, \mathbb{Z}/3) \longleftrightarrow \left\{ \begin{array}{c} F\text{-isomorphism classes of} \\ \text{the } F\text{-cubic pairs which are} \\ \text{isomorphic to } (A, V)_{F_{\mathsf{sep}}} \text{ over } F_{\mathsf{sep}} \end{array} \right\}.$$

As $\mathsf{H}^1(F, \mathbb{Z}/3)$ classifies the Galois $\mathbb{Z}/3$-algebras over F (see (28.15) in [Knus *et al.*, 1998]), there is a one to one correspondence between the isomorphism classes of F-cubic pairs (A', V') which are isomorphic to $(A, V)_{F_{\mathsf{sep}}}$ over F_{sep}, and the isomorphism classes of Galois $\mathbb{Z}/3$-algebras over F. The bijection is defined as follows: let $m \in A^\times$ be such that

$$\mathsf{Aut}(A, V)(F_{\mathsf{sep}}) = \{F_{\mathsf{sep}}^\times, mF_{\mathsf{sep}}^\times, m^2 F_{\mathsf{sep}}^\times\};$$

for an F-cubic pair (A', V') which is isomorphic to $(A, V)_{F_{\mathsf{sep}}}$ over F_{sep}, let $(a_\sigma F_{\mathsf{sep}}^\times)$ be the corresponding 1-cocycle with values in $\mathsf{Aut}(A, V)(F_{\mathsf{sep}})$. Let

$$H = \{\sigma \in \Gamma \mid a_\sigma \in F_{\mathsf{sep}}^\times\}.$$

Then H is an open-closed subgroup of Γ and so there exists a field extension L/F with $L \subset F_{\mathsf{sep}}$ such that $H = \mathsf{Gal}(F_{\mathsf{sep}}/L)$. If $H = \Gamma$ (i.e. (A', V') is F-isomorphic to (A, V)), then $[(F^3, \rho)]$ where ρ is the automorphism of F^3 defined by $\rho(x, y, z) = (y, z, x)$, is the corresponding isomorphism class of Galois $\mathbb{Z}/3$-algebra. If $H \neq \Gamma$, let σ_0 be such that $a_{\sigma_0} F_{\mathsf{sep}}^\times = mF_{\mathsf{sep}}^\times$. Since $\Gamma = H \cup \sigma_0 H \cup \sigma_0^2 H$ and

$$\tau H = \{\sigma \in \Gamma \mid a_\sigma F_{\mathsf{sep}}^\times = a_\tau F_{\mathsf{sep}}^\times\} = H\tau$$

for all $\tau \in \Gamma$, the extension L/F is Galois of degree 3. Then $[(L, \sigma_0|_L)]$ is the isomorphism class of Galois $\mathbb{Z}/3$-algebra corresponding to $[(A', V')]$. Conversely, for a non-trivial Galois $\mathbb{Z}/3$-algebra (L, ρ), we define

$$a_\sigma = \begin{cases} 1 & \text{if } \sigma|_L = \mathsf{id}_L, \\ m & \text{if } \sigma|_L = \rho, \\ m^2 & \text{if } \sigma|_L = \rho^2. \end{cases}$$

Then the isomorphism class of F-cubic pair corresponding to $[(L,\rho)]$, is the one associated to $[a_\sigma]$. By Theorem 4.2.7 and Theorem 4.3.14, a non-singular cubic pair (A,V) of the first kind is isomorphic over F_{sep} to the pair $(\mathsf{M}_3(F_{\mathsf{sep}}),V_\alpha)$ for some $\alpha\in F_{\mathsf{sep}}\setminus\{0,\frac{1}{8},\frac{1}{9},\frac{1}{6}\}$. In general, α may not be in F. First, we classify the cubic pairs (A,V) of the first kind such that there exists $\alpha\in F$ so that $(A,V)_{F_{\mathsf{sep}}}\cong(\mathsf{M}_3(F_{\mathsf{sep}}),V_\alpha)$ and next we classify the ones which do not have this property.

First we fix two notations: for $a,b\in F(\omega)^\times$, let $(a,b)_{\omega,F(\omega)}$ denote the symbol F-algebra generated by ξ_0 and η_0 such that $\xi_0^3=a$, $\eta_0^3=b$ and $\xi_0\eta_0=\omega\eta_0\xi_0$; and for $a\in F^\times$, L/F a cyclic extension of degree 3 and ρ a generator of the Galois group $\mathsf{Gal}(L/F)$, let $(a,L/F,\rho)$ denote the cyclic algebra $\bigoplus_{i=0}^{2}Le^i$ with multiplication defined by $e^3=a$ and $e\xi=\rho(\xi)e$ for all $\xi\in L$.

Case 1: Let $\alpha\in F\setminus\{0,\frac{1}{8},\frac{1}{9},\frac{1}{6}\}$. We want to describe all the F-cubic pairs (A,V) such that

$$(A,V)_{F_{\mathsf{sep}}}\cong\big(\mathsf{M}_3(F_{\mathsf{sep}}),V_\alpha\big).$$

Put $A:=\mathsf{M}_3(F)$, $V:=\mathsf{span}_F\langle u,v,w_1(\alpha)\rangle$ and

$$m:=\begin{pmatrix}\alpha & 0 & -1\\ \alpha & -2\alpha & 0\\ 3\alpha^2 & -\alpha & \alpha\end{pmatrix}$$

so that $\mathsf{Aut}(A,V)(F_{\mathsf{sep}})=\{F_{\mathsf{sep}}^\times,mF_{\mathsf{sep}}^\times,m^2F_{\mathsf{sep}}^\times\}$. Note that Γ acts trivially on $\mathsf{Aut}(A,V)(F_{\mathsf{sep}})$ so $\mathsf{Aut}(A,V)(F_{\mathsf{sep}})$ and $\mathbb{Z}/3$ are isomorphic as Γ-groups. Let (L,ρ) be a non-trivial Galois $\mathbb{Z}/3$-algebra. We define, for $\sigma\in\Gamma$,

$$a_\sigma=\begin{cases}1 & \text{if }\sigma|_L=\mathsf{id}_L,\\ m & \text{if }\sigma|_L=\rho,\\ m^2 & \text{if }\sigma|_L=\rho^2.\end{cases}$$

The isomorphism class of cubic pair corresponding to $[(L,\rho)]$ is $[(A',V')]$ with

$$A'=\{\xi\in A_{\mathsf{sep}}\mid a_\sigma\sigma(\xi)a_\sigma^{-1}=\xi\ \text{ for all }\sigma\in\Gamma\}$$

and

$$V'=\{\xi\in V_{\mathsf{sep}}\mid a_\sigma\sigma(\xi)a_\sigma^{-1}=\xi\ \text{ for all }\sigma\in\Gamma\}.$$

To determine V', it is sufficient to find three linearly independent vectors ξ in V_L such that $m\rho(\xi)m^{-1}=\xi$. By Corollary 4.3.5, the group homomorphism

$$\mathsf{Aut}(A,V)(F_{\mathsf{sep}})\to\mathsf{GL}(V)\colon gF_{\mathsf{sep}}^\times\mapsto(\hat{g}\colon V\to V\colon\xi\mapsto g\xi g^{-1})$$

is injective. Since the order of $mF_{\mathsf{sep}}^{\times}$ divides 3, so does the order of $\hat{m}$. Therefore, the minimal polynomial of $\hat{m}$ divides x^3-1, so $\hat{m}$ is diagonalizable and its eigenvalues are cube roots of unity. We observe that $m^2 \in V$ thus m^2 is an eigenvector of $\hat{m}$ with eigenvalue 1 and in particular $m^2 \in V'$.

First we assume that F contains a primitive cube root of unity. Then $\mathbb{Z}/3 = \mu_3$ and the isomorphism classes of the Galois $\mathbb{Z}/3$-algebras are in one to one correspondence with the elements of $F^\times/F^{\times 3}$: the class of (L,ρ) corresponds to $dF^{\times 3} \in F^\times/F^{\times 3}$, such that there exists $\theta \in L$ with $\theta^3 = d \in F$ and $\rho(\theta) = \omega\theta$. To determine V', it is sufficient to find the eigenvectors of $\hat{m}$. Indeed, if $\xi_0 \in V$ is an eigenvector of $\hat{m}$ with eigenvalue ω^i then $\theta^{-i}\xi_0 \in V'$. Put

$$\begin{aligned}
\xi_0 &:= \alpha(6\alpha-1)v - 2w_1(\alpha),\\
\eta_0 &:= \frac{1}{2}(\omega-\omega^2)(8\alpha-1)\theta u + \alpha(1-9\alpha)\theta v - \theta w_1(\alpha),\\
\zeta_0 &:= \frac{1}{2}(\omega^2-\omega)(8\alpha-1)\theta^2 u + \alpha(1-9\alpha)\theta^2 v - \theta^2 w_1(\alpha),
\end{aligned}$$

then $\xi_0, \eta_0, \zeta_0 \in V'$ are linearly independent and

$$\xi_0^3 = \alpha(8\alpha-1)^2, \quad \eta_0^3 = d\alpha(8\alpha-1)^2(9\alpha-1), \quad \xi_0\eta_0 = \omega\eta_0\xi_0$$

(we have $\xi_0\eta_0 = \omega\eta_0\xi_0$ because $\xi_0 F = m^2 F$). Hence V' is the vector subspace of A_{sep} spanned by ξ_0, η_0, ζ_0 and A' is the symbol algebra

$$\big(\alpha(8\alpha-1)^2, d\alpha(8\alpha-1)^2(9\alpha-1)\big)_{\omega,F}$$

generated by ξ_0 and η_0. Replacing θ by 1, we obtain that the algebra

$$\big(\alpha(8\alpha-1)^2, \alpha(8\alpha-1)^2(9\alpha-1)\big)_{\omega,F}$$

is trivial since it is generated by $\xi_0, \eta_0 \in \mathsf{M}_3(F)$. So A' is Brauer equivalent to $(\alpha(8\alpha-1)^2, d)_{\omega,F}$. We have

$$\zeta_0 = \frac{3\omega^2}{(8\alpha-1)(9\alpha-1)}\xi_0\eta_0^2 - \frac{\omega(6\alpha-1)}{\alpha(8\alpha-1)^2(9\alpha-1)}\xi_0^2\eta_0^2$$

and

$$f_{A',V'} = a\xi_0^{\star 3} + b\eta_0^{\star 3} + c\zeta_0^{\star 3} - 3\lambda\xi_0^\star\eta_0^\star\zeta_0^\star$$

where $(\xi_0^\star, \eta_0^\star, \zeta_0^\star)$ denotes the dual basis of (ξ_0, η_0, ζ_0) and

$$\begin{gathered}
a = \alpha(8\alpha-1)^2, \quad b = d\alpha(8\alpha-1)^2(9\alpha-1),\\
c = d^2\alpha(8\alpha-1)^2(9\alpha-1), \quad \lambda = d\alpha(8\alpha-1)^2(1-6\alpha).
\end{gathered}$$

The scalars a, b, c, λ satisfy the relation[2]

$$\frac{abc - \lambda^3}{a^2} = \big(3d\alpha(8\alpha - 1)\big)^3 \in F^{\times 3}.$$

Thus we already proved:

Theorem 4.4.1 *Assume F contains a primitive cube root of unity. Let $\alpha \in F \setminus \{0, \frac{1}{8}, \frac{1}{9}, \frac{1}{6}\}$. Then, up to F-isomorphism, the F-cubic pairs which are isomorphic to $(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha)$ over F_{sep}, are the pairs*

$$\Big(\big(\alpha(8\alpha - 1)^2, d\alpha(8\alpha - 1)^2(9\alpha - 1)\big)_{\omega, F}, \mathsf{span}_F\langle \xi_0, \eta_0, \zeta_0\rangle\Big),$$

for all $dF^{\times 3} \in F^\times / F^{\times 3}$, where ξ_0, η_0 are generators of the symbol algebra such that $\xi_0^3 = \alpha(8\alpha-1)^2$, $\eta_0^3 = d\alpha(8\alpha - 1)^2(9\alpha - 1)$, $\xi_0\eta_0 = \omega\eta_0\xi_0$ and

$$\zeta_0 = 3\omega^2\alpha(8\alpha - 1)\xi_0\eta_0^2 - \omega(6\alpha - 1)\xi_0^2\eta_0^2.$$

The associated cubic forms are semi-diagonal.

Note that, by Theorem 3.1.3 the cubic curve $\{(\xi_0^\star\eta_0^\star\zeta_0^\star)(\xi) = 0\}$ is an inflexional triangle of $f_{A',V'}$ whose lines are defined over F.

Now we assume that F does not contain a primitive cube root of unity and F is infinite. By Proposition (18.32) in [Knus *et al.*, 1998], there exists $\theta \in L$ such that $L = F(\theta)$ and the minimal polynomial of θ over F is $x^3 - 3x + \lambda$ for some $\lambda \in F \setminus \{2, -2\}$. Let $\theta' = \rho(\theta)$ and $\theta'' = \rho^2(\theta)$ be the other roots of $x^3 - 3x + \lambda$ in F_{sep}. Since $\theta + \theta' + \theta'' = 0$ and $\theta\theta' + \theta\theta'' + \theta'\theta'' = -3$, we have

$$\theta' = \frac{-\theta + \delta}{2} \quad \text{and} \quad \theta'' = \frac{-\theta - \delta}{2},$$

where

$$\delta^2 = 12 - 3\theta^2 = \frac{3}{4 - \lambda^2}(2\theta^2 + \lambda\theta - 4)^2 \in F(\theta)^{\times 2}.$$

So $\delta = x_0^{-1}(2\theta^2 + \lambda\theta - 4)$, where $x_0 \in F$ is a square root of $(4 - \lambda^2)/3$. Using Cardano's method, we may write

$$\theta = -\phi - \phi^{-1}$$

where $\phi \in F_{\mathsf{sep}}$ is a cube root of $(\lambda + (\omega - \omega^2)x_0)/2$, and then

$$\theta' = -\omega\phi - \omega^2\phi^{-1} \quad \text{and} \quad \theta'' = -\omega^2\phi - \omega\phi^{-1}.$$

[2]The details of these computations are given in Section A.3 of the appendix.

Put

$$\begin{aligned}\xi_0 &:= \alpha(6\alpha-1)v-2w_1(\alpha),\\ \eta_0 &:= \frac{1}{2}(1-8\alpha)\delta u+\alpha(9\alpha-1)\theta v+\theta w_1(\alpha),\\ \zeta_0 &:= \frac{3}{2}(8\alpha-1)\theta u+\alpha(9\alpha-1)\delta v+\delta w_1(\alpha),\end{aligned}$$

then ξ_0, η_0, ζ_0 are linearly independent vectors of V', so V' is the F-vector subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$ spanned by ξ_0, η_0 and ζ_0. Put

$$\eta_1 := \frac{1}{2}\eta_0+\frac{\omega^2-\omega}{6}\zeta_0 = \frac{1}{2}(\omega-\omega^2)(8\alpha-1)\phi u+\alpha(1-9\alpha)\phi v-\phi w_1(\alpha).$$

Then $\xi_0, \eta_1 \in A'\otimes_F F(\omega)$ are such that

$$\xi_0^3=\alpha(8\alpha-1)^2,\ \eta_1^3=\phi^3\alpha(8\alpha-1)^2(9\alpha-1) \text{ and } \xi_0\eta_1=\omega\eta_1\xi_0.$$

So $A'\otimes_F F(\omega)=\big(\alpha(8\alpha-1)^2,\phi^3\alpha(8\alpha-1)^2(9\alpha-1)\big)_{\omega,F(\omega)}$ is the symbol $F(\omega)$-algebra generated by ξ_0 and η_1. We shall find a subfield of A' which is a Galois extension of degree 3 over F. We use the following notation: for $a,b\in F$, we write

$$\overline{a+\omega b}=a+\omega^2 b$$

and, for $\xi=\sum \xi_i\otimes x_i\in A'\otimes_F F(\omega)$, we write

$$\bar{\xi}=\sum \xi_i\otimes\overline{x_i}.$$

Put

$$\eta_2 := \frac{3}{(8\alpha-1)(9\alpha-1)}\xi_0\eta_1-\frac{6\alpha-1}{\alpha(8\alpha-1)^2(9\alpha-1)}\xi_0^2\eta_1,$$

then $\eta_2^3=\phi^3\in F(\omega)$. Let τ,τ' be the F-automorphisms of $F(\eta_2)$ defined by $\tau(\eta_2)=\omega\eta_2$ and $\tau'(\eta_2)=\eta_2^{-1}$. Then $\tau'(\eta_2)=\overline{\eta_2}$ since $\eta_2\overline{\eta_2}=1$ and we have $\tau'\tau=\tau\tau'$. So the subfield

$$L'=\{\xi\in F(\eta_2)\mid \bar{\xi}=\xi\}$$

of $F(\eta_2)$ is a Galois extension of degree 3 over F with Galois group generated by $\tau|_{L'}$, and it is contained in

$$A'=\{\xi\in A'_{F(\omega)}\mid \bar{\xi}=\xi\}.$$

Put $\eta_3:=-\eta_2-\eta_2^{-1}$, then $L'=F(\eta_3)$. Moreover we have $\xi_0\eta_3=\tau(\eta_3)\xi_0$ because $\xi_0\eta_2=\omega\eta_2\xi_0$. Hence A' is the cyclic algebra

$$\big(\alpha(8\alpha-1)^2,L'/F,\tau|_{L'}\big)$$

generated by ξ_0 and η_3. Observe that the mapping $\eta_3 \mapsto \theta$ defines an isomorphism between the Galois $\mathbb{Z}/3$-algebras $(L', \tau|_{L'})$ and (L, ρ). We shall write η_0 and ζ_0 in function of ξ_0 and η_3. We have

$$\eta_0 = \eta_1 + \overline{\eta_1} \quad \text{and} \quad \zeta_0 = (\omega - \omega^2)(\eta_1 - \overline{\eta_1}).$$

But

$$\eta_1 = \Big(\frac{3}{(8\alpha-1)(9\alpha-1)}\xi_0 - \frac{6\alpha-1}{\alpha(8\alpha-1)^2(9\alpha-1)}\xi_0^2\Big)^{-1}\eta_2$$

and since $\tau(\eta_3) = -\omega\eta_2 - \omega^2\eta_2^{-1}$ we get

$$\eta_2 = \frac{1}{\omega^2 - \omega}\big(-\omega^2\eta_3 + \tau(\eta_3)\big).$$

Therefore

$$\begin{cases} \eta_0 = \frac{1}{1-9\alpha}\big(3\alpha(6\alpha-1)(8\alpha-1) + (6\alpha-1)^2\xi_0 + 9\alpha\xi_0^2\big)\eta_3, \\ \zeta_0 = \frac{1}{1-9\alpha}\big(3\alpha(6\alpha-1)(8\alpha-1) + (6\alpha-1)^2\xi_0 + 9\alpha\xi_0^2\big)\big(\eta_3 + 2\tau(\eta_3)\big). \end{cases}$$

We shall give explanations over these computations in the appendix. Now we describe the cubic form $f_{A',V'}$. Let $(\xi_0^\star, \eta_1^\star, \overline{\eta_1}^\star)$ denote the dual basis of $(\xi_0, \eta_1, \overline{\eta_1})$. We observe that the cubic curve $\{(\xi_0^\star\eta_1^\star\overline{\eta_1}^\star)(\xi) = 0\}$ is an inflexional triangle of $f_{A',V'}$. The triangle $\{(\xi_0^\star\eta_1^\star\overline{\eta_1}^\star)(\xi) = 0\}$ is a priori defined over F_{sep}. But the line $\{\xi_0^\star(\xi) = 0\}$ is defined over F because $\eta_0 F$ and $\zeta_0 F$ are distinct F-points of $\{\xi_0^\star(\xi) = 0\}$. Hence the triangle $\{(\xi_0^\star\eta_1^\star\overline{\eta_1}^\star)(\xi) = 0\}$ is defined over F and the cubic form $f_{A',V'}$ is a semi-trace form. We have

$$\begin{aligned} f(x\xi_0 + y\eta_0 + z\zeta_0) &= \Big(x\xi_0 + y'\eta_1 + z'\overline{\eta_1}\Big)^3 \\ &= a_1x^3 + b_2y'^3 + b_3z'^3 - 3\mu xy'z' \end{aligned}$$

where $y' = y + (\omega - \omega^2)z$, $z' = y - (\omega - \omega^2)z$ and

$$a_1 = \alpha(8\alpha-1)^2, \ \ b_2 = \phi^3\alpha(8\alpha-1)^2(9\alpha-1),$$
$$b_3 = \phi^{-3}\alpha(8\alpha-1)^2(9\alpha-1), \ \ \mu = \alpha(8\alpha-1)(1-6\alpha).$$

Put $K = F \times F(\omega)$ and define $\Theta\colon V' \to K$ by

$$\Theta(x\xi_0 + y\eta_0 + z\zeta_0) = \big(x, y + (\omega - \omega^2)z\big).$$

The scalars $a_2 = \frac{1}{2}\lambda\alpha(8\alpha-1)^2(9\alpha-1)$ and $a_3 = \frac{1}{2}x_0\alpha(8\alpha-1)^2(9\alpha-1)$ are such that

$$\begin{cases} a_2 + (\omega - \omega^2)a_3 = b_2, \\ a_2 - (\omega - \omega^2)a_3 = b_3 \end{cases}$$

thus we have

$$f(\xi) = \mathsf{Tr}_{K/F}\big(a\Theta(\xi)^3\big) - 3\mu \mathsf{N}_{K/F}\big(\Theta(\xi)\big)$$

where $a = \big(a_1, a_2 + (\omega - \omega^2)a_3\big)$. Also the elements a and μ satisfy the relation

$$\frac{\mathsf{N}_{K/F}(a) - \mu^3}{a_1^2} = \frac{a_1 b_2 b_3 - \mu^3}{a_1^2} \in F^{\times 3}.$$

We shall prove that $f_{A',V'}$ is not semi-diagonal. The line $\{\eta_1^\star(\xi) = 0\}$ is not defined over F because otherwise the intersection point $\overline{\eta_1} F_{\mathsf{sep}}$ of this line and $\{\xi_0^\star(\xi) = 0\}$ would be defined over F and it would contradict the assumption that F does not contain a primitive cube root of unity. We consider the action of Γ on A'_{sep} defined by

$$\sigma \star \xi = a_\sigma \sigma(\xi) a_\sigma^{-1}.$$

The points uF_{sep}, $mum^{-1}F_{\mathsf{sep}}$ and $m^2um^{-2}F_{\mathsf{sep}}$ are flexes of the curve $\{f_{A',V'}(\xi) = 0\}$ and they lie on the line $\{\xi_0^\star(\xi) = 0\}$. Let T be an inflexional triangle of $f_{A',V'}$ distinct from $\{(\xi_0^\star \eta_1^\star \overline{\eta_1}^\star)(\xi) = 0\}$. Then a line of the triangle T passes through one and only one point among uF_{sep}, $mum^{-1}F_{\mathsf{sep}}$ and $m^2um^{-2}F_{\mathsf{sep}}$; thus it is not preserved under the action of Γ and it is not defined over F. Hence there does not exist an inflexional triangle of $f_{A',V'}$ whose lines are defined over F; so $f_{A',V'}$ is not semi-diagonal.

We shall prove that also $f_{A,V}$ is semi-trace but not semi-diagonal. Observe that V is spanned by

$$\begin{aligned}
\xi_0 &:= \alpha(6\alpha - 1)v - 2w_1(\alpha),\\
\eta_0 &:= \frac{1}{2}(8\alpha - 1)\delta u + \alpha(9\alpha - 1)\theta v + \theta w_1(\alpha),\\
\zeta_0 &:= \frac{3}{2}(8\alpha - 1)\theta u + \alpha(9\alpha - 1)\delta v + \delta w_1(\alpha)
\end{aligned}$$

where $\theta = -2$ and $\delta = 0$. The $F(\omega)$-algebra $A \otimes F(\omega)$ is generated by ξ_0 and η_1 where

$$\eta_1 := \frac{1}{2}\eta_0 + \frac{\omega^2 - \omega}{6}\zeta_0 = \frac{1}{2}(\omega - \omega^2)(8\alpha - 1)\phi u + \alpha(1 - 9\alpha)\phi v - \phi w_1(\alpha)$$

for $\phi = 1$. Note that $\phi^3 = \big(\lambda + (\omega - \omega^2)x_0\big)/2$ with $x_0 = 0$ and $\lambda = 2$, and

$$\theta = -\phi - \phi^{-1}, \quad \frac{-\theta + \delta}{2} = -\omega\phi - \omega^2\phi^{-1}, \quad \frac{-\theta - \delta}{2} = -\omega^2\phi - \omega\phi^{-1}$$

are the roots of the polynomial $x^3 - 3x + \lambda$. The matrix

$$\eta_2 := \frac{3}{(8\alpha-1)(9\alpha-1)}\xi_0\eta_1 - \frac{6\alpha-1}{\alpha(8\alpha-1)^2(9\alpha-1)}\xi_0^2\eta_1$$

is such that $\eta_2^3 = \phi^3$. We have

$$\begin{cases} \eta_0 = \frac{1}{1-9\alpha}\big(3\alpha(6\alpha-1)(8\alpha-1) + (6\alpha-1)^2\xi_0 + 9\alpha\xi_0\big)\eta_3, \\ \zeta_0 = \frac{1}{1-9\alpha}\big(3\alpha(6\alpha-1)(8\alpha-1) + (6\alpha-1)^2\xi_0 + 9\alpha\xi_0\big)(\eta_3 + 2\eta_3') \end{cases}$$

where $\eta_3 = -\eta_2 - \eta_2^{-1}$ and $\eta_3' = -\omega\eta_2 - \omega^2\eta_2^{-1}$. Put $L' := F \oplus F\eta_3 \oplus F\eta_3'$ and let ρ be the F-algebra automorphism of L' defined by $\rho(\eta_3) = \eta_3'$ and $\rho(\eta_3') = -\eta_3 - \eta_3'$. We note that the map $\Psi\colon L' \to F^3$ defined by

$$\Psi(x + y\eta_3 + z\eta_3') = (x+y+z, x-2y+z, x+y-2z)$$

is an F-algebra isomorphism such that $\rho\big(\Psi^{-1}(1,0,0)\big) = \Psi^{-1}(0,1,0)$ and $\rho\big(\Psi^{-1}(0,1,0)\big) = \Psi^{-1}(0,0,1)$. Also $A = \bigoplus_{i=0}^{2} L'\xi_0^i$ with $\xi_0\xi = \rho(\xi)\xi_0$ for all $\xi \in L'$ and $1, \eta_3, \rho(\eta_3)$ span L' such that

$$(x-\eta_3)\big(x-\rho(\eta_3)\big)\big(x-\rho^2(\eta_3)\big) = x^3 - 3x + \lambda.$$

We can prove that $f_{A,V}$ is a semi-trace form which is not semi-diagonal by letting $\theta = -2$, $\delta = 0 = x_0$, $\lambda = 2$ and $\phi = 1$ in the relations that we found for (A', V').

We proved:

Theorem 4.4.2 *Assume that F is an infinite field which does not contain a primitive cube root of unity and let $\alpha \in F \setminus \{0, \frac{1}{8}, \frac{1}{9}, \frac{1}{6}\}$. Then, up to F-isomorphism, the cubic pairs over F which are isomorphic to $\big(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha\big)$ over F_{sep}, are either $\big(\mathsf{M}_3(F), \mathsf{span}_F\langle u, v, w_1(\alpha)\rangle\big)$ or the pairs*

$$\Big(\big(\alpha(8\alpha-1)^2, L/F, \rho\big), \mathsf{span}_F\langle \xi_0, \eta_0, \zeta_0\rangle\Big),$$

for all non-trivial isomorphism classes $[(L,\rho)]$ of Galois $\mathbb{Z}/3$-algebras, where ξ_0 and $L = F(\theta)$ generate the cyclic algebra such that $\xi_0\theta = \rho(\theta)\xi_0$, $\xi_0^3 = \alpha(8\alpha-1)^2$, $\theta^3 - 3\theta \in F$, and

$$\begin{aligned} \eta_0 &= \big(3\alpha(6\alpha-1)(8\alpha-1) + (6\alpha-1)^2\xi_0 + 9\alpha\xi_0^2\big)\theta, \\ \zeta_0 &= \big(3\alpha(6\alpha-1)(8\alpha-1) + (6\alpha-1)^2\xi_0 + 9\alpha\xi_0^2\big)\rho(\theta). \end{aligned}$$

The associated cubic forms are semi-trace forms and they are not semi-diagonal.

Case 2: Now we classify the F-cubic pairs (A,V) of the first kind such that $(A,V)_{F_{\mathsf{sep}}}$ is not isomorphic to $(\mathsf{M}_3(F_{\mathsf{sep}}),V_\beta)$ for any $\beta \in F$. Let (A,V) be such a cubic pair. Then there exists an F_{sep}-isomorphism

$$\Theta\colon (A,V)_{F_{\mathsf{sep}}} \to \big(\mathsf{M}_3(F_{\mathsf{sep}}),V_\alpha\big)$$

for some $\alpha \in F_{\mathsf{sep}} \setminus \{0,\frac{1}{8},\frac{1}{9},\frac{1}{6}\}$. For $\sigma \in \Gamma$, the composition

$$\Theta \circ (\mathsf{id}_A \otimes \sigma) \circ \Theta^{-1} \circ \sigma^{-1}$$

is an F_{sep}-algebra automorphism of $\mathsf{M}_3(F_{\mathsf{sep}})$. Thus, by the Skolem-Noether Theorem, there exists $a_\sigma \in \mathsf{GL}_3(F_{\mathsf{sep}})$ such that

$$\Theta \circ (\mathsf{id}_A \otimes \sigma) \circ \Theta^{-1} \circ \sigma^{-1} = \mathsf{int}(a_\sigma).$$

Put

$$\sigma \star \xi := \Theta \circ (\mathsf{id}_A \otimes \sigma) \circ \Theta^{-1}(\xi) = a_\sigma \sigma(\xi) a_\sigma^{-1}$$

for $\sigma \in \Gamma$ and $\xi \in \mathsf{M}_3(F_{\mathsf{sep}})$. Then we obtain a continuous action of Γ on $\mathsf{M}_3(F_{\mathsf{sep}})$ by semi-linear F_{sep}-algebra automorphism and we have

$$\Theta|_A\colon (A,V) \cong \big(\mathsf{M}_3(F_{\mathsf{sep}})^\Gamma, V_\alpha^\Gamma\big).$$

For all $\sigma \in \Gamma$ and $\xi \in V_\alpha$, we have $\sigma \star \xi \in V_\alpha$ and $\sigma(\xi) \in V_{\sigma(\alpha)}$. Thus $a_\sigma V_{\sigma(\alpha)} a_\sigma^{-1} = V_\alpha$ and V_α is conjugate to $V_{\sigma(\alpha)}$. Therefore we need to know whenever V_β is isomorphic to $V_{\beta'}$.

For $\alpha \in F_{\mathsf{sep}} \setminus \{0,\frac{1}{8},\frac{1}{9}\}$, we fix ρ a square root of $1-8\alpha$ in F_{sep} and we put

$$\begin{aligned}
\alpha' &:= \frac{(18\alpha-1)(8\alpha-1)+(6\alpha-1)\rho}{16(9\alpha-1)^2},\\
\alpha'' &:= \frac{(18\alpha-1)(8\alpha-1)-(6\alpha-1)\rho}{16(9\alpha-1)^2}.
\end{aligned}$$

Lemma 4.4.3 *Let $\alpha \in F_{\mathsf{sep}} \setminus \{0,\frac{1}{8},\frac{1}{9},\frac{1}{6}\}$. There are exactly three distinct values for $\beta \in F_{\mathsf{sep}}$ such that $V_\alpha \cong V_\beta$, namely α,α' and α''.*

Proof: By Lemma 4.3.4, there are exactly 27 distinct elements $nF_{\mathsf{sep}}^\times$ in $\mathsf{PGL}_3(F_{\mathsf{sep}})$ such that $nV_\alpha n^{-1}$ is a special subspace. We already know that the automorphism group $\mathsf{Aut}(A,V_\alpha)(F_{\mathsf{sep}})$ contains 3 elements. Let

$mF_{\mathsf{sep}}^{\times} \in \mathsf{PGL}_3(F_{\mathsf{sep}})$ be a generator of the group $\mathsf{Aut}(A, V_\alpha)(F_{\mathsf{sep}})$. Put

$$\begin{aligned} m' &:= \begin{pmatrix} \frac{-72\alpha^2+16\alpha-1+(12\alpha-1)\rho}{2(8\alpha-1)} & \frac{-6\alpha+1-(18\alpha-1)\rho}{8\alpha-1} & \frac{36\alpha-5+3\rho}{8\alpha-1} \\ 0 & \frac{-8\alpha+1-(12\alpha-1)\rho}{2(8\alpha-1)} & \frac{3(8\alpha-1)+\rho}{8\alpha-1} \\ 0 & 0 & 1 \end{pmatrix}, \\ m'' &:= \begin{pmatrix} \frac{-72\alpha^2+16\alpha-1-(12\alpha-1)\rho}{2(8\alpha-1)} & \frac{-6\alpha+1+(18\alpha-1)\rho}{8\alpha-1} & \frac{36\alpha-5-3\rho}{8\alpha-1} \\ 0 & \frac{-8\alpha+1+(12\alpha-1)\rho}{2(8\alpha-1)} & \frac{3(8\alpha-1)-\rho}{8\alpha-1} \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

then $m' V_\alpha m'^{-1} = V_{\alpha'}$ and $m'' V_\alpha m''^{-1} = V_{\alpha''}$. By the proof of Theorem 4.2.7, the subspace $\mathsf{span}_{F_{\mathsf{sep}}}\langle u, v, w_2(\beta)\rangle$ is conjugate to $V_{\beta(9\beta+\omega^2)^{-1}}$ for all $\beta \in F_{\mathsf{sep}} \backslash \{0, -\frac{\omega^2}{3}, -\frac{\omega^2}{9}\}$. So V_α is conjugate to $\mathsf{span}_{F_{\mathsf{sep}}}\langle u, v, w_2(\beta)\rangle$ with $\beta = \frac{-\omega^2\alpha}{9\alpha-1}$. Let $m_2 \in \mathsf{GL}_3(F_{\mathsf{sep}})$ be such that

$$m_2 V_\alpha m_2^{-1} = \mathsf{span}_{F_{\mathsf{sep}}}\langle u, v, w_2(\beta)\rangle.$$

Suppose that an invertible matrix n is such that the subspace

$$n \cdot \mathsf{span}_{F_{\mathsf{sep}}}\langle u, v, w_2(\beta)\rangle \cdot n^{-1}$$

is special and $n \star uF_{\mathsf{sep}} = uF_{\mathsf{sep}}$. Then there exists $\lambda \in F_{\mathsf{sep}}^{\times}$ such that $nun^{-1} = \lambda u$ and it implies that

$$nF_{\mathsf{sep}}^{\times} = \begin{pmatrix} \lambda^2 & \lambda a & b \\ 0 & \lambda & a \\ 0 & 0 & 1 \end{pmatrix} F_{\mathsf{sep}}^{\times}$$

for some $a, b \in F_{\mathsf{sep}}$. We deduce that

$$n w_2(\beta) n^{-1} = \begin{pmatrix} \beta & \star & \star \\ 0 & \omega\beta & \star \\ 0 & 0 & \omega^2\beta \end{pmatrix}$$

and $n \cdot \mathsf{span}_{F_{\mathsf{sep}}}\langle u, v, w_2(\beta)\rangle \cdot n^{-1} = \mathsf{span}_{F_{\mathsf{sep}}}\langle u, v, w_2(\beta')\rangle$ for some scalar $\beta' \in F_{\mathsf{sep}}$. Therefore there exist matrices $m_2', m_2'' \in \mathsf{GL}_3(F_{\mathsf{sep}})$ such that

$$\begin{aligned} m_2' V_\alpha m_2'^{-1} &= \mathsf{span}_{F_{\mathsf{sep}}}\langle u, v, w_2(\beta')\rangle, \\ m_2'' V_\alpha m_2''^{-1} &= \mathsf{span}_{F_{\mathsf{sep}}}\langle u, v, w_2(\beta'')\rangle \end{aligned}$$

for some $\beta', \beta'' \in F_{\mathsf{sep}}$ and $m_2 F_{\mathsf{sep}}^{\times}, m_2' F_{\mathsf{sep}}^{\times}, m_2'' F_{\mathsf{sep}}^{\times}$ are distinct pairwise. In the same way, we can prove that there exist $m_3, m_3', m_3'' \in \mathsf{GL}_3(F_{\mathsf{sep}})$

such that

$$\begin{aligned} m_3 V_\alpha m_3^{-1} &= \mathsf{span}_{F_{\mathsf{sep}}}\langle u, v, w_3(\gamma)\rangle, \\ m_3' V_\alpha m_3'^{-1} &= \mathsf{span}_{F_{\mathsf{sep}}}\langle u, v, w_3(\gamma')\rangle, \\ m_3'' V_\alpha m_3''^{-1} &= \mathsf{span}_{F_{\mathsf{sep}}}\langle u, v, w_3(\gamma'')\rangle. \end{aligned}$$

for some $\gamma, \gamma', \gamma'' \in F_{\mathsf{sep}}$ and $m_3 F_{\mathsf{sep}}^\times, m_3' F_{\mathsf{sep}}^\times, m_3'' F_{\mathsf{sep}}^\times$ are distinct pairwise. We put $m_1 := 1$, $m_1' := m'$ and $m_1'' := m''$ so that the set

$$\{m_i m^j F_{\mathsf{sep}}^\times, m_i' m^j F_{\mathsf{sep}}^\times, m_i'' m^j F_{\mathsf{sep}}^\times \mid i, j = 1, 2, 3\}$$

consists of the 27 elements $nF_{\mathsf{sep}}^\times$ such that $nV_\alpha n^{-1}$ is a special subspace. Thus V_α is conjugate to V_a if and only if a is equal to α, α' or α''. □

Lemma 4.4.4 *Let $\alpha \in F_{\mathsf{sep}} \setminus \{0, \frac{1}{8}, \frac{1}{9}, \frac{1}{6}\}$. There exists an F-cubic pair (A, V) such that $(A, V)_{F_{\mathsf{sep}}} \cong \big(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha\big)$ and $(A, V)_{F_{\mathsf{sep}}}$ is not isomorphic to $(\mathsf{M}_3(F_{\mathsf{sep}}), V_\beta)$ for all $\beta \in F$, if and only if the minimal polynomial of α over F is equal to*

$$x^3 - tx^2 + \frac{8t-1}{36}x - \frac{8t-1}{648}$$

for some $t \in F$.

Proof : Suppose that (A, V) is an F-cubic pair which is isomorphic to $\big(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha\big)$ over F_{sep} and such that $(A, V)_{F_{\mathsf{sep}}}$ is not isomorphic to $(\mathsf{M}_3(F_{\mathsf{sep}}), V_\beta)$ for all $\beta \in F$. By the previous lemma the scalars $\alpha, \alpha', \alpha''$ are not in F. The action of Γ on F_{sep} restricts to $\{\alpha, \alpha', \alpha''\}$. Indeed, for all $\sigma \in \Gamma$, the subspaces V_α and $V_{\sigma(\alpha)}$ are conjugate. Thus by the previous lemma $\sigma(\alpha) \in \{\alpha, \alpha', \alpha''\}$. There exist matrices $a, b \in \mathsf{GL}_3(F_{\mathsf{sep}})$ such that $aV_\alpha a^{-1} = V_{\alpha'}$ and $bV_\alpha b^{-1} = V_{\alpha''}$. Since

$$\sigma(a) V_{\sigma(\alpha)} \sigma(a)^{-1} = V_{\sigma(\alpha')} \quad \text{and} \quad \sigma(b) V_{\sigma(\alpha)} \sigma(b)^{-1} = V_{\sigma(\alpha'')}$$

the subspaces $V_{\sigma(\alpha')}$ and $V_{\sigma(\alpha'')}$ are both conjugate to V_α and we have $\sigma(\alpha'), \sigma(\alpha'') \in \{\alpha, \alpha', \alpha''\}$ for all $\sigma \in \Gamma$. Because $\alpha, \alpha', \alpha'' \notin F$ the minimal polynomial of α over F is equal to

$$(x - \alpha)(x - \alpha')(x - \alpha'') = x^3 - tx^2 + \frac{8t-1}{36}x - \frac{8t-1}{648},$$

where

$$t = \frac{648\alpha^3 - 18\alpha + 1}{8(9\alpha - 1)^2} \in F.$$

Conversely, suppose that the minimal polynomial of α over F is equal to

$$x^3 - tx^2 + \frac{8t-1}{36}x - \frac{8t-1}{648}$$

for some $t \in F$. Then

$$t = \frac{648\alpha^3 - 18\alpha + 1}{8(9\alpha - 1)^2}$$

and the other roots of the minimal polynomial are α' and α''. Put

$$a_\sigma = \begin{cases} 1 & \text{if } \sigma(\alpha) = \alpha, \\ m' & \text{if } \sigma(\alpha) = \alpha', \\ m'' & \text{if } \sigma(\alpha) = \alpha'' \end{cases}$$

where m', m'' are the matrices introduced in the proof of Lemma 4.4.3. Then $(a_\sigma F_{\mathsf{sep}}^\times)_{\sigma\in\Gamma}$ is a 1-cocycle with values in $\mathsf{PGL}_3(F_{\mathsf{sep}})$ and by Galois Descent, the pair (A, V) with

$$\begin{array}{rcl} A & = & \{\xi \in \mathsf{M}_3(F_{\mathsf{sep}}) \mid a_\sigma\sigma(\xi)a_\sigma^{-1} = \xi \;\text{ for all } \sigma \in \Gamma\}, \\ V & = & \{\xi \in V_\alpha \mid a_\sigma\sigma(\xi)a_\sigma^{-1} = \xi \;\text{ for all } \sigma \in \Gamma\} \end{array}$$

is an F-cubic pair which is isomorphic to $(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha)$ over F_{sep}. □

Let $\alpha \in F_{\mathsf{sep}}$ with minimal polynomial over F equal to

$$x^3 - tx^2 + \frac{8t-1}{36}x - \frac{8t-1}{648}$$

for some $t \in F$. We observe that $t \neq \frac{1}{2}$ since otherwise $\alpha = \frac{1}{6}$. Similarly we have $t \neq \frac{1}{8}$. Put

$$\xi_0 := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; \eta_0 := \begin{pmatrix} 2(2t-1) & \frac{1}{2} & \frac{1}{4} \\ 0 & -4(2t-1) & -\frac{1}{2} \\ -48(2t-1)^2 & 0 & 2(2t-1) \end{pmatrix}$$

$$\text{and } \zeta_0 := \begin{pmatrix} 0 & 0 & \frac{1}{24(2t-1)} \\ 2(2t-1) & 0 & 0 \\ 0 & -2(2t-1) & 0 \end{pmatrix},$$

then $\big(\mathsf{M}_3(F), \mathsf{span}_F\langle\xi_0, \eta_0, \zeta_0\rangle\big)_{F_{\mathsf{sep}}} \cong \big(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha\big)$. Indeed, put

$$a := \begin{pmatrix} 18(6\alpha-1)^4 & 6(6\alpha-1)^2(9\alpha-1) & (9\alpha-1)^2 \\ 0 & 18\alpha(6\alpha-1)^2(9\alpha-1) & 6\alpha(9\alpha-1)^2 \\ 0 & 0 & 18\alpha^2(9\alpha-1)^2 \end{pmatrix}$$

then $a \in \mathsf{GL}_3(F_{\mathsf{sep}})$ and

$$\begin{aligned} a\xi_0 a^{-1} &= \frac{(6\alpha-1)^2}{\alpha(9\alpha-1)}u, \\ a\eta_0 a^{-1} &= -\frac{27\alpha^2(6\alpha-1)^2}{(9\alpha-1)^2}v + \frac{3(6\alpha-1)^2(3\alpha-1)}{\alpha(9\alpha-1)^2}w_1(\alpha), \\ a\zeta_0 a^{-1} &= \frac{6\alpha-1}{2\alpha(9\alpha-1)}w_1(\alpha). \end{aligned}$$

We put $A := \mathsf{M}_3(F)$ and $V := \mathsf{span}_F\langle \xi_0, \eta_0, \zeta_0\rangle$ so that (A, V) is a cubic pair over F which is isomorphic to $(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha)$ over F_{sep}. Put

$$m := \begin{pmatrix} 2t-1 & \frac{1}{2} & -\frac{1}{8} \\ 0 & -2(2t-1) & -\frac{1}{2} \\ 24(2t-1)^2 & 0 & 2t-1 \end{pmatrix},$$

then $m \in \mathsf{GL}_3(F)$ and

$$\mathsf{Aut}(A, V)(F_{\mathsf{sep}}) = \{F_{\mathsf{sep}}^\times, mF_{\mathsf{sep}}^\times, m^2 F_{\mathsf{sep}}\}.$$

Since Γ acts trivially on $mF_{\mathsf{sep}}^\times$, the Γ-group $\mathsf{Aut}(A, V)(F_{\mathsf{sep}})$ is isomorphic to $\mathbb{Z}/3$.

The isomorphism classes of F-cubic pairs which are isomorphic to $(A, V)_{F_{\mathsf{sep}}}$ over F_{sep} are in bijection with the isomorphism classes of Galois $\mathbb{Z}/3$-algebras over F. Let (L, ρ) be a non-trivial Galois $\mathbb{Z}/3$-algebra over F. For $\sigma \in \Gamma$, we put

$$a_\sigma := \begin{cases} 1 & \text{if } \sigma|_L = \mathsf{id}_L, \\ m & \text{if } \sigma|_L = \rho, \\ m^2 & \text{if } \sigma|_L = \rho^2. \end{cases}$$

Then the isomorphism class of F-cubic pair corresponding to $[(L, \rho)]$ is $[(A', V')]$ with

$$\begin{aligned} A' &= \{\xi \in A_{\mathsf{sep}} \mid a_\sigma \sigma(\xi) a_\sigma^{-1} = \xi \ \text{ for all } \sigma \in \Gamma\}, \\ V' &= \{\xi \in V_{\mathsf{sep}} \mid a_\sigma \sigma(\xi) a_\sigma^{-1} = \xi \ \text{ for all } \sigma \in \Gamma\}. \end{aligned}$$

As in the first case, the endomorphism $\hat{m}\colon V \to V : \xi \mapsto m\xi m^{-1}$ is diagonalizable and its eigenvalues are cube roots of unity. Since $m^2 \in V$, we deduce that m^2 is an eigenvector with eigenvalue 1 and in particular $m^2 \in V'$.

Suppose that F contains a primitive cube root of unity. Let $\theta \in L$ be such that $\theta^3 = d \in F$ and $\rho(\theta) = \omega\theta$. Put

$$\begin{aligned}
\xi_1 &:= \eta_0 + 6\zeta_0, \\
\eta_1 &:= \Big(\frac{1-8t}{4(\omega-\omega^2)}\xi_0 + \frac{1}{2}\eta_0 - 4(2t-1)\zeta_0\Big)\theta, \\
\zeta_1 &:= \Big(\frac{8t-1}{4(\omega-\omega^2)}\xi_0 + \frac{1}{2}\eta_0 - 4(2t-1)\zeta_0\Big)\theta^2.
\end{aligned}$$

Then ξ_1, η_1, ζ_1 are linearly independent vectors of V' such that

$$\xi_1^3 = -4(8t-1)^2(2t-1),\ \eta_1^3 = \frac{2d}{3}(8t-1)^2(2t-1)^2,\ \xi_1\eta_1 = \omega\eta_1\xi_1$$

(we have $\xi_1\eta_1 = \omega\eta_1\xi_1$ because $\xi_1 F = m^2 F$). Therefore V' is the subspace of A_{sep} spanned by ξ_1, η_1, ζ_1 and A' is the symbol F-algebra

$$\Big(-4(8t-1)^2(2t-1), \frac{2d}{3}(8t-1)^2(2t-1)^2\Big)_{\omega,F}.$$

We have

$$\zeta_1 = -\frac{\omega^2}{2(8t-1)(2t-1)}\xi_1\eta_1^2 - \frac{\omega}{2(8t-1)^2(2t-1)}\xi_1^2\eta_1^2$$

and the cubic form $f_{A',V'}$ is semi-diagonal:

$$f_{A',V'} = a\xi_1^{\star 3} + b\eta_1^{\star 3} + c\zeta_1^{\star 3} - 3\lambda\xi_1^\star\eta_1^\star\zeta_1^\star$$

for some scalars $a, b, c, \lambda \in F$ such that $(abc - \lambda^2)a^{-2} \in F^{\times 3}$, where $(\xi_1^\star, \eta_1^\star, \zeta_1^\star)$ denotes the dual basis of (ξ_1, η_1, ζ_1).

We have thus shown:

Theorem 4.4.5 *Suppose that F contains a primitive cube root of unity. Let $\alpha \in F_{\mathsf{sep}}$ be such that its minimal polynomial over F is equal to*

$$x^3 - tx^2 + \frac{8t-1}{36}x - \frac{8t-1}{648}$$

for some $t \in F$. Up to F-isomorphism, the F-cubic pairs which are isomorphic to $\big(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha\big)$ over F_{sep} are the pairs

$$\Big(\Big(-4(8t-1)^2(2t-1), \frac{2d}{3}(8t-1)^2(2t-1)^2\Big)_{\omega,F}, \mathsf{span}_F\langle\xi_1, \eta_1, \zeta_1\rangle\Big)$$

for all $dF^{\times 3} \in F^\times/F^{\times 3}$, where ξ_1, η_1 are generators of the symbol algebra such that

$$\xi_1^3 = -4(8t-1)^2(2t-1),\ \eta_1^3 = \frac{2d}{3}(8t-1)^2(2t-1)^2,\ \xi_1\eta_1 = \omega\eta_1\xi_1$$

and $\zeta_1 = \omega^2(8t-1)\xi_1\eta_1^2 + \omega\xi_1^2\eta_1^2$. *The cubic forms associated to these cubic pairs are semi-diagonal.*

Now we assume that F does not contain a primitive cube root of unity and F is infinite. Let $\theta \in L$ be such that the minimal polynomial of θ over F is $x^3 - 3x + \lambda$ for some $\lambda \in F \setminus \{2, -2\}$. Put $\theta' := \rho(\theta)$ and $\theta'' := \rho^2(\theta)$. Then

$$\theta' = \frac{-\theta + \delta}{2}, \ \theta'' = \frac{-\theta - \delta}{2},$$

where $\delta = x_0^{-1}(2\theta^2 + \lambda\theta - 4)$ and $x_0 \in F$ is a square root of $(4-\lambda^2)/3$. Also, there exists a cube root ϕ of $(\lambda + (\omega - \omega^2)x_0)/2$ in F_{sep} such that

$$\theta = -\phi - \phi^{-1}, \ \theta' = -\omega\phi - \omega^2\phi^{-1}, \ \theta'' = -\omega^2\phi - \omega\phi^{-1}.$$

Put

$$\begin{aligned}
\xi_1 &:= \eta_0 + 6\zeta_0, \\
\eta_1 &:= \frac{1}{12}(8t-1)\delta\xi_0 + \frac{1}{2}\theta\eta_0 - 4(2t-1)\theta\zeta_0, \\
\zeta_1 &:= \frac{1}{4}(1-8t)\theta\xi_0 + \frac{1}{2}\delta\eta_0 - 4(2t-1)\delta\zeta_0.
\end{aligned}$$

Then ξ_1, η_1, ζ_1 are linearly independent vectors of V', thus V' is the span of ξ_1, η_1, ζ_1. Put

$$\eta_2 := -\frac{1}{2}\eta_1 + \frac{\omega - \omega^2}{6}\zeta_1 = \Big(\frac{1-8t}{4(\omega-\omega^2)}\xi_0 + \frac{1}{2}\eta_0 - 4(2t-1)\zeta_0)\Big)\phi$$

then $\xi_1, \eta_2 \in A' \otimes_F F(\omega)$ are such that

$$\xi_1^3 = -4(8t-1)^2(2t-1), \ \eta_2^3 = \frac{2}{3}\phi^3(8t-1)^2(2t-1)^2, \ \xi_1\eta_2 = \omega\eta_2\xi_1.$$

Therefore $A' \otimes_F F(\omega)$ is the symbol algebra

$$\Big(-4(8t-1)^2(2t-1), \frac{2}{3}\phi^3(8t-1)^2(2t-1)^2\Big)_{\omega, F(\omega)}$$

generated by ξ_1 and η_2. Put

$$\eta_3 := -\frac{1}{2(8t-1)(2t-1)}\xi_1\eta_2 - \frac{1}{2(8t-1)^2(2t-1)}\xi_1^2\eta_2$$

then $\eta_3^3 = \phi^3$ and $\overline{\eta_3}\eta_3 = 1$. Put $\eta_4 := -\eta_3 - \eta_3^{-1}$, then $L' := F(\eta_4)$ is a Galois extension of degree 3 over F with Galois group generated by τ where

$$\tau(\eta_4) = -\omega\eta_3 - \omega^2\eta_3^{-1}$$

and L' is contained in A'. Since $\xi_1\eta_4 = \tau(\eta_4)\xi_1$, the algebra A' is the cyclic algebra $\big(-4(8t-1)^2(2t-1), L'/F, \tau\big)$ generated by ξ_1 and η_4, where $(L',\tau) \cong (L,\rho)\colon \eta_4 \mapsto \theta$. We can write η_1 and ζ_1 in function of ξ_1 and η_4:

$$\begin{aligned}
\eta_1 &= \Big(\frac{2}{3}(8t-1)(2t-1) - \frac{2}{3}(2t-1)\xi_1 + \frac{1}{6}\xi_1^2\Big)\eta_4,\\
\zeta_1 &= \Big(\frac{2}{3}(8t-1)(2t-1) - \frac{2}{3}(2t-1)\xi_1 + \frac{1}{6}\xi_1^2\Big)\big(\eta_4 + 2\rho(\eta_4)\big).
\end{aligned}$$

Again $f_{A',V'}$ is a semi-trace form. Let $(\xi_1^\star, \eta_2^\star, \overline{\eta_2}^\star)$ denote the dual basis of $(\xi_1, \eta_2, \overline{\eta_2})$. The cubic curve $\{(\xi_1^\star\eta_2^\star\overline{\eta_2}^\star)(\xi) = 0\}$ is an inflexional triangle of $f_{A',V'}$ and the line $\{\xi_1^\star(\xi) = 0\}$ is defined over F. Hence $\{(\xi_1^\star\eta_2^\star\overline{\eta_2}^\star)(\xi) = 0\}$ is defined over F and $f_{A',V'}$ is a semi-trace form:

$$f_{A',V'}(\xi) = \mathsf{Tr}_{K/F}\big(a\Theta(\xi)^3\big) - 3\mu\mathsf{N}_{K/F}\big(\Theta(\xi)\big)$$

where $K = F \times F(\omega)$, the map $\Theta\colon V' \to K$ is defined by

$$\Theta(x\xi_1 + y\eta_1 + z\zeta_1) = \big(x, y + (\omega - \omega^2)z\big),$$

$a = \big(a_1, a_2 + (\omega - \omega^2)a_3\big)$ with

$$\begin{aligned}
a_1 &= -4(8t-1)^2(2t-1),\\
a_2 &= -\frac{\lambda}{3}(8t-1)^2(2t-1)^2,\\
a_3 &= -\frac{x_0}{3}(8t-1)^2(2t-1)^2
\end{aligned}$$

and $\mu = \frac{4}{3}(8t-1)^2(2t-1)^2$. The elements a and μ satisfy the relation

$$\frac{\mathsf{N}_{K/F}(a) - \mu^3}{a_1^2} = \Big(-\frac{1}{3}(8t-1)(2t-1)\Big)^3 \in F^{\times 3}.$$

The line $\{\xi_1^\star(\xi) = 0\}$ passes through the flexes $\xi_0 F_{\mathsf{sep}}$, $m\xi_0 m^{-1}F_{\mathsf{sep}}$ and $m^2\xi_0 m^{-2}F_{\mathsf{sep}}$. Therefore a line of an inflexional triangle distinct from $\{(\xi_1^\star\eta_2^\star\overline{\eta_2}^\star)(\xi) = 0\}$ is not defined over F. Since the line $\{\eta_2^\star(\xi) = 0\}$ is not defined over F the form $f_{A',V'}$ is not semi-diagonal.

We note that the vector space V is spanned by

$$\begin{aligned}
\xi_1 &:= \eta_0 + 6\zeta_0,\\
\eta_1 &:= \frac{1}{12}(8t-1)\delta\xi_0 + \frac{1}{2}\theta\eta_0 - 4(2t-1)\theta\zeta_0,\\
\zeta_1 &:= \frac{1}{4}(1-8t)\theta\xi_0 + \frac{1}{2}\delta\eta_0 - 4(2t-1)\delta\zeta_0
\end{aligned}$$

with $\delta = 0$ and $\theta = -2$. We put

$$\begin{aligned}
\eta_2 &:= -\frac{1}{2}\eta_1 + \frac{\omega - \omega^2}{6}\zeta_1, \\
\eta_3 &:= -\frac{1}{2(8t-1)(2t-1)}\xi_1\eta_2 - \frac{1}{2(8t-1)^2(2t-1)}\xi_1^2\eta_2, \\
\eta_4 &:= -\eta_3 - \eta_3^{-1}, \\
\eta_4' &:= -\omega\eta_3 - \omega^2\eta_3^{-1}.
\end{aligned}$$

Then $\eta_3^3 = \phi^3$ for $\phi = 1$ and

$$\begin{aligned}
\eta_1 &= \Big(\frac{2}{3}(8t-1)(2t-1) - \frac{2}{3}(2t-1)\xi_1 + \frac{1}{6}\xi_1^2\Big)\eta_4, \\
\zeta_1 &= \Big(\frac{2}{3}(8t-1)(2t-1) - \frac{2}{3}(2t-1)\xi_1 + \frac{1}{6}\xi_1^2\Big)\big(\eta_4 + 2\eta_4'\big).
\end{aligned}$$

Put $L' := F \oplus F\eta_4 \oplus F\eta_4'$ and let ρ be the F-algebra automorphism of L' defined by $\rho(\eta_4) = \eta_4'$ and $\rho(\eta_4') = -\eta_4 - \eta_4'$. Then $1, \eta_4, \rho(\eta_4)$ span L' and

$$(x - \eta_4)\big(x - \rho(\eta_4)\big)\big(x - \rho^2(\eta_4)\big) = x^3 - 3x + \lambda$$

for $\lambda = 2$. There exists an F-algebra isomorphism $\Psi\colon L' \to F^3$ such that

$$\rho\big(\Psi^{-1}(1,0,0)\big) = \Psi^{-1}(0,1,0) \quad \text{and} \quad \rho\big(\Psi^{-1}(0,1,0)\big) = \Psi^{-1}(0,0,1).$$

We have $A = \bigoplus_{i=0}^{2} L'\xi_1^i$ with $\xi_1\xi = \rho(\xi)\xi_1$ for all $\xi \in L'$. We can prove that $f_{A,V}$ is semi-trace but not semi-diagonal by letting $\theta = -2$, $\delta = 0 = x_0$, $\lambda = 2$ and $\phi = 1$ in the relations we proved for (A', V').

Theorem 4.4.6 *Suppose that F is infinite and does not contain a primitive cube root of unity. Let $\alpha \in F_{\mathsf{sep}}$ be such that its minimal polynomial over F is equal to $x^3 - tx^2 + (8t-1)/36x - (8t-1)/648$ for some $t \in F$. Then, up to F-isomorphism, the F-cubic pairs which are isomorphic to $(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha)$ over F_{sep}, are either $\big(\mathsf{M}_3(F), \mathsf{span}_F\langle \xi_0, \eta_0, \zeta_0\rangle\big)$, where*

$$\xi_0 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ \eta_0 := \begin{pmatrix} 2(2t-1) & \frac{1}{2} & \frac{1}{4} \\ 0 & -4(2t-1) & -\frac{1}{2} \\ -48(2t-1)^2 & 0 & 2(2t-1) \end{pmatrix}$$

$$\textit{and } \zeta_0 := \begin{pmatrix} 0 & 0 & \frac{1}{24(2t-1)} \\ 2(2t-1) & 0 & 0 \\ 0 & -2(2t-1) & 0 \end{pmatrix},$$

or the pairs

$$\Big(\big(-4(8t-1)^2(2t-1), L/F, \rho\big), \mathsf{span}_F\langle \xi_1, \eta_1, \zeta_1\rangle\Big)$$

for all non-trivial isomorphism classes $[(L,\rho)]$ *of Galois* $\mathbb{Z}/3$*-algebras, where* ξ_1 *and* $L = F(\theta)$ *generates the cyclic algebra in such a way that* $\xi_1\theta = \rho(\theta)\xi_1$, $\theta^3 - 3\theta \in F$ *and* $\xi_1^3 = -4(8t-1)^2(2t-1)$, *and*

$$\begin{aligned} \eta_1 &= \Big(\frac{2}{3}(8t-1)(2t-1) - \frac{2}{3}(2t-1)\xi_1 + \frac{1}{6}\xi_1^2\Big)\theta, \\ \zeta_1 &= \Big(\frac{2}{3}(8t-1)(2t-1) - \frac{2}{3}(2t-1)\xi_1 + \frac{1}{6}\xi_1^2\Big)\rho(\theta). \end{aligned}$$

The associated cubic forms are semi-trace forms and not semi-diagonal.

4.5 Classification of cubic pairs of the second kind

We classify up to F-isomorphism the non-singular cubic pairs (A,V) over F such that $\mathsf{Aut}(A,V)(F_{\mathsf{sep}}) \cong \mathbb{Z}/3 \times \mathbb{Z}/3$ as an abstract group, i.e. the F-cubic pairs (A,V) such that $(A,V)_{F_{\mathsf{sep}}} \cong (\mathsf{M}_3(F_{\mathsf{sep}}), V_{\frac{1}{6}})$. Put $w := w_1(\frac{1}{6})$, $A := \mathsf{M}_3(F)$ and $V := \mathsf{span}_F\langle u, v, w\rangle$. Then

$$\mathsf{Aut}(A,V)(F_{\mathsf{sep}}) = \{m_1^i m_2^j F_{\mathsf{sep}}^\times \mid i,j = 0,1,2\}$$

with

$$m_1 = \begin{pmatrix} 1 & 0 & -6 \\ 1 & -2 & 0 \\ \frac{1}{2} & -1 & 1 \end{pmatrix} \quad \text{and} \quad m_2 = \begin{pmatrix} 1 & 2(\omega-1) & -6\omega \\ 0 & \omega & 2(\omega^2-\omega) \\ 0 & 0 & \omega^2 \end{pmatrix}$$

and the mappings $m_1F_{\mathsf{sep}}^\times \mapsto (1+3\mathbb{Z}, 1)$ and $m_2F_{\mathsf{sep}}^\times \mapsto (3\mathbb{Z}, \omega)$ define a Γ-group isomorphism from $\mathsf{Aut}(A,V)(F_{\mathsf{sep}})$ to $\mathbb{Z}/3 \times \mu_3$. By Theorem 4.1.2, there is a bijection

$$\mathsf{H}^1(F, \mathbb{Z}/3 \times \mu_3) \longleftrightarrow \left\{ \begin{array}{c} F\text{-isomorphism classes of} \\ \text{the } F\text{-cubic pairs which are} \\ \text{isomorphic to } (A,V)_{F_{\mathsf{sep}}} \text{ over } F_{\mathsf{sep}} \end{array} \right\}.$$

Since the action of Γ on $\mathbb{Z}/3 \times \mu_3$ restricts to $\mathbb{Z}/3$ and μ_3, we have

$$\mathsf{H}^1(F, \mathbb{Z}/3 \times \mu_3) \cong \mathsf{H}^1(F, \mathbb{Z}/3) \times \mathsf{H}^1(F, \mu_3).$$

The characteristic of F is different from 3, thus

$$\mathsf{H}^1(F, \mu_3) \cong F^\times / F^{\times 3}$$

(see (30.1) in [Knus *et al.*, 1998]). So, there is a one to one correspondence between the F-isomorphism classes of non-singular F-cubic pairs of the second kind, and the product $\mathsf{Isom}(\mathbb{Z}/3\text{-}\mathsf{Gal}_F) \times \big(F^\times/F^{\times 3}\big)$, where $\mathsf{Isom}(\mathbb{Z}/3\text{-}\mathsf{Gal}_F)$ denotes the set of isomorphism classes of Galois $\mathbb{Z}/3$-algebras over F.

Let (L_1, ρ_1) be a Galois $\mathbb{Z}/3$-algebra and $d_2F^{\times 3} \in F^\times/F^{\times 3}$. Let $(a_{i,\sigma}F^\times_{\mathsf{sep}})_{\sigma\in\Gamma}$ be the 1-cocycle with values in $\{F^\times_{\mathsf{sep}}, m_iF^\times_{\mathsf{sep}}, m_i^2F^\times_{\mathsf{sep}}\}$ corresponding to (L_1, ρ_1) for $i = 1$, and $d_2F^{\times 3}$ for $i = 2$. Let $\theta_2 \in F_{\mathsf{sep}}$ be such that $\theta_2^3 = d_2$ and put $a_\sigma F^\times_{\mathsf{sep}} := a_{1,\sigma}a_{2,\sigma}F^\times_{\mathsf{sep}}$. The F-cubic pair corresponding to the 1-cocycle $(a_\sigma F^\times_{\mathsf{sep}})_{\sigma\in\Gamma}$ is the pair (A', V') with

$$\begin{aligned} A' &= \{\xi \in A_{\mathsf{sep}} \mid a_\sigma\sigma(\xi)a_\sigma^{-1} = \xi \ \text{ for all } \sigma \in \Gamma\}, \\ V' &= \{\xi \in V_{\mathsf{sep}} \mid a_\sigma\sigma(\xi)a_\sigma^{-1} = \xi \ \text{ for all } \sigma \in \Gamma\}. \end{aligned}$$

First we assume that F contains a primitive cube root of unity. If L_1 is a field then there exists $\theta_1 \in L_1$ such that $L_1 = F(\theta_1)$, $\theta_1^3 = d_1 \in F$ and $\rho_1(\theta_1) = \omega\theta_1$. If $L_1 = F_1^3$ we put $\theta_1 := 1$. Suppose that $L_1 \cong F(\theta_2) \neq F$ and $d_2F^{\times 3} = d_1F^{\times 3}$. Then we may assume that $d_1 = d_2$, $\theta_1 = \theta_2$ and

$$a_\sigma = \begin{cases} 1 & \text{if } \sigma(\theta_1) = \theta_1, \\ m_1m_2 & \text{if } \sigma(\theta_1) = \omega\theta_1, \\ m_1^2m_2^2 & \text{if } \sigma(\theta_1) = \omega^2\theta_1. \end{cases}$$

If $L_1 \cong F(\theta_2) \neq F$ and $d_2F^{\times 3} = d_1^2F^{\times 3}$, then we may assume that $d_2 = d_1^2$, $\theta_2 = \theta_1^2$ and

$$a_\sigma = \begin{cases} 1 & \text{if } \sigma(\theta_1) = \theta_1, \\ m_1m_2^2 & \text{if } \sigma(\theta_1) = \omega\theta_1, \\ m_1^2m_2 & \text{if } \sigma(\theta_1) = \omega^2\theta_1. \end{cases}$$

Now we suppose that $L_1 \not\cong F(\theta_2)$. Then we may assume that $a_{i,\sigma} = 1$ if $d_i \in F^{\times 3}$ and

$$a_{i,\sigma} = \begin{cases} 1 & \text{if } \sigma(\theta_i) = \theta_i, \\ m_i & \text{if } \sigma(\theta_i) = \omega\theta_i, \\ m_i^2 & \text{if } \sigma(\theta_i) = \omega^2\theta_i, \end{cases}$$

if $d_i \notin F^{\times 3}$. Put

$$\begin{aligned} \xi_0 &:= 12w\theta_2^2, \\ \eta_0 &:= \Big((\omega^2 - \omega)u + \frac{1}{2}v + 6w\Big)\theta_1\theta_2, \\ \zeta_0 &:= \Big((\omega - \omega^2)u + \frac{1}{2}v + 6w)\Big)\theta_1^2\theta_2. \end{aligned}$$

Then ξ_0, η_0, ζ_0 are linearly independent vectors of V' such that

$$\xi_0^3 = -4d_2^2, \ \ \eta_0^3 = -2d_1d_2, \ \ \xi_0\eta_0 = \omega\eta_0\xi_0.$$

So $V' = \mathsf{span}_F\langle \xi_0, \eta_0, \zeta_0\rangle$ and $A' = (-4d_2^2, -2d_1d_2)_{\omega,F}$ is the F-algebra generated by ξ_0 and η_0. We have

$$\zeta_0 = \frac{\omega^2}{2d_2}\xi_0\eta_0^2$$

and the associated cubic form $f_{A',V'}$ is diagonal:

$$f_{A',V'} = a\xi_0^{\star 3} + b\eta_0^{\star 3} + c\zeta_0^{\star 3}$$

where $a = -4d_2^2$, $b = -2d_1d_2$ and $c = -2d_1^2d_2$ and $(\xi_0^\star, \eta_0^\star, \zeta_0^\star)$ denotes the dual basis of (ξ_0, η_0, ζ_0). The scalars a, b, c satisfy the relation

$$\frac{abc}{a^2} = (-d_1)^3 \in F^{\times 3}.$$

We proved the following:

Theorem 4.5.1 *Assume that F contains a primitive cube root of unity. Then, up to F-isomorphism, the non-singular F-cubic pairs of the second kind are the pairs*

$$\big((-4d_2^2, -2d_1d_2)_{\omega,F}, \mathsf{span}_F\langle \xi_0, \eta_0, \zeta_0\rangle\big),$$

for all $d_1F^{\times 3}, d_2F^{\times 3} \in F^\times/F^{\times 3}$ where ξ_0, η_0 are generators of the symbol algebra such that $\xi_0^3 = -4d_2^2$, $\eta_0^3 = -2d_1d_2$, $\xi_0\eta_0 = \omega\eta_0\xi_0$, and $\zeta_0 = \xi_0\eta_0^2$. The cubic forms associated to these cubic pairs are diagonal.

Now we assume that F is an infinite field which does not contain a primitive cube root of unity. In the case where L_1/F is a Galois extension of degree 3, there exists $\theta_1 \in L_1$ such that $L_1 = F(\theta_1)$ and the minimal polynomial of θ_1 over F is $x^3 - 3x + \lambda_1$ for some $\lambda_1 \in F \setminus \{2, -2\}$ with $(4 - \lambda_1^2)/3 \in F^2$. Let $\theta_1' = \rho_1(\theta_1)$ and $\theta_1'' = \rho_1^2(\theta_1)$ be the other roots of $x^3 - 3x + \lambda_1$. We may choose $x_1 \in F$ with $x_1^2 = (4 - \lambda_1^2)/3$ such that

$$\begin{aligned}
\theta_1 &= -\phi_1 - \phi_1^{-1} \\
\theta_1' &= -\omega\phi_1 - \omega^2\phi_1^{-1} = \frac{-\theta_1 + \delta_1}{2} \\
\theta_1'' &= -\omega^2\phi_1 - \omega\phi_1^{-1} = \frac{-\theta_1 - \delta_1}{2},
\end{aligned}$$

where $\delta_1 = x_1^{-1}(2\theta_1^2 + \lambda_1\theta_1 - 4)$ and ϕ_1 is a cube root of $(\lambda_1 + (\omega - \omega^2)x_1)/2$ in F_{sep}. In the case where $L_1 = F^3$, we put $\theta_1 := -2$, $\delta_1 := 0$ and $\phi_1 := 1$

(then we have $\theta_1^3 - 3\theta_1 + \lambda_1 = 0$ with $\lambda_1 = 2$). We can observe that $L_1 \not\cong F(\theta_2)$ since either L_1/F is an Galois extension of degree 3 or $L_1 = F^3$ and $F(\theta_2)$ is a field which is not a Galois extension of degree 3 over F. If $d_2 \notin F^{\times 3}$, let $\rho_2 \in \mathsf{Gal}(F(\theta_2)/F)$ be defined by $\rho_2(\theta_2) = \omega\theta_2$. Then, we may assume that $a_{i,\sigma} = 1$ if $\theta_i \in F$, and

$$a_{i,\sigma} = \begin{cases} 1 & \text{if } \sigma|_{L_i} = \mathsf{id}_{L_i}, \\ m_i & \text{if } \sigma|_{L_i} = \rho_i, \\ m_i^2 & \text{if } \sigma|_{L_i} = \rho_i^2, \end{cases}$$

otherwise. Put

$$\begin{aligned} \xi_0 &:= 12\theta_2^2 w, \\ \eta_0 &:= -\delta_1\theta_2 u + \frac{1}{2}\theta_1\theta_2 v + 6\theta_1\theta_2 w, \\ \zeta_0 &:= 3\theta_1\theta_2 u + \frac{1}{2}\delta_1\theta_2 v + 6\delta_1\theta_2 w. \end{aligned}$$

Then ξ_0, η_0, ζ_0 are linearly independent vectors of V', so V' is the span of ξ_0, η_0, ζ_0. Put

$$\eta_1 := -\frac{1}{2}\eta_0 + \frac{\omega - \omega^2}{6}\zeta_0 = \Big((\omega^2 - \omega)u + \frac{1}{2}v + 6w\Big)\phi_1\theta_2,$$

then

$$\xi_0^3 = -4d_2^2,\ \eta_1^3 = -2\phi_1^3 d_2,\ \xi_0\eta_1 = \omega\eta_1\xi_0.$$

So $A'_{F(\omega)}$ is the $F(\omega)$-algebra $(-4d_2^2, -2\phi_1^3 d_2)_{\omega,F(\omega)}$ generated by ξ_0 and η_1. Put $\eta_2 := (2d_2)^{-1}\xi_0\eta_1$, then $\eta_2^3 = \phi_1^3$.

Suppose that $L_1 = F^3$, then $A'_{F(\omega)}$ is split. But $F(\omega)/F$ is a field extension of degree 2, thus A' is also split. The vector space V' is spanned by $12\theta_2^2 w, \theta_2 u$ and $\theta_2(v + 12w)$. Put

$$n := \begin{pmatrix} \theta_2^{-2} & -2\theta_2^{-2} & 2\theta_2^{-2} \\ 0 & \theta_2^{-1} & -2\theta_2^{-1} \\ 0 & 0 & 1 \end{pmatrix}$$

then n is an invertible matrix such that

$$n(\theta_2^2 u)n^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ n(12\theta_2^2 w)n^{-1} = \begin{pmatrix} 0 & 0 & 4 \\ d_2 & 0 & 0 \\ 0 & -d_2 & 0 \end{pmatrix},$$

$$\text{and } n\big(\theta_2(v + 12w)\big)n^{-1} = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ d_2 & 0 & 0 \end{pmatrix}.$$

Thus (A', V') is isomorphic to $\bigl(\mathsf{M}_3(F), \mathsf{span}_F\langle \xi_0', \eta_0', \zeta_0'\rangle\bigr)$ with

$$\xi_0' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},\ \eta_0' = \begin{pmatrix} 0 & 0 & 4 \\ d_2 & 0 & 0 \\ 0 & -d_2 & 0 \end{pmatrix},\ \zeta_0' = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ d_2 & 0 & 0 \end{pmatrix}.$$

Now we assume that L_1 is a field. Put $\eta_3 := -\eta_2 - \eta_2^{-1}$ then $L' = F(\eta_3)$ is a Galois extension of degree 3 over F with Galois group generated by τ where $\tau(\eta_3) = -\omega\eta_2 - \omega^2\eta_2^{-1}$. Moreover L' is contained in A' because $\eta_2\overline{\eta_2} = 1$. Since $\xi_0\eta_2 = \omega\eta_2\xi_0$, we have $\xi_0\xi = \tau(\xi)\xi_0$ for all $\xi \in L'$. So A' is the cyclic F-algebra $(-4d_2^2, L'/F, \tau)$ generated by ξ_0 and η_3. Note that the Galois $Z/3$-algebras (L', τ) and (L_1, ρ_1) are isomorphic. Also we have

$$\begin{aligned} \eta_0 &= -\frac{1}{2d_2}\xi_0^2\eta_3, \\ \zeta_0 &= -\frac{1}{2d_2}\xi_0^2\bigl(\eta_3 + 2\rho(\eta_3)\bigr). \end{aligned}$$

The form $f_{A',V'}$ is a semi-trace form:

$$f_{A',V'}(\xi) = \mathsf{Tr}_{K/F}\bigl(a\Theta(\xi)^3\bigr)$$

where $K = F \times F(\omega)$, the map $\Theta\colon V' \to K$ is defined by

$$\Theta(x\xi_0 + y\eta_0 + z\zeta_0) = \bigl(x, y + (\omega - \omega^2)z\bigr),$$

$a = (-4d_2^2, 2\phi_1^3 d_2)$. We have the relation

$$\frac{\mathsf{N}_{K/F}(a)}{(-4d_2^2)^2} = (-1)^3 \in F^{\times 3}$$

and again $f_{A',V'}$ is not semi-diagonal.

We observe that, if $L_1 = F^3$, then

$$\eta_0 = -\frac{1}{2d2}\xi_0^2\eta_3,\ \ \zeta_0 = -\frac{1}{2d2}\xi_0^2(\eta_3 + 2\eta_3')$$

where $\eta_3 = -\eta_2 - \eta_2^{-1}$ and $\eta_3' = -\omega\eta_2 - \omega^2\eta_2^{-1}$. Put $L' := F \oplus F\eta_3 \oplus F\eta_3'$ and let ρ be the F-algebra automorphism of L' defined by $\rho(\eta_3) = \eta_3'$ and $\rho(\eta_3') = -\eta_3 - \eta_3'$. Then $1, \eta_3$ and $\rho(\eta_3)$ span L' such that

$$(x - \eta_3)\bigl(x - \rho(\eta_3)\bigr)\bigl(x - \rho^2(\eta_3)\bigr) = x^3 - 3x + \lambda$$

for $\lambda = 2$, and there exists an F-algebra isomorphism Ψ from L' to F^3 such that

$$\rho\bigl(\Psi^{-1}(1,0,0)\bigr) = \Psi^{-1}(0,1,0) \quad \text{and} \quad \rho\bigl(\Psi^{-1}(0,1,0)\bigr) = \Psi^{-1}(0,0,1).$$

We also have $A = \bigoplus_{i=0}^{2} L'\xi_0^i$ with $\xi_0\xi = \rho(\xi)\xi_0$ for all $\xi \in L'$. We can prove that $f_{A',V'}$ is a semi-trace form and is not semi-diagonal if $L_1 = F^3$ by letting $\theta_1 = -2$, $\delta_1 = 0 = x_1$, $\lambda_1 = 2$ and $\phi_1 = 1$ in the relations that we found in the case where L_1 is a field.

Thus we arrive at:

Theorem 4.5.2 *Suppose that F is infinite and does not contain a primitive cube root of unity. Then, up to F-isomorphism, the non-singular F-cubic pairs of the second kind are either the pairs*

$$\big(\mathsf{M}_3(F), \mathsf{span}_F\langle \xi_0', \eta_0', \zeta_0'\rangle\big)$$

for all $d_2F^{\times 3} \in F^\times/F^{\times 3}$, where

$$\xi_0' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ \eta_0' = \begin{pmatrix} 0 & 0 & 4 \\ d_2 & 0 & 0 \\ 0 & -d_2 & 0 \end{pmatrix}, \ \zeta_0' = \begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 2 \\ d_2 & 0 & 0 \end{pmatrix},$$

or the pairs

$$\big((-4d_2^2, L_1/F, \rho_1), \mathsf{span}_F\langle \xi_0, \eta_0, \zeta_0\rangle\big)$$

for all $d_2F^{\times 3} \in F^\times/F^{\times 3}$ and for all non trivial isomorphism classes $[(L_1, \rho_1)]$ of Galois $\mathbb{Z}/3$-algebras, where ξ_0 and $L_1 = F(\theta_1)$ generate the cyclic algebra such that

$$\xi_0^3 = -4d_2^2, \ \theta_1^3 - 3\theta_1 \in F, \ \xi_0\theta_1 = \rho_1(\theta_1)\xi_0,$$

and $\eta_0 = \xi_0^2\theta_1$, $\zeta_0 = \xi_0^2\rho_1(\theta_1)$. The associated cubic forms are semi-trace forms and are not semi-diagonal.

5

Classification of singular cubic pairs

We classify the isomorphism classes of singular cubic pairs. We split the classification into nine parts corresponding to the zero cubic curve and the eight different kinds of non-zero singular cubic curves. In the case of the triangle we use Galois cohomology but for the rest we use another method.

5.1 A useful proposition

To classify non-singular cubic pairs, we shall use an easier method than the one used for the classification of non-singular cubic pairs. Clearly, if two non-singular cubic pairs are isomorphic, their associated cubic curves are singular curves of the same kind. Thus we may split the classification of non-singular cubic pairs into nine parts. Before we give the new method, we need preliminaries.

We have a continuous action of Γ on $\mathbb{P}(\mathsf{M}_3(F_{\mathsf{sep}}))$ induced by the action on $\mathsf{M}_3(F_{\mathsf{sep}})$:

$$\sigma(uF_{\mathsf{sep}}) = \sigma(u)F_{\mathsf{sep}}.$$

Lemma 5.1.1 *We have that* $\mathbb{P}(\mathsf{M}_3(F_{\mathsf{sep}}))^\Gamma = \mathbb{P}(\mathsf{M}_3(F))$.

Proof : Let $p = uF_{\mathsf{sep}} \in \mathbb{P}(\mathsf{M}_3(F_{\mathsf{sep}}))^\Gamma$. Then $\sigma(uF_{\mathsf{sep}}) = uF_{\mathsf{sep}}$ for all $\sigma \in \Gamma$; thus there exists a scalar $\lambda_\sigma \in F_{\mathsf{sep}}^\times$ such that $\sigma(u) = \lambda_\sigma u$. Because $\sigma\tau(u) = \sigma(\tau(u))$, we have

$$\lambda_{\sigma\tau} = \lambda_\sigma \sigma(\lambda_\tau).$$

Hence $(\lambda_\sigma)_{\sigma\in\Gamma}$ is a 1-cocycle with values in $F_{\mathsf{sep}}^\times$. By Hilbert's Theorem 90, there exists $\mu \in F_{\mathsf{sep}}^\times$ such that $\lambda_\sigma = \mu\sigma(\mu)^{-1}$. Hence we can deduce that $\mu u \in \mathsf{M}_3(F)$:

$$\sigma(\mu u) = \sigma(\mu)\lambda_\sigma u = \mu u;$$

and therefore $p \in \mathbb{P}(\mathsf{M}_3(F))$. $\square$

Similarly if $\varphi \in V^\star_{\mathsf{sep}}$ is such that ${}^\sigma\varphi = \lambda_\sigma\varphi$ for all $\sigma \in \Gamma$ then there exists $\mu \in F^\times_{\mathsf{sep}}$ such that $\mu\varphi \in V^\star$.

By section 1.6, the number of singular points of a cubic curve with finitely many singular points is less than or equal to three and the singular points are defined over F_{sep}. Note that the action of Γ on $\mathbb{P}\big(\mathsf{M}_3(F_{\mathsf{sep}})\big)$ permutes the singular F_{sep}-points of a cubic curve over F.

We deduce the following proposition from the previous lemma.

Proposition 5.1.2 *Suppose that (A, V) is a singular cubic pair over F where A is a division algebra. Then the associated cubic curve is a triangle.*

Proof : Since A is division, there are no F-points on the cubic curve $\{f_{A,V}(\xi) = 0\}$. Indeed, if uF is a F-point of $\{f_{A,V}(\xi) = 0\}$, then $u \in A$ and $u^3 = 0$ which is impossible in a division algebra. In particular the cubic curve $f_{A,V}$ is non-zero.

Suppose that the cubic curve $\{f_{A,V}(\xi) = 0\}$ has one singular point in $\mathbb{P}(V_{\mathsf{sep}})$. Then the action of Γ on $\mathbb{P}(V_{\mathsf{sep}})$ leaves this singular point invariant. By Lemma 5.1.1 the singular point is defined over F and in particular there exists an F-point of $\{f_{A,V}(\xi) = 0\}$; this is impossible since A is division. Suppose that $\{f_{A,V}(\xi) = 0\}$ has two singular points in $\mathbb{P}(V_{\mathsf{sep}})$. Then the action of Γ on the singular points is not trivial since otherwise there would exist an F-point of $\{f_{A,V}(\xi) = 0\}$. Thus the subgroup of Γ which leaves the singular points invariant, has index two. So, there exists a point of $\{f_{A,V}(\xi) = 0\}$ over a quadratic extension of F. But A remains division after extending the scalars to a quadratic extension, thus we get a contradiction. Now if $f_{A,V} = l_1^2 \cdot l_2$ for some $l_1, l_2 \in V^\star_{\mathsf{sep}}$, then Γ leaves the line $\{l_1(\xi) = 0\}$ invariant. Thus it is defined over F and in particular there is a point of $\{f_{A,V}(\xi) = 0\}$ over F; this is impossible. Therefore, by Section 1.6, the curve $\{f_{A,V}(\xi) = 0\}$ is a triangle. $\square$

Hence, if (A, V) is a singular cubic pair over F such that the associated cubic curve is not a triangle, then $A \cong \mathsf{M}_3(F)$. To classify such cubic pairs up to isomorphism we may thus assume that the algebra of the cubic pair is $\mathsf{M}_3(F)$. In the case where the associated cubic curve is a triangle, we use Theorem 4.1.2.

The following lemma shall be useful to classify singular cubic pairs.

Lemma 5.1.3 *Let V be a cubic subspace of $\mathsf{M}_3(F)$ and (e_1, e_2, e_3) a basis of V. Then $f_V(x_1e_1 + x_2e_2 + x_3e_3)$ is equal to*

$$\sum_{i=1}^{3} e_i^3 x_i^3 + \sum_{i \neq j = 1}^{3} \mathsf{tr}(e_i^2 e_j) x_i^2 x_j + \mathsf{tr}(e_1e_2e_3 + e_1e_3e_2)x_1x_2x_3.$$

Proof : We have $f_V(x_1e_2 + x_2e_2 + x_3e_3) = (x_1e_1 + x_2e_2 + x_3e_3)^3$. The fact that

$$(x_1e_1 + x_2e_2 + x_3e_3)^3 = \frac{1}{3}\mathsf{tr}\Big((x_1e_2 + x_2e_2 + x_3e_3)^3\Big)$$

implies the result. □

5.2 Zero projective curve

We shall describe up to F-isomorphism the cubic pairs over F with the zero projective curve as associated cubic curve.

Suppose that V is a cubic subspace of $\mathsf{M}_3(F)$ such that $f_V = 0$. We want to describe V up to conjugacy. Suppose that $\xi^2 = 0$ for all $\xi \in V$ then we may assume that

$$u := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in V.$$

For $\xi \in \mathsf{M}_3(F_{\mathsf{sep}})$, let ξ_{ij} denote[1] the scalar on row i and column j in ξ. Since $f_{A,V} = 0$, we have $\mathsf{tr}(\xi\eta\zeta + \xi\zeta\eta) = 0$ for all $\xi, \eta, \zeta \in V$. Let $v, w \in V$ be such that (u, v, w) is a basis of V. Because $\mathsf{tr}(v) = 0$ and $\mathsf{tr}(uv) = 0$ we have $v_{33} = -v_{11} - v_{22} = 0$ and $v_{31} = 0$. Since $v^3 = (u + v)^3 = 0$ we have $v^2 = 0$ and $uv + vu = 0$. But $uv + vu = 0$ implies $v_{32} = 0$, $v_{22} = 0$, $v_{21} = 0$, and $v^2 = 0$ implies $v_{11} = 0$ and $v_{12}v_{23} = 0$. Replacing v by $v - v_{13}u$ if necessary, we may assume that $v_{13} = 0$, hence

$$v = \begin{pmatrix} 0 & v_{12} & 0 \\ 0 & 0 & v_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

[1]This notation shall be used in the remainder of this chapter.

In the same way, we can prove that

$$w = \begin{pmatrix} 0 & w_{12} & 0 \\ 0 & 0 & w_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

with $w_{12}w_{23} = 0$. Since v and w are linearly independent there exists a matrix $\xi \in V$ such that $\xi^2 \neq 0$ and it contradicts the assumption. Therefore there exists a matrix $u \in V$ such that $u^2 \neq 0$. Because $u^3 = 0$ we may assume that

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Suppose that u^2 and v^2 are linearly independent. We need the following lemma.

Lemma 5.2.1 *Let V be a cubic subspace of $\mathsf{M}_3(F)$ and $u, v \in V$. Suppose that $(xu + yv)^3 = 0$ for all $x, y \in F$ and u^2, v^2 are linearly independent, then there exist $m \in \mathsf{GL}_3(F)$ and non-zero $\lambda, \mu \in F$ such that*

$$mum^{-1} = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad mvm^{-1} = \mu \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Proof: Since $u^3 = 0$ and $u^2 \neq 0$, we may assume that

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Because $\mathsf{tr}(v) = 0$, $\mathsf{tr}(uv) = 0$ and $\mathsf{tr}(u^2v) = 0$, we have

$$v_{33} = -v_{11} - v_{22}, \; v_{32} = -v_{21}, \; v_{31} = 0.$$

Then $\mathsf{tr}(uv^2) = 0$ implies $v_{21}(2v_{11} + v_{22}) = 0$. If $v_{21} = 0$, then $\mathsf{tr}(v^2) = 0$ and $v^3 = 0$ imply $v_{11} = v_{22} = 0$; this contradicts the fact that u^2 and v^2 are linearly independent. So $v_{21} \neq 0$ and $v_{22} = -2v_{11}$. Replacing v by $v_{21}^{-1}v$ if necessary, we may assume that $v_{21} = 1$. We have $v_{23} = v_{12} + 3v_{11}^2$ and $v_{13} = v_{11}^3$ because $\mathsf{tr}(v^2) = 0$ and $v^3 = 0$. Put $\alpha := v_{11}$, $\beta := v_{12}$ and

$$m := \begin{pmatrix} 1 & -\alpha & 2\alpha^2 + \beta \\ 0 & 1 & -\alpha \\ 0 & 0 & 1 \end{pmatrix}$$

then $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$, $mum^{-1} = u$ and

$$mvm^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

which concludes the proof. □

By this lemma, we may now assume that

$$v = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Since $\mathsf{tr}(w) = 0$, $\mathsf{tr}(uw) = 0$, $\mathsf{tr}(u^2w) = 0$, $\mathsf{tr}(vw) = 0$ and $\mathsf{tr}(v^2w) = 0$ we have

$$w_{33} = -w_{11} - w_{22},\ w_{32} = -w_{21},\ w_{31} = 0,\ w_{23} = w_{12},\ w_{13} = 0;$$

thus

$$w = \begin{pmatrix} w_{11} & w_{12} & 0 \\ w_{21} & w_{22} & w_{12} \\ 0 & -w_{21} & -w_{11} - w_{22} \end{pmatrix}.$$

Replacing w by $w - w_{12}u - w_{21}v$ if necessary, we may assume that $w_{12} = 0$ and $w_{21} = 0$. So $\mathsf{tr}(w^2)$ implies $w_{22} = \rho w_{11}$ for some primitive cube root $\rho \in F_{\mathsf{sep}}$ of unity and $w^3 = 0$ implies $w = 0$. We get a contradiction, thus u^2 and v^2 are linearly dependent. Since $\mathsf{tr}(v) = 0$, $\mathsf{tr}(uv) = 0$ and $\mathsf{tr}(u^2v) = 0$ we have

$$v_{33} = -v_{11} - v_{22},\ v_{32} = -v_{21},\ v_{31} = 0.$$

But $v^2 = \lambda u^2$ for some $\lambda \in F$, so

$$v_{11} = 0,\ v_{21} = 0,\ v_{22} = 0$$

and v is an upper triangular matrix. Similarly, since u^2 and w^2 are linearly dependent, we can prove that w is an upper triangular matrix. Hence V is the subspace of $\mathsf{M}_3(F)$ of upper triangular matrices.

Theorem 5.2.2 *Up to F-isomorphism there exists one F-cubic pair such that the associated cubic curve is the zero projective curve, namely the pair $\big(\mathsf{M}_3(F), V\big)$ where V is the subspace of $\mathsf{M}_3(F)$ of upper triangular matrices.*

5.3 Triple line

We want to describe up to F-isomorphism the cubic pairs over F with a triple line as associated cubic curve. To do this it is sufficient to describe, up to conjugacy, the singular cubic subspaces of $\mathsf{M}_3(F)$, such that the associated cubic curve is a triple line.

Suppose that V is a singular cubic subspace of $\mathsf{M}_3(F)$ such that $\{f_V(\xi)=0\}$ is a triple line: $f_V = l^3$ for some non-zero $l \in V^\star_{\mathsf{sep}}$. Clearly, the group Γ acts trivially on $\{l(\xi)=0\}$, so $\{l(\xi)=0\}$ is defined over F. Hence $l = \mu\varphi$ for some $\mu \in F_{\mathsf{sep}}$ and $\varphi \in V^\star$. Thus $f_V = \lambda\varphi^3$ with $\lambda = \mu^3 \in F^\times$ since $f_V \in \mathsf{S}^3(V^\star)$ and $\varphi \in V^\star$. We describe V up to conjugacy.

Case 1: Suppose that there exists $u \in V$ such that $u^3 = 0$ and $u^2 \neq 0$. Then we may assume that

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Let $v, w \in V$ be such that u and v span the kernel of φ and $\varphi(w) = 1$. Then (u, v, w) is a basis of V and

$$(xu + yv + zw)^3 = \lambda z^3.$$

By Lemma 5.1.3, we have

$$\begin{cases} u^3 = v^3 = \mathsf{tr}(u^2v) = \mathsf{tr}(uv^2) = \mathsf{tr}(u^2w) = \mathsf{tr}(uw^2) = 0, \\ \mathsf{tr}(v^2w) = \mathsf{tr}(vw^2) = \mathsf{tr}(uvw + uwv) = 0 \text{ and } w^3 \neq 0. \end{cases}$$

For all $\xi \in V$, we have

$$\xi_{33} = -\xi_{11} - \xi_{22},\ \xi_{32} = -\xi_{21},\ \xi_{31} = 0,\ \xi_{21}(2\xi_{11} + \xi_{22}) = 0$$

since $\mathsf{tr}(\xi) = 0$, $\mathsf{tr}(u\xi) = 0$, $\mathsf{tr}(u^2\xi) = 0$ and $\mathsf{tr}(u\xi^2) = 0$. Replacing v by $v - v_{12}u$ and w by $w - w_{12}u$, we may assume that $v_{12} = 0$ and $w_{12} = 0$. Thus

$$v = \begin{pmatrix} v_{11} & 0 & v_{13} \\ v_{21} & v_{22} & v_{23} \\ 0 & -v_{21} & -v_{11} - v_{22} \end{pmatrix}, w = \begin{pmatrix} w_{11} & 0 & w_{13} \\ w_{21} & w_{22} & w_{23} \\ 0 & -w_{21} & -w_{11} - w_{22} \end{pmatrix}$$

and $v_{21}(2v_{11} + v_{22}) = 0 = w_{21}(2w_{11} + w_{22})$. Suppose $v_{21} \neq 0$, then $v_{22} = -2v_{11}$ and replacing v by $\frac{1}{v_{21}}v$ if necessary, we may assume that

$v_{21}=1$. Since $\mathsf{tr}(v^2)=0$ and $v^3=0$, we have $v_{23}=3v_{11}^2$ and $v_{13}=v_{11}^3$. Put $\alpha := v_{11}$, then

$$v=\begin{pmatrix} \alpha & 0 & \alpha^3 \\ 1 & -2\alpha & 3\alpha^2 \\ 0 & -1 & \alpha \end{pmatrix}.$$

If $w_{21}=0$, then $\mathsf{tr}(w^2)$ implies $w_{11}^2+w_{11}w_{22}+w_{22}^2=0$. So $w_{22}=\rho w_{11}$ for some $\rho\in F_{\mathsf{sep}}$ with $\rho^2+\rho+1=0$ and

$$w=\begin{pmatrix} w_{11} & 0 & w_{13} \\ 0 & \rho w_{11} & w_{23} \\ 0 & 0 & \rho^2 w_{11} \end{pmatrix}$$

with $w_{11}\neq 0$ because $w^3=\lambda\neq 0$. But $\mathsf{tr}(uvw+uwv)=0$ implies $(1-\rho^2)w_{11}=0$ thus we get a contradiction. Now if $w_{21}\neq 0$, then $w_{22}=-2w_{11}$. We have

$$w_{23}=3w_{11}^2w_{21}^{-1},\ w_{11}=\alpha w_{21},\ w_{13}=\alpha^3 w_{21}$$

because $\mathsf{tr}(w^2)=0$, $\mathsf{tr}(vw)=0$ and $\mathsf{tr}(v^2w)=0$. So

$$w=\begin{pmatrix} w_{21}\alpha & 0 & w_{21}\alpha^3 \\ w_{21} & -2w_{21}\alpha & 3w_{21}\alpha^2 \\ 0 & -w_{21} & w_{21}\alpha \end{pmatrix}=w_{21}v$$

and we have a contradiction; therefore $v_{21}=0$. Since $\mathsf{tr}(v^2)=0$ and $v^3=0$, we have $v_{22}=v_{11}=0$. So

$$v=\begin{pmatrix} 0 & 0 & v_{13} \\ 0 & 0 & v_{23} \\ 0 & 0 & 0 \end{pmatrix}.$$

If $v_{23}=0$, then we may assume that $v_{13}=1$. Replacing w by $w-w_{13}v$ if necessary, we may assume that $w_{13}=0$. Since $\mathsf{tr}(w^2v)=0$, $\mathsf{tr}(w^2)=0$ and $w^3\neq 0$, we have

$$w_{21}=0,\ w_{22}=\rho w_{11},\ w_{11}\neq 0$$

where $\rho\in F_{\mathsf{sep}}$ is a primitive cube root of unity. Note that $\rho\in F$ because $w_{11}\in F$, $\rho w_{11}\in F$ and $w_{11}\neq 0$. Put

$$m:=\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{w_{23}}{(\rho-\rho^2)w_{11}} \\ 0 & 0 & 1 \end{pmatrix}$$

then $m \in \mathsf{GL}_3(F)$ is such that $mum^{-1}, mvm^{-1} \in \mathsf{span}_F\langle u, v\rangle$ and

$$mwm^{-1} = w_{11}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}.$$

Hence V is conjugate to the subspace of $\mathsf{M}_3(F)$ spanned by

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}.$$

If $v_{23} \neq 0$, then we may assume that $v_{13} = 0$ and $v_{23} = 1$. Indeed, the invertible matrix

$$m = \begin{pmatrix} 1 & -v_{13}v_{23}^{-1} & 0 \\ 0 & 1 & -v_{13}v_{23}^{-1} \\ 0 & 0 & 1 \end{pmatrix}$$

is such that $mum^{-1} = u$ and

$$mvm^{-1} = v_{23}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Replacing w by $w - w_{23}v$, we may assume that $w_{23} = 0$. Then $\mathsf{tr}(w^2) = 0$, $\mathsf{tr}(vw) = 0$ and $w^3 \neq 0$ imply

$$w_{22} = \rho w_{11},\ w_{21} = 0,\ w_{11} \neq 0$$

where $\rho \in F$ is a primitive cube root of unity. We may assume that $w_{11} = 1$ and $w_{13} = 0$ because

$$m = \begin{pmatrix} 1 & 0 & \frac{w_{13}}{(1-\rho^2)w_{11}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \in \mathsf{GL}_3(F)$$

is such that $mum^{-1} = u$, $mvm^{-1} = v$ and

$$mwm^{-1} = w_{11}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}.$$

Theorem 5.3.1 *Suppose that F contains a primitive cube root of unity. The F-cubic pairs (A, V) such that the associated cubic curve is a triple* *line and there exists an element $\xi_0 \in V$ such that $\xi_0^3 = 0$ and $\xi_0^2 \neq 0$, are F-isomorphic to the pair $\big(\mathsf{M}_3(F), \mathsf{span}_F\langle u, v, w\rangle\big)$ where*

$$\textit{either } u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, v = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}$$

$$\textit{or } u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}.$$

for some primitive cube root of unity ρ. There are, up to isomorphism, four such cubic pairs.

Proof: Put

$$v_1 := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ v_2 := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ w_i = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho_i & 0 \\ 0 & 0 & \rho_i^2 \end{pmatrix},$$

for $i = 1, 2$, where ρ_i is a primitive cube root of unity. If $\mathsf{span}_F\langle u, v_1, w_1\rangle$ is conjugate to $\mathsf{span}_F\langle u, v_2, w_2\rangle$ then $\mathsf{span}_F\langle u, v_1\rangle$ and $\mathsf{span}_F\langle u, v_2\rangle$ are conjugate. We observe that $(\alpha u + \beta v_1)^2 = 0$ if and only if $\alpha = 0$ and $(\alpha u + \beta v_2)^2 = 0$ if and only if $\alpha = 0$ or $\alpha = -\beta$. Thus $\mathsf{span}_F\langle u, v_1\rangle$ is not conjugate to $\mathsf{span}_F\langle u, v_2\rangle$. By straightforward computations one can check that $\mathsf{span}_F\langle u, v_i, w_1\rangle$ is not conjugate to $\mathsf{span}_F\langle u, v_i, w_2\rangle$ if $\rho_1 \neq \rho_2$. □

Case 2: Assume $\xi^3 = 0$ implies $\xi^2 = 0$ for all $\xi \in V$. Let $u, v, w \in V$ be such that u and v span the kernel of φ and $\varphi(w) = 1$. Since $u^3 = 0$, we have $u^2 = 0$ and we may assume that

$$u = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Replacing v by $v - v_{13}u$ and w by $w - w_{13}u$, we may assume that $v_{13} = 0$ and $w_{13} = 0$. For $\xi \in V$, we have $\mathsf{tr}(\xi) = 0$ and $\mathsf{tr}(u\xi) = 0$, thus

$$\xi_{33} = -\xi_{11} - \xi_{22}, \ \xi_{31} = 0.$$

Since $v^3 = 0$ and $(u + v)^3 = 0$, we deduce that $v^2 = 0$ and $uv + vu = 0$. But $uv + vu = 0$ implies $v_{32} = v_{21} = v_{22} = 0$ and $v^2 = 0$ implies $v_{11} = 0$

and $v_{12}v_{23} = 0$. Hence

$$v = \begin{pmatrix} 0 & v_{12} & 0 \\ 0 & 0 & v_{23} \\ 0 & 0 & 0 \end{pmatrix}$$

and either $v_{12} = 0$ or $v_{23} = 0$.

If $v_{23} = 0$, then we may assume that $v_{12} = 1$. Replacing w by $w - w_{12}v$, we may assume that $w_{12} = 0$. Since $\mathsf{tr}(vw) = 0$ we have $w_{21} = 0$, thus

$$w = \begin{pmatrix} w_{11} & 0 & 0 \\ 0 & w_{22} & w_{23} \\ 0 & w_{32} & -w_{11} - w_{22} \end{pmatrix}.$$

Suppose that F contains a primitive cube root of unity. If $w_{32} \neq 0$ then $\mathsf{tr}(w^2) = 0$ implies $w_{23} = -w_{32}^{-1}(w_{11}^2 + w_{22}^2 + w_{11}w_{22})$. Put $\alpha := w_{11}w_{32}^{-1}$ and $\beta := w_{22}w_{32}^{-1}$, then $\alpha \neq 0$ because $w^3 \neq 0$. The matrix

$$m = \begin{pmatrix} (\omega - \omega^2)\alpha & 0 & 0 \\ 0 & 1 & \omega\alpha - \beta \\ 0 & 1 & \omega^2\alpha - \beta \end{pmatrix} \in \mathsf{GL}_3(F)$$

is such that $mum^{-1}, mvm^{-1} \in \mathsf{span}_F\langle u, v\rangle$ and

$$mwm^{-1} = \alpha w_{32} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}.$$

Now suppose that $w_{32} = 0$. We have $w_{22} = \rho w_{11}$ for some primitive cube root ρ of unity because $\mathsf{tr}(w^2) = 0$. Put $\alpha := w_{11}$, $\beta := w_{23}$ then $\alpha \neq 0$ because $w^3 \neq 0$. The invertible matrix

$$m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{\beta}{(\rho - \rho^2)\alpha} \\ 0 & 0 & 1 \end{pmatrix}$$

is such that $mum^{-1}, mvm^{-1} \in \mathsf{span}_F\langle u, v\rangle$ and

$$mwm^{-1} = \alpha \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}.$$

Hence we may assume that

$$w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}.$$

Since the invertible matrix

$$m' = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

is such that $m'um'^{-1}, m'vm'^{-1} \in \mathsf{span}_F\langle u, v\rangle$ and

$$m' \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} m'^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix},$$

the vector space V is conjugate to the subspace of $\mathsf{M}_3(F)$ spanned by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}.$$

Now suppose that F does not contain a primitive cube root of unity. If $w_{32} = 0$, then $\mathsf{tr}(w^2) = 0$ implies $w_{22} = \rho w_{11}$ for some primitive cube root of unity $\rho \in F_{\mathsf{sep}}$. Since $w^3 \neq 0$ we have $w_{11} \neq 0$ and it contradicts the fact that F does not contain ρ; thus $w_{32} \neq 0$. Because $\mathsf{tr}(w^2) = 0$, we have

$$w_{23} = -w_{32}^{-1}(w_{11}^2 + w_{22}^2 + w_{11}w_{22}).$$

Put $\alpha := w_{11}w_{32}^{-1}$ and $\beta := w_{22}w_{32}^{-1}$ then $\alpha \neq 0$ because $w^3 \neq 0$. The invertible matrix

$$m = \begin{pmatrix} -\alpha & 0 & 0 \\ 0 & \alpha^{-1} & -1-\alpha^{-1}\beta \\ 0 & \alpha^{-1} & -\alpha^{-1}\beta \end{pmatrix}$$

is such that $mum^{-1}, mwm^{-1} \in \mathsf{span}_F\langle u, v\rangle$ and

$$mwm^{-1} = \alpha w_{32} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

Therefore V is conjugate to the subspace of $\mathsf{M}_3(F)$ spanned by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}.$$

If $v_{12} = 0$, then we may assume that $v_{23} = 1$. Replacing w by $w - w_{23}v$, we may assume that $w_{23} = 0$. We have

$$w_{32} = 0,\ w_{11} + w_{22} \neq 0$$

because $\mathsf{tr}(vw) = 0$ and $w^3 \neq 0$. Suppose that F contains a primitive cube root of unity. If $w_{21} \neq 0$, then $\mathsf{tr}(w^2) = 0$ implies

$$w_{12} = -\frac{1}{w_{21}}(w_{11}^2 + w_{22}^2 + w_{11}w_{22}).$$

Put $\alpha := w_{11}w_{21}^{-1}$, $\beta := w_{22}w_{21}^{-1}$ and

$$m := \begin{pmatrix} 1 & \omega^2\alpha - \omega\beta & 0 \\ 1 & \omega\alpha - \omega^2\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then $m \in \mathsf{GL}_3(F)$, $mum^{-1}, mvm^{-1} \in \mathsf{span}_F\langle u, v\rangle$ and

$$mwm^{-1} = -w_{21}(\alpha+\beta)\begin{pmatrix} \omega & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Thus we may assume that $w_{21} = 0$. Since $\mathsf{tr}(w^2) = 0$, there exists a primitive cube root of unity ρ such that $w_{22} = \rho w_{11}$. Put $\alpha := w_{11}$ and $\beta := w_{12}$ then $\alpha \neq 0$ because $w^3 \neq 0$. The invertible matrix

$$m = \begin{pmatrix} 1 & \frac{\beta}{(1-\rho)\alpha} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is such that $m \in \mathsf{GL}_3(F)$, $mum^{-1}, mvm^{-1} \in \mathsf{span}_F\langle u, v\rangle$ and

$$mwm^{-1} = \alpha\begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}.$$

Therefore V is conjugate to the span of

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}.$$

We may assume that $\rho = \omega$ since

$$m' = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is such that $m'um'^{-1}, m'vm'^{-1} \in \mathsf{span}_F\langle u, v\rangle$ and

$$m' \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} m'^{-1} = \omega \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

Now assume that F does not contain a primitive cube root of unity, then $w_{21} \neq 0$ and we have $w_{12} = -w_{21}(w_{11}^2 + w_{22}^2 + w_{11}w_{22})$ since $\mathsf{tr}(w^2) = 0$. Put $\alpha := w_{11}w_{21}^{-1}$ and $\beta := w_{22}w_{21}^{-1}$ then $\alpha + \beta \neq 0$ because $w^3 \neq 0$. The invertible matrix

$$m = \begin{pmatrix} 0 & 1 & 0 \\ -(\alpha+\beta)^{-1} & \alpha(\alpha+\beta)^{-1} & 0 \\ 0 & 0 & \alpha+\beta \end{pmatrix}$$

is such that $mum^{-1}, mvm^{-1} \in \mathsf{span}_F\langle u, v\rangle$ and

$$mwm^{-1} = w_{21}(\alpha+\beta) \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Therefore V is conjugate to the subspace of $\mathsf{M}_3(F)$ spanned by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

We note that the subspace of $\mathsf{M}_3(F)$ spanned by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is not conjugate to the one spanned by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Thus, we have the following theorems:

Theorem 5.3.2 *Suppose that F contains a primitive cube root of unity. The cubic pairs (A, V) over F with a triple line as associated cubic curve and such that $\xi^3 = 0$ implies $\xi^2 = 0$ for all $\xi \in V$, are F-isomorphic to the pair $\big(\mathsf{M}_3(F), V\big)$ where V is the subspace of $\mathsf{M}_3(F)$ spanned by*

- *either* $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$
- *or* $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}.$

There are exactly two such cubic pairs up to isomorphism.

Theorem 5.3.3 *Suppose that F does not contain a primitive cube root of unity. The cubic pairs over F with a triple line as associated cubic curve are F-isomorphic to the pair $\big(\mathsf{M}_3(F), V\big)$ where V is the subspace of $\mathsf{M}_3(F)$ spanned by*

- *either* $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{pmatrix}$
- *or* $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$

There are exactly two such cubic pairs up to isomorphism.

5.4 Double line plus simple line

We want to describe up to conjugacy the F-cubic subspaces of $\mathsf{M}_3(F)$ such that the associated cubic curve is a double line plus simple line.

Thereto, suppose that V is a cubic subspace of $\mathsf{M}_3(F)$ such that the cubic curve $\{f_V(\xi) = 0\}$ is a double line plus simple line. There exist linearly independent $l_1, l_2 \in V^\star_{\mathsf{sep}}$ such that $f_V = l_1^2 l_2$. Clearly the lines $\{l_1(\xi) = 0\}$ and $\{l_2(\xi) = 0\}$ are invariant under the action of Γ. So there exist linearly independent $\varphi_i \in V^\star$ and $\mu_i \in F^\times_{\mathsf{sep}}$ such that $l_i = \mu_i \varphi_i$. Hence $f_V = \lambda \varphi_1^2 \varphi_2$ with $\lambda = \mu_1^2 \mu_2 \in F$. Replacing φ_2 by $\lambda^{-1}\varphi_2$ if necessary, we may assume that $\lambda = 1$. Let (u, v, w) be a basis of V such that $\varphi_1(u) = \varphi_2(v) = 1$, the vectors u, w span the kernel of φ_2 and v, w span the kernel of φ_1. Because $\mathsf{tr}(u^2 v) \neq 0$ we have $u^2 \neq 0$. Since $u^3 = 0$ we may assume that

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

For all $\xi \in V$, we have $\mathsf{tr}(\xi) = 0$ and $\mathsf{tr}(u\xi) = 0$, so

$$\xi_{33} = -\xi_{11} - \xi_{22},\ \xi_{32} = -\xi_{21}.$$

Case 1: Suppose that $v^2 \neq 0$. By Lemma 4.2.4 and since u and v are determinant zero matrices of V such that $v^2 \neq 0$, $\mathsf{tr}(u^2v) \neq 0$ and $\mathsf{tr}(uv^2) = 0$, we may assume that

$$v = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

Because $\mathsf{tr}(vw) = 0$, $\mathsf{tr}(u^2w) = 0$ and $\mathsf{tr}(v^2w) = 0$, we have

$$w_{23} = w_{12},\ w_{31} = 0,\ w_{13} = 0.$$

So $\mathsf{tr}(w^2) = 0$ implies $w_{22} = \rho w_{11}$ for some primitive cube root of unity $\rho \in F_{\mathsf{sep}}$, next $\mathsf{tr}(uvw+uwv) = 0$ implies $w_{12} = \frac{\rho^2-1}{2}w_{11}$ and $\mathsf{tr}(vw^2) = 0$ implies $w_{11} = 0$. Hence

$$w = w_{21}\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

and V is the subspace of $\mathsf{M}_3(F)$ spanned by

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Case 2: Suppose that $v^2 = 0$. We use the following lemma:

Lemma 5.4.1 *Let V be a cubic subspace of $\mathsf{M}_3(F)$ and $u, v \in V$ determinant zero matrices such that $v^2 = 0$ and $\mathsf{tr}(u^2v) \neq 0$. Then there exist $m \in \mathsf{GL}_3(F)$ and scalars $\lambda, \mu \in F^\times$ such that*

$$mum^{-1} = \lambda \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \textit{and} \quad mvm^{-1} = \mu \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Proof : Because $u^3 = 0$ and $u^2 \neq 0$, we may assume that

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\mathsf{tr}(v) = 0$, $\mathsf{tr}(uv) = 0$, $\mathsf{tr}(u^2 v) \neq 0$, we have

$$v_{33} = -v_{11} - v_{22},\ v_{32} = -v_{21},\ v_{31} \neq 0.$$

Put $\alpha := v_{11} v_{31}^{-1}$ and $\beta := v_{21} v_{31}^{-1}$ then $v^2 = 0$ implies

$$v = v_{31} \begin{pmatrix} \alpha & -\alpha\beta & \alpha(-\alpha + \beta^2) \\ \beta & -\beta^2 & \beta(-\alpha + \beta^2) \\ 1 & -\beta & -\alpha + \beta^2 \end{pmatrix}.$$

The invertible matrix

$$m = \begin{pmatrix} 1 & -\beta & -\alpha + \beta^2 \\ 0 & 1 & -\beta \\ 0 & 0 & 1 \end{pmatrix}$$

is such that $mum^{-1} = u$ and

$$mvm^{-1} = v_{31} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

□

Since $u, v \in V$ are determinant zero matrices such that $v^2 = 0$ and $\mathsf{tr}(u^2 v) \neq 0$, we may assume that

$$v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

We have

$$w_{31} = 0,\ w_{13} = 0,\ w_{23} = -w_{12}$$

because $\mathsf{tr}(u^2 w) = 0$, $\mathsf{tr}(vw) = 0$ and $\mathsf{tr}(uvw + uwv) = 0$. Then the relations $\mathsf{tr}(vw^2) = 0$, $\mathsf{tr}(w^2) = 0$ and $w^3 = 0$ imply $w_{12}, w_{11}, w_{22} = 0$. Thus

$$w = w_{21} \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}$$

and we obtain the same subspace as in the first case. So we obtain the following theorem:

Theorem 5.4.2 *Up to F-isomorphism, there is one F-cubic pair such that the associated cubic curve is a double line plus simple line, namely the pair* $(\mathsf{M}_3(F), \mathsf{span}_F\langle u, v, w\rangle)$ *where*

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; w = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

5.5 Three concurrent lines

We want to describe up to F-isomorphism the F-cubic pairs such that the associated cubic curve is three concurrent lines.

Suppose that V is a cubic subspace of $\mathsf{M}_3(F)$ such that the curve $\{f_V(\xi) = 0\}$ is three concurrent lines. Then there exists a basis (u, v, w) of V_{sep} such that

$$f_V(xu + yv + zw) = x^2y + xy^2.$$

Since $\mathsf{tr}(u^2v) = 1$ we have $u^2 \neq 0$. But $u^3 = 0$ so we may assume that

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

If $\xi \in V_{\mathsf{sep}}$, then

$$\xi_{33} = -\xi_{11} - \xi_{22}, \; \xi_{32} = -\xi_{21}$$

because $\mathsf{tr}(\xi) = 0$ and $\mathsf{tr}(u\xi) = 0$. We have

$$w_{31} = 0, \; w_{21}(2w_{11} + w_{22}) = 0$$

since $\mathsf{tr}(u^2w) = 0$ and $\mathsf{tr}(uw^2) = 0$. Suppose that $w_{21} = 0$, then we have $w_{22} = \rho w_{11}$ for some primitive cube root of unity $\rho \in F_{\mathsf{sep}}$ because $\mathsf{tr}(w^2) = 0$, and $w^3 = 0$ implies $w_{11} = 0$. Therefore $w^2 = w_{12}w_{23}u^2$ and $uw + wu = (w_{12} + w_{23})u^2$. But $\mathsf{tr}(u^2v) \neq 0$, $\mathsf{tr}(w^2v) = 0$ and $\mathsf{tr}(uwv + wuv) = 0$, so $w_{12} = w_{23} = 0$ and $w = w_{13}u^2$. Since $\mathsf{tr}(u^2v) \neq 0$ and $\mathsf{tr}(vw) = 0$, we have $w_{13} = 0$ and so $w = 0$; we get a contradiction. Thus we have $w_{21} \neq 0$, and u^2 and w^2 are linearly independent. By Lemma 5.2.1, we may assume that

$$w = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

We have

$$v_{31} = 1,\ v_{23} = v_{12},\ v_{13} = 0$$

since $\mathsf{tr}(u^2v) = 1$, $\mathsf{tr}(vw) = 0$ and $\mathsf{tr}(vw^2) = 0$. Then $\mathsf{tr}(v^2) = 0$, $\mathsf{tr}(uvw + uwv) = 0$ and $v^3 = 0$ imply

$$v = \begin{pmatrix} 0 & 0 & 0 \\ v_{21} & 0 & 0 \\ 1 & -v_{21} & 0 \end{pmatrix}$$

and this contradicts the fact that $\mathsf{tr}(uv^2) = 1$.

Therefore we obtain the following result:

Theorem 5.5.1 *There is no cubic pair over F such that the associated cubic curve is three concurrent lines.*

5.6 Conic plus tangent

We want to classify up to F-isomorphism the F-cubic pairs such that the associated cubic curve is a conic plus tangent.

Suppose that V is a cubic subspace of $\mathsf{M}_3(F)$ such that $\{f_V(\xi) = 0\}$ is a conic plus tangent. Then there exists a basis (u, v, w) of V_{sep} such that

$$f_V(xu + yv + zw) = (x^2 - yz)z.$$

Since $\mathsf{tr}(vw^2) \neq 0$ we have $w^2 \neq 0$. So u and w are determinant zero matrices such that $w^2 \neq 0$, $\mathsf{tr}(u^2w) \neq 0$ and $\mathsf{tr}(uw^2) = 0$ and by Lemma 4.2.4, we may assume that

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

Because $\mathsf{tr}(v) = 0$, $\mathsf{tr}(uv) = 0$, $\mathsf{tr}(u^2v) = 0$, $\mathsf{tr}(vw) = 0$ and $\mathsf{tr}(vw^2) = -1$, we deduce that

$$v = \begin{pmatrix} v_{11} & v_{12} & 1 \\ v_{21} & v_{22} & v_{12} + 1 \\ 0 & -v_{21} & -v_{11} - v_{22} \end{pmatrix}.$$

Then $\mathsf{tr}(uv^2) = 0$ implies $v_{21}(2v_{11} + v_{22}) = 0$. Suppose that $v_{21} = 0$, then $v_{11} = 0$, $v_{22} = 0$ and $v_{12} = -\frac{1}{2}$ because $\mathsf{tr}(v^2) = 0$, $v^3 = 0$ and $\mathsf{tr}(uvw+uwv) = 0$; it contradicts the fact that $\mathsf{tr}(v^2w) = 0$. Now suppose that $v_{21} \neq 0$, then $v_{22} = -2v_{11}$. We have $v_{12} = -\frac{1}{2}$ and $v_{21} = 3v_{11}^2$ because $\mathsf{tr}(uvw + uwv) = 0$ and $\mathsf{tr}(v^2) = 0$. Since $\mathsf{tr}(v^2w) = 0$ and $v^3 = 0$, we have

$$6v_{11}^2 - 3v_{11} + \frac{1}{4} = 0 \quad \text{and} \quad v_{11}^3(1 - 9v_{11}) = 0;$$

which is impossible.

We proved the following:

Theorem 5.6.1 *There is no F-cubic pair with a conic plus tangent as associated cubic curve.*

5.7 Conic plus chord

We want to classify the F-cubic pairs such that the associated cubic curve is a conic plus chord. To do this we need to describe a cubic form over V with a conic plus chord as associated cubic curve.

Proposition 5.7.1 *Let $f \in \mathsf{S}^3(V^\star)$ be a singular cubic form such that the cubic curve $\{f(\xi) = 0\}$ is a conic plus chord. Then there exist a basis (u, v, w) of V and non-zero scalars $a, b \in F$ such that*

$$f(xu + yv + zw) = (x^2 - ay^2 + bz^2)z.$$

Proof: There exist $q \in \mathsf{S}^2(V^\star_{\mathsf{sep}})$ and $l \in V^\star_{\mathsf{sep}}$ such that $f = q \cdot l$. Since Γ acts trivially on the line $\{l(\xi) = 0\}$ we may assume that $l \in V^\star$. Let (u, v, w) be a basis of V such that $l(w) = 1$. We know that the curve $\{f(\xi) = 0\}$ has two distinct singular points p_1 and p_2. Thus the subgroup of Γ which leaves the singular points invariant has index less than or equal to 2 and the singular points are defined over a quadratic extension of F. Let $a \in F^\times$ be such that the singular points are defined over $F(\sqrt{a})$ (if Γ acts trivially on the singular points, we may choose $a \in F^{\times 2}$). We know that the line $\{l(\xi) = 0\}$ passes through the singular points and is not the tangent at these points. Changing the basis if necessary we may assume that $p_1 = (\sqrt{a}u + v)\overline{F}$ and $p_2 = (-\sqrt{a}u + v)\overline{F}$ are the singular points and the tangents at these points intersect at $w\overline{F}$. Let $\lambda_{i,j,k} \in F$ be such that $q(xu + yv + zw)$ is equal to

$$\lambda_{2,0,0}x^2 + \lambda_{0,2,0}y^2 + \lambda_{0,0,2}z^2 + \lambda_{1,1,0}xy + \lambda_{1,0,1}xz + \lambda_{0,1,1}yz.$$

Since p_1 and p_2 lies on the conic $\{q(\xi) = 0\}$, we deduce that $\lambda_{1,1,0} = 0$ and $\lambda_{0,2,0} = -a\lambda_{2,0,0}$. The $\overline{F}$-points of the tangent to $\{q(\xi) = 0\}$ at

p_1 are the points $(xu+yv+zw)\overline{F}$ for all $x,y,z\in\overline{F}$ not all zero such that $x=\sqrt{a}y$, so $\sqrt{a}\lambda_{1,0,1}+\lambda_{0,1,1}=0$. Similarly, using the tangent to $\{q(\xi)=0\}$ at p_2, we deduce that $-\sqrt{a}\lambda_{1,0,1}+\lambda_{0,1,1}=0$, and thus $\lambda_{1,0,1}=\lambda_{0,1,1}=0$. Hence

$$f(xu+yv+zw)=(\lambda_{2,0,0}x^2-a\lambda_{2,0,0}y^2+\lambda_{0,0,2}z^2)z$$

with $\lambda_{2,0,0},\lambda_{0,0,2}\neq 0$ since q is irreducible. Replacing w by $\lambda_{2,0,0}^{-1}w$ we may assume that $\lambda_{2,0,0}=1$. Thus

$$f(ux+yv+zw)=(x^2-ay^2+bz^2)z$$

where $a,b\neq 0$. □

Let V be a cubic subspace of $\mathsf{M}_3(F)$ such that the curve $\{f_V(\xi)=0\}$ is a conic plus chord. By the previous lemma, there exist a basis (u,v,w) of V and non-zero $a,b\in F$ such that

$$f_V(xu+yv+zw)=(x^2-ay^2+bz^2)z.$$

Since $u^3=0$ and $\mathsf{tr}(u^2w)\neq 0$, we have $u^2\neq 0$ and we may assume that

$$u=\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}.$$

Then for $\xi\in V$, we have

$$\xi_{33}=-\xi_{11}-\xi_{22},\ \xi_{32}=-\xi_{21}$$

since $\mathsf{tr}(\xi)=0$ and $\mathsf{tr}(u\xi)=0$. Because $\mathsf{tr}(u^2v)=0$ and $\mathsf{tr}(uv^2)=0$, we deduce that

$$v_{31}=0,\ v_{21}(2v_{11}+v_{22}).$$

Suppose that $v_{21}=0$, then $\mathsf{tr}(v^2)=0$ and $v^3=0$ imply

$$v=\begin{pmatrix}0&v_{12}&v_{13}\\0&0&v_{23}\\0&0&0\end{pmatrix}.$$

So $v^2=v_{12}v_{23}u^2$ and $uv+vu=(v_{12}+v_{23})u^2$. But $\mathsf{tr}(u^2w)=1$, $\mathsf{tr}(v^2w)=-a$ and $\mathsf{tr}(uvw+uwv)=0$, thus $v_{23}=-v_{12}$ and $v_{12}^2=a$. Hence $a\in F^{\times 2}$ and we may assume that $a=1$ and $v_{12}=1$. Put

$$m:=\begin{pmatrix}1&\frac{v_{13}}{2}&0\\0&1&\frac{v_{13}}{2}\\0&0&1\end{pmatrix}$$

then m is an invertible matrix such that $mum^{-1} = u$ and

$$mvm^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

So we may assume that $v_{13} = 0$. Since $\mathsf{tr}(vw) = 0$, $\mathsf{tr}(u^2w) = 1$, $\mathsf{tr}(uw^2) = 0$, $\mathsf{tr}(vw^2) = 0$ and $\mathsf{tr}(w^2) = 0$, we have

$$w_{21} = 0,\ w_{31} = 1,\ w_{12} = 0,\ w_{23} = 0,\ w_{13} = -w_{11}^2 - w_{22}^2 - w_{11}w_{22}.$$

We have $w_{22} \neq 0$ because $w^3 \neq 0$. The invertible matrix

$$m = \begin{pmatrix} 1 & 0 & -w_{11} \\ 0 & 1 & 0 \\ 0 & 0 & w_{22} \end{pmatrix}$$

is such that $mum^{-1}, mvm^{-1} \in \mathsf{span}_F\langle u, v\rangle$ and

$$mwm^{-1} = w_{22} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

So V is conjugate to the subspace of $\mathsf{M}_3(F)$ spanned by

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}.$$

Now suppose that $v_{21} \neq 0$, then u^2 and v^2 are linearly independent. Since $(xu + yv)^3 = 0$ for all $x, y \in F_{\mathsf{sep}}$, by Lemma 5.2.1 we may assume that

$$v = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Since we know that $\mathsf{tr}(u^2w) = 1$, $\mathsf{tr}(vw) = 0$, $\mathsf{tr}(uvw + uwv) = 0$, $\mathsf{tr}(uw^2) = 0$, $\mathsf{tr}(w^2) = 0$, $\mathsf{tr}(vw^2) = 0$ and $w^3 \neq 0$, we have

$$w = \begin{pmatrix} w_{11} & 0 & -3w_{11}^2 \\ 0 & -2w_{11} & 0 \\ 1 & 0 & w_{11} \end{pmatrix}$$

with $w_{11} \neq 0$. Since $a = -\mathsf{tr}(v^2w) = -3w_{11}^2$ and $b = w^3 = -8w_{11}^3$, we deduce that $a \in F(\omega)^{\times 2}$ and $b \in F^{\times 3}$. Suppose that F contains a primitive cube root of unity, then we can change the basis (u, v, w) of V so that

$$f_V(xu + yv + zw) = (x^2 + 3y^2 - 8z^2)z.$$

Thus we may assume that

$$w = \begin{pmatrix} 1 & 0 & -3 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Therefore, the vector space V is spanned by

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -3 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

We proved the following:

Theorem 5.7.2 *Suppose that F contains a primitive cube root of unity. The F-cubic pairs such that the associated cubic curve is a conic plus chord are F-isomorphic to the pair $\big(\mathsf{M}_3(F), V\big)$ where V is the subspace of $\mathsf{M}_3(F)$ spanned by*

- *either* $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$
- *or* $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & -3 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$

Theorem 5.7.3 *Suppose that F does not contain a primitive cube root of unity. The cubic pairs over F with a conic plus chord as associated cubic curve are F-isomorphic to the pair $\big(\mathsf{M}_3(F), V\big)$ where V is the subspace of $\mathsf{M}_3(F)$ spanned by*

- *either* $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}$
- *or* $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \begin{pmatrix} \alpha & 0 & -3\alpha^2 \\ 0 & -2\alpha & 0 \\ 1 & 0 & \alpha \end{pmatrix},$

for some $\alpha \in F$.

5.8 Cuspidal curve

We want to classify up to F-isomorphism the F-cubic pairs such that the associated cubic curve is cuspidal. First we need to describe a cubic form over V with a cuspidal associated cubic curve.

Lemma 5.8.1 *Let $f \in \mathsf{S}^3(V^\star)$ be such that the curve $\{f(\xi) = 0\}$ is cuspidal. Then there exist a basis (u, v, w) of V and a non-zero $\lambda \in F$ such that*

$$f(xu + yv + zw) = \lambda(z^3 + x^2 y).$$

Proof: The cubic curve $\{f(\xi) = 0\}$ has one flex, one singular point and one tangent at the singular point which are all defined over F_{sep}. Thus these points and their tangents are invariant under the action of Γ and so there are defined over F. Let $u, v, w \in V$ be such that $u\overline{F}$ is the flex, $v\overline{F}$ is the singular point and $w\overline{F}$ is the intersection $\overline{F}$-point of the tangents. Then (u, v, w) is a basis of V. Let $\lambda_{i,j,k} \in F$ be such that

$$f(xu + yv + zw) = \sum \lambda_{i,j,k} x^i y^j z^k$$

where the sum runs over all the positive integers i, j and k such that $i + j + k = 3$. Since $u\overline{F}$ is a flex of $\{f(\xi) = 0\}$ and the $\overline{F}$-points of its tangent are the $(\alpha u + \beta w)\overline{F}$ for all $\alpha, \beta \in \overline{F}$ not both zero, the root $z = 0$ of the polynomial

$$f(u + zw) = \lambda_{3,0,0} + \lambda_{2,0,1} z + \lambda_{1,0,2} z^2 + \lambda_{0,0,3} z^3$$

has a multiplicity equal to 3. Therefore $\lambda_{3,0,0} = \lambda_{2,0,1} = \lambda_{1,0,2} = 0$ and $\lambda_{0,0,3} \neq 0$. Because $v\overline{F}$ is a singular point and the $\overline{F}$-points of its double tangent are the $(\alpha v + \beta w)\overline{F}$ for all $\alpha, \beta \in \overline{F}$ not both zero, we have

$$\lambda_{0,3,0} = \lambda_{0,1,2} = \lambda_{1,2,0} = \lambda_{1,1,1} = \lambda_{0,2,1} = 0 \text{ and } \lambda_{2,1,0} \neq 0.$$

Hence

$$f(xu + yv + zw) = \lambda_{0,0,3} z^3 + \lambda_{2,1,0} x^2 y.$$

Replacing v by $\lambda_{2,1,0}^{-1} \lambda_{0,0,3} v$, we may assume that $\lambda_{2,1,0} = \lambda_{0,0,3}$ and

$$f(xu + yv + zw) = \lambda_{0,0,3}(z^3 + x^2 y)$$

as wanted. □

Suppose that V is a cubic subspace of $\mathsf{M}_3(F)$ such that the curve $\{f_V(\xi) = 0\}$ is cuspidal. We shall describe V up to conjugacy. By the previous lemma, there exist a basis (u, v, w) of V and a non-zero $\lambda \in F$ such that

$$f_V(xu + yv + zw) = \lambda(z^3 + x^2y).$$

Suppose that $v^2 \neq 0$. Since $u, v \in V$ are determinant zero matrices such that $\mathsf{tr}(u^2v) = \lambda$ and $\mathsf{tr}(uv^2) = 0$, by Lemma 4.2.4 we may assume that

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad v = \lambda \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}.$$

Since $\mathsf{tr}(w) = 0$, $\mathsf{tr}(uw) = 0$, $\mathsf{tr}(u^2w) = 0$, $\mathsf{tr}(vw) = 0$, $\mathsf{tr}(v^2w) = 0$, $\mathsf{tr}(uvw + uwv) = 0$ and $\mathsf{tr}(vw^2) = 0$, we deduce that

$$w = \begin{pmatrix} w_{11} & 0 & 0 \\ w_{21} & -2w_{11} & 0 \\ 0 & -w_{21} & w_{11} \end{pmatrix}.$$

Then $\mathsf{tr}(w^2) = 0$ imply $w_{11} = 0$ and it contradicts the fact that $w^3 = 0$. Suppose that $v^2 = 0$, since $\mathsf{tr}(u^2v) = \lambda \neq 0$, by Lemma 5.4.1 we may assume that

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad v = \lambda \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Since $\mathsf{tr}(w) = 0$, $\mathsf{tr}(uw) = 0$, $\mathsf{tr}(u^2w) = 0$, $\mathsf{tr}(vw) = 0$, $\mathsf{tr}(uvw + uwv) = 0$ and $\mathsf{tr}(vw^2) = 0$, we have

$$w = \begin{pmatrix} w_{11} & 0 & 0 \\ w_{21} & w_{22} & 0 \\ 0 & -w_{21} & -w_{11} - w_{22} \end{pmatrix}.$$

So $\mathsf{tr}(w^2) = 0$ implies $w_{22} = \rho w_{11}$ for some primitive cube root of unity $\rho \in F_{\mathsf{sep}}$. Since $w^3 = \lambda \neq 0$ we have $w_{11} \neq 0$ and the cube root ρ is in F. Then $\mathsf{tr}(uw^2) = 0$ imply $w_{21} = 0$. Therefore V is the subspace of $\mathsf{M}_3(F)$ spanned by

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}.$$

So we obtain:

Theorem 5.8.2 *Suppose that F does not contain a primitive cube root of unity, then there is no F-cubic pair with a cuspidal associated cubic curve.*

Theorem 5.8.3 *Suppose that F contains a primitive cube root of unity and (A, V) is a F-cubic pair such that $\{f_{A,V}(\xi) = 0\}$ is cuspidal, then (A, V) is F-isomorphic to the pair $(\mathsf{M}_3(F), \mathsf{span}_F\langle u, v, w\rangle)$, where*

$$u = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ v = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \ w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & \rho^2 \end{pmatrix}$$

for some $\rho \in F$ such that $\rho^2 + \rho + 1 = 0$. There are two such cubic pairs up to isomorphism.

Proof: Let W be the subspace of $\mathsf{M}_3(F)$ spanned by u, v and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}$$

and W' the one spanned by u, v and

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}.$$

Suppose that $m \in \mathsf{GL}_3(F)$ is such that $mWm^{-1} = W'$. Since uF_{sep} is a flex of $\{f_W(\xi) = 0\}$ the point $m \star uF_{\mathsf{sep}}$ is a flex of $\{f_{W'}(\xi) = 0\}$. Therefore $m \star uF_{\mathsf{sep}} = uF_{\mathsf{sep}}$. We can also prove that $m \star vF_{\mathsf{sep}} = vF_{\mathsf{sep}}$. By straightforward computations we deduce that

$$mF_{\mathsf{sep}}^{\times} = \begin{pmatrix} \lambda^2 & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & 1 \end{pmatrix} F_{\mathsf{sep}}^{\times}$$

for some $\lambda \in F_{\mathsf{sep}}^{\times}$ and $mwm^{-1} = w$; so we get a contradiction. □

5.9 Nodal curve

Suppose that V is a cubic subspace of $\mathsf{M}_3(F)$ such that $\{f_V(\xi) = 0\}$ is nodal. We want to describe V up to conjugacy. Thereto we first describe those cubic forms over F which have a nodal associated cubic curve.

Lemma 5.9.1 *Let $f \in \mathsf{S}^3(V^\star)$ be such that $\{f(\xi) = 0\}$ is nodal. Then there exist a basis (u, v, w) of V and $\lambda, \mu, \nu \in F$ with $\mu \neq 0$ such that*

$$f(xu + yv + zw) = \lambda z^3 + xy^2 - \mu xz^2 + \nu yz^2.$$

Moreover uF_{sep} is the unique singular point of $\{f(\xi) = 0\}$.

Proof : The curve $\{f(\xi) = 0\}$ has a unique singular point, thus it is defined over F. Let $u \in V$ be such that $u\overline{F}$ is the singular point. The three flexes of $\{f(\xi) = 0\}$ are collinear, hence there is a unique line which passes through the flexes and it is defined over F. At the singular point, there are two simple tangents, so these tangents are defined at least over a quadratic extension of F. Let $v, w \in V$ be linearly independent and $b \in F^\times$ such that $v\overline{F}$ and $w\overline{F}$ lie on the line passing through the flexes and the $\overline{F}$-points of the tangents at the singular point are the points $(xu + yv + zw)\overline{F}$ for all $x, y, z \in \overline{F}$ not all zero such that $y^2 - bz^2 = 0$. Let $\lambda_{i,j,k} \in F$ be such that

$$f(xu + yv + zw) = \sum_{i+j+k=3} \lambda_{i,j,k} x^i y^j z^k = c(x, y, z).$$

Then the first partial derivatives of c cancel at $(1, 0, 0)$, so

$$\lambda_{3,0,0} = \lambda_{2,1,0} = \lambda_{2,0,1}.$$

Using the tangents at $u\overline{F}$, we have

$$\lambda_{1,1,1} = 0, \ \lambda_{1,0,2} = -b\lambda_{1,2,0}, \ \lambda_{1,2,0} \neq 0.$$

Let $h \in \mathsf{S}^3(V^\star)$ be defined by

$$h(a_1u + a_2v + a_3w) = \det \begin{pmatrix} \frac{\partial^2 c}{\partial x^2}(a) & \frac{\partial^2 c}{\partial x \partial y}(a) & \frac{\partial^2 c}{\partial x \partial z}(a) \\ \frac{\partial^2 c}{\partial x \partial y}(a) & \frac{\partial^2 c}{\partial y^2}(a) & \frac{\partial^2 c}{\partial y \partial z}(a) \\ \frac{\partial^2 c}{\partial x \partial z}(a) & \frac{\partial^2 c}{\partial y \partial z}(a) & \frac{\partial^2 c}{\partial z^2}(a) \end{pmatrix}$$

for all $a = (a_1, a_2, a_3)$, so that $\{h(\xi) = 0\}$ is the Hessian curve H_f. Then $h(yv + zw)$ is a multiple of

$$\lambda_{0,1,2}y^3 + b^2\lambda_{0,2,1}z^3 + (2b\lambda_{0,2,1} + 3\lambda_{0,0,3})y^2z + (2b\lambda_{0,1,2} + 3b^2\lambda_{0,3,0})yz^2.$$

Since the $\overline{F}$-points of the line passing through the flexes are the points $(xu + yv + zw)\overline{F}$ for all $x, y, z \in \overline{F}$ not all zero such that $x = 0$, we deduce that $h(yv+zw)$ is a multiple of $f(yv+zw)$. Thus there exists $\alpha \in \overline{F}^{\times}$ such that

$$\begin{cases} \lambda_{0,3,0} = \alpha\lambda_{0,1,2}, \\ \lambda_{0,0,3} = \alpha b^2 \lambda_{0,2,1}, \\ \lambda_{0,2,1} = \alpha(2b\lambda_{0,2,1} + 3\lambda_{0,0,3}), \\ \lambda_{0,1,2} = \alpha(2b\lambda_{0,1,2} + 3b^2\lambda_{0,3,0}). \end{cases}$$

If $\lambda_{0,2,1} = \lambda_{0,1,2} = 0$ then $\lambda_{0,3,0} = \lambda_{0,0,3} = 0$ and f is reducible. Thus either $\lambda_{0,2,1} \neq 0$ or $\lambda_{0,1,2} \neq 0$ and we obtain that $3b^2\alpha^2 + 2b\alpha - 1 = 0$. Hence either $\alpha = -\frac{1}{b}$ or $\alpha = \frac{1}{3b}$. If $\alpha = -\frac{1}{b}$, then

$$f(xu+yv+zw) = \Big(-\frac{\lambda_{0,1,2}}{b}y + \lambda_{0,2,1}z + \lambda_{1,2,0}x\Big)(y^2 - bz^2);$$

this is impossible because f is irreducible. Thus $\alpha = \frac{1}{3b}$ and we obtain that $f(xu+yv+zw)$ is equal to

$$\frac{\lambda_{0,1,2}}{3b}y^3 + \frac{b\lambda_{0,2,1}}{3}z^3 + \lambda_{1,2,0}xy^2 + \lambda_{0,2,1}y^2z - b\lambda_{1,2,0}xz^2 + \lambda_{0,1,2}yz^2.$$

Replacing u by $\lambda_{1,2,0}^{-1}u$ we may assume that $\lambda_{1,2,0} = 1$. Now we put $u' := u$, $v' := -\frac{\lambda_{0,1,2}}{3b}u + v$, $w' := -\lambda_{0,2,1}u + w$, then

$$f(xu'+yv'+zw') = \frac{4b\lambda_{0,2,1}}{3}z^3 + xy^2 - bxz^2 + \frac{4\lambda_{0,1,2}}{3}yz^2$$

in which indeed $b \neq 0$. □

We keep the notation as in the statement of the lemma.

Suppose that $u^2 \neq 0$. Since $\mathsf{tr}(u^2v) = 0$ and $\mathsf{tr}(uv^2) = 1$, we may assume that

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Since $\mathsf{tr}(w) = 0$, $\mathsf{tr}(uw) = 0$, $\mathsf{tr}(u^2w) = 0$, $\mathsf{tr}(vw) = 0$ and $\mathsf{tr}(v^2w) = 0$, we have

$$w = \begin{pmatrix} w_{11} & w_{12} & 0 \\ w_{21} & w_{22} & w_{12} \\ 0 & -w_{21} & -w_{11} - w_{22} \end{pmatrix}.$$

Then $\mathsf{tr}(w^2)$ implies $w_{22} = \rho w_{11}$ for some $\rho \in F_{\mathsf{sep}}$ such that $\rho^2+\rho+1 = 0$, and $\mathsf{tr}(uvw+uwv) = 0$ implies $w_{12} = \frac{\rho^2-1}{2}w_{11}$. We have $w_{11} \neq 0$ because

$w^3 \neq 0$ and in particular $\rho \in F$. Hence V is conjugate to the span of

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ \alpha & 0 & \rho^2 \end{pmatrix}$$

for some $\alpha \in F$. So we obtain:

Theorem 5.9.2 *Suppose that F does not contain a primitive cube root of unity. Then there is no F-cubic pair with a nodal associated cubic curve and such that the unique singular point $\tilde{u}\overline{F}$ satisfies $\tilde{u}^2 \neq 0$.*

Theorem 5.9.3 *Suppose that F contains a primitive cube root of unity and (A, V) is an F-cubic pair such that $\{f_{A,V}(\xi) = 0\}$ is nodal with $\tilde{u}\overline{F}$ as the unique singular point. If $\tilde{u}^2 \neq 0$, then (A, V) is F-isomorphic to the pair $(\mathsf{M}_3(F), \mathsf{span}_F\langle u, v, w\rangle)$ where*

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 0 \end{pmatrix}, \; v = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \; w = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho & 0 \\ \alpha & 0 & \rho^2 \end{pmatrix}$$

for some $\alpha \in F$ and some primitive cube root of unity ρ.

Now we suppose that $u^2 = 0$. Since $\mathsf{tr}(uv^2) = 1$, we may assume that

$$u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

Because $\mathsf{tr}(w) = 0$, $\mathsf{tr}(uw) = 0$, $\mathsf{tr}(uvw + uwv) = 0$, $\mathsf{tr}(vw) = 0$ and $\mathsf{tr}(v^2w) = 0$, we deduce that

$$w = \begin{pmatrix} w_{11} & w_{12} & 0 \\ w_{21} & w_{22} & -w_{12} \\ 0 & -w_{21} & -w_{11} - w_{22} \end{pmatrix}.$$

Since $\mathsf{tr}(uw^2) = -\mu$, we have $w_{12}^2 = \mu$ and in particular $\mu \in F^{\times 2}$. Replacing w by $w_{12}^{-1}w$ we may assume that $w_{12} = 1$ and $\mu = 1$. Put $\alpha := w_{11}$ and $\beta := w_{22}$, then $\mathsf{tr}(w^2) = 0$ implies $w_{21} = -\frac{1}{2}(\alpha^2 + \beta^2 + \alpha\beta)$; so

$$w = \begin{pmatrix} \alpha & 1 & 0 \\ -\frac{1}{2}(\alpha^2 + \beta^2 + \alpha\beta) & \beta & -1 \\ 0 & \frac{1}{2}(\alpha^2 + \beta^2 + \alpha\beta) & -\alpha - \beta \end{pmatrix}.$$

We proved:

Theorem 5.9.4 *Let (A,V) be an F-cubic pair such that $\{f_{A,V}(\xi)=0\}$ is nodal and $\tilde{u}\overline{F}$ its unique singular point. Suppose that $\tilde{u}^2=0$ then (A,V) is F-isomorphic to the pair $(\mathsf{M}_3(F), \mathsf{span}_F\langle u,v,w\rangle)$ where*

$$u=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ v=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\text{and } w=\begin{pmatrix} \alpha & 1 & 0 \\ -\frac{1}{2}(\alpha^2+\beta^2+\alpha\beta) & \beta & -1 \\ 0 & \frac{1}{2}(\alpha^2+\beta^2+\alpha\beta) & -\alpha-\beta \end{pmatrix}$$

for some $\alpha,\beta\in F$.

5.10 Triangle

We shall classify the singular cubic pairs over F such that the associated cubic curve is a triangle. To do this we follow the method described in Section 4.1 which uses Theorem 4.1.2.

Triangles over the separable closure

Suppose that V is a cubic subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$ such that the curve $\{f_V(\xi)=0\}$ is a triangle: there exists a basis (u,v,w) of V such that

$$f_V(xu+yv+zw)=xyz.$$

We shall describe V up to conjugacy.

First suppose that $u^2\neq 0$, then we may assume that

$$u=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

For all $\xi\in V$, we have $\mathsf{tr}(\xi)=0$, $\mathsf{tr}(u\xi)=0$ and $\mathsf{tr}(u^2\xi)=0$, so

$$\xi_{33}=-\xi_{11}-\xi_{22},\ \xi_{32}=-\xi_{21},\ \xi_{31}=0.$$

Since $\mathsf{tr}(uv^2)=0$, we deduce that $v_{21}(2v_{11}+v_{22})=0$. If $v_{21}=0$, then $\mathsf{tr}(v^2)=0$ and $v^3=0$ imply $v_{11}=v_{22}=0$, thus $uv+vu=(v_{12}+v_{23})u^2$. But $\mathsf{tr}(u^2w)=0$ and $\mathsf{tr}(uvw+uwv)=1$, so we get a contradiction. If $v_{21}\neq 0$ then $u,v\in V$ are such that $(xu+yv)^3=0$ for all $x,y\in F_{\mathsf{sep}}$ and

u^2, v^2 are linearly independent, thus by Lemma 5.2.1 we may assume that

$$v = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

Since $\mathsf{tr}(vw) = 0$, $\mathsf{tr}(v^2w) = 0$, $\mathsf{tr}(uvw + uwv) = 0$ and $\mathsf{tr}(w^2) = 0$, we have

$$w = \begin{pmatrix} 0 & w_{12} & 0 \\ w_{21} & 0 & w_{12} \\ 0 & -w_{21} & 0 \end{pmatrix},$$

and it contradicts the fact that u, v, w are linearly independent.

Next suppose that $u^2 = 0$, then we may assume that

$$u = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

For $\xi \in V$, we have $\mathsf{tr}(\xi) = 0$ and $\mathsf{tr}(u\xi) = 0$, hence

$$\xi_{33} = -\xi_{11} - \xi_{22} \text{ and } \xi_{31} = 0.$$

Using the first case we know that $v^2 = 0$ and $w^2 = 0$; so $v_{21}v_{32} = 0$ and $w_{21}w_{32} = 0$. Since $\mathsf{tr}(uvw + uwv) = 1$, we have $v_{32}w_{21} + v_{21}w_{32} = 1$, hence either $v_{32} = w_{21} = 0$ and $v_{21}, w_{32} \neq 0$ or $v_{21} = w_{32} = 0$ and $v_{32}, w_{21} \neq 0$. By symmetry, we may assume that $v_{32} = w_{21} = 0$ and $v_{21}, w_{32} \neq 0$. Replacing v by $v_{21}^{-1}v$ if necessary, we may assume that $v_{21} = 1$, and then $w_{32} = 1$. Because $v^2 = 0$, we deduce that

$$v_{22} = -v_{11},\ v_{12} = -v_{11}^2,\ v_{13} = v_{11}v_{23}.$$

Put

$$m := \begin{pmatrix} 1 & -v_{11} & v_{23} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

then $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$, $mum^{-1} = u$ and

$$mvm^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So we may assume that $v_{11} = 0$ and $v_{23} = 0$. Since $\mathsf{tr}(vw) = 0$ and $w^2 = 0$ we have

$$w_{12} = 0,\ w_{11} = 0,\ w_{23} = -w_{22}^2,\ w_{13} = 0.$$

Put

$$m := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -w_{22} \\ 0 & 0 & 1 \end{pmatrix}$$

then $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$, $mum^{-1} = u$, $mvm^{-1} = v$ and

$$mwm^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

So we proved the following:

Theorem 5.10.1 *Suppose that (A, V) is a cubic pair over F_{sep} such that the curve $\{f_{A,V}(\xi) = 0\}$ is a triangle, then (A, V) is isomorphic to the pair $(\mathsf{M}_3(F_{\mathsf{sep}}), \mathsf{span}_F\langle u, v, w\rangle)$ where*

$$u = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; v = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; w = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Automorphism group

The cubic pairs over F_{sep} with a triangle as associated cubic curve being classified, we compute the automorphism group of such cubic pairs. Let A be the matrix algebra $\mathsf{M}_3(F_{\mathsf{sep}})$ and V the subspace of A spanned by u, v, w where

$$u = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; v = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; w = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Suppose that $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$ is such that $mVm^{-1} = V$. The singular points of the triangle $\{f_{A,V}(\xi) = 0\}$ are the points uF_{sep}, vF_{sep} and wF_{sep}. Thus $m \star uF_{\mathsf{sep}}$ is equal to uF_{sep}, vF_{sep} or wF_{sep}.

Case 1: Suppose that $m \star uF_{\mathsf{sep}} = uF_{\mathsf{sep}}$, There exists $\lambda \in F_{\mathsf{sep}}^{\times}$ such that $mu = \lambda um$, thus

$$m = \begin{pmatrix} \lambda m_{33} & m_{12} & m_{13} \\ 0 & m_{22} & m_{23} \\ 0 & 0 & m_{33} \end{pmatrix}.$$

Since $mvm^{-1} \in V$ we have $m_{12} = m_{13} = 0$ and then $mwm^{-1} \in V$ implies $m_{23} = 0$. Thus m is diagonal matrix, and conversely, if n is an invertible diagonal matrix then we can check that $nVn^{-1} = V$ and $n \star uF_{\mathsf{sep}} = uF_{\mathsf{sep}}$. Let G_0 denote the subgroup of $\mathsf{PGL}_3(F_{\mathsf{sep}})$ which consists of the elements $nF_{\mathsf{sep}}^{\times}$ where n is an invertible diagonal matrix.

Case 2: Suppose that $m \star uF_{\mathsf{sep}} = vF_{\mathsf{sep}}$. Put

$$m' := \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

then $m'Vm'^{-1} = V$ and $m'^{-1}m \star uF_{\mathsf{sep}} = uF_{\mathsf{sep}}$. Thus $m'^{-1}mF_{\mathsf{sep}}^{\times} \in G_0$ and

$$mF_{\mathsf{sep}}^{\times} \in m'F_{\mathsf{sep}}^{\times}G_0.$$

Conversely, it is easy to check that $nVn^{-1} = V$ and $n \star uF_{\mathsf{sep}} = vF_{\mathsf{sep}}$ if $nF_{\mathsf{sep}}^{\times} \in m'F_{\mathsf{sep}}^{\times}G_0$.

Case 3: Suppose that $m \star uF_{\mathsf{sep}} = wF_{\mathsf{sep}}$. Put

$$m'' := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

then $m''Vm''^{-1} = V$ and $m''^{-1}m \star uF_{\mathsf{sep}} = uF_{\mathsf{sep}}$. Thus

$$mF_{\mathsf{sep}}^{\times} \in m''F_{\mathsf{sep}}^{\times}G_0.$$

Conversely, if $nF_{\mathsf{sep}}^{\times} \in m''F_{\mathsf{sep}}^{\times}G_0$ then one can check that $nVn^{-1} = V$ and $n \star uF_{\mathsf{sep}} = wF_{\mathsf{sep}}$.

We proved that $\mathsf{Aut}(A, V)(F_{\mathsf{sep}})$ is equal to

$$G_0 \cup \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} F_{\mathsf{sep}}^{\times} \cdot G_0 \cup \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} F_{\mathsf{sep}}^{\times} \cdot G_0.$$

First cohomology set

Let A be the matrix algebra $\mathsf{M}_3(F)$ and V the subspace of A spanned by

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We want to describe the elements of the first set $\mathsf{H}^1(F, \mathsf{Aut}(A, V))$ of cohomology.

First we prove that $\mathsf{H}^1(\Gamma, G_0) = 1$. We consider the split exact sequence of abelian groups

$$\begin{tikzcd} 1 \arrow[r] & F_{\mathsf{sep}}^{\times} \arrow[r] & D_3 \arrow[r] & G_0 \arrow[r] \arrow[l, bend right, "s"'] & 1 \end{tikzcd}$$

where D_3 is the subgroup of $\mathsf{GL}_3(F_{\mathsf{sep}})$ of diagonal matrices and

$$s\left(\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix} F_{\mathsf{sep}}^{\times}\right) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & ba^{-1} & 0 \\ 0 & 0 & ca^{-1} \end{pmatrix}.$$

It induces the following exact sequence

$$\mathsf{H}^1(\Gamma, D_3) \longrightarrow \mathsf{H}^1(\Gamma, G_0) \xrightarrow{\delta_1} \mathsf{H}^2(\Gamma, F_{\mathsf{sep}}^{\times}).$$

But $\mathsf{H}^1(\Gamma, D_3) = 1$ and Remark (28.7) in [Knus *et al.*, 1998] says that δ_1 is trivial. Thus $\mathsf{H}^1(\Gamma, G_0) = 1$.

Next we put $G := \mathsf{Aut}(A, V)(F_{\mathsf{sep}})$ and we consider the exact sequence of groups

$$1 \longrightarrow G_0 \longrightarrow G \longrightarrow G/G_0 \longrightarrow 1.$$

It induces an exact sequence

$$1 = \mathsf{H}^1(\Gamma, G_0) \longrightarrow \mathsf{H}^1(\Gamma, G) \xrightarrow{f} \mathsf{H}^1(\Gamma, G/G_0)$$

where $f([a_\sigma]) = [a_\sigma G_0]$ for a 1-cocycle $(a_\sigma)_{\sigma\in\Gamma}$ with values in G. The mapping

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} F_{\mathsf{sep}}^{\times} G_0 \mapsto 1 + 3\mathbb{Z}$$

defines a Γ-group isomorphism between G/G_0 and $\mathbb{Z}/3$. Suppose that $[a_\sigma] \in \mathsf{H}^1(\Gamma, G)$ is non-trivial, then there exists a Galois extension L over F of degree 3 such that

$$\{\sigma \in \Gamma \mid a_\sigma \in G_0\} = \mathsf{Gal}(F_{\mathsf{sep}}/L).$$

Put $\Gamma' := \mathsf{Gal}(F_{\mathsf{sep}}/L)$, then $(a_\sigma)_{\sigma\in\Gamma'}$ is a 1-cocycle with values in G_0. Since $\mathsf{H}^1(\Gamma', G_0) = 1$, there exists $b \in G_0$, such that

$$a_\sigma = b\sigma(b)^{-1}$$

for all $\sigma \in \Gamma'$. Replacing $(a_\sigma)_{\sigma\in\Gamma}$ by $\big(b^{-1}a_\sigma\sigma(b)\big)_{\sigma\in\Gamma}$ if necessary, we may assume that $a_\sigma = 1$ for all $\sigma \in \Gamma'$. Let $\sigma_0 \in \Gamma$ be such that

$$a_{\sigma_0}G_0 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} F_{\mathsf{sep}}^\times G_0.$$

Since L is a Galois extension of F, $\sigma\Gamma' = \Gamma'\sigma$ for all $\sigma \in \Gamma$. So for all $\tau \in \Gamma'$ there exists $\tau' \in \Gamma'$ such that $\tau\sigma_0 = \sigma_0\tau'$. Let $\tau \in \Gamma'$, then

$$\tau(a_{\sigma_0}) = a_\tau\tau(a_{\sigma_0}) = a_{\tau\sigma_0} = a_{\sigma_0\tau'} = a_{\sigma_0}\sigma_0(a_{\tau'}) = a_{\sigma_0}$$

where $\tau' \in \Gamma'$ is such that $\tau\sigma_0 = \sigma_0\tau'$. Thus there exist $\lambda, \mu \in L$ such that $a_{\sigma_0} = mF_{\mathsf{sep}}^\times$ and

$$m = \begin{pmatrix} 0 & 0 & 1 \\ \lambda & 0 & 0 \\ 0 & \mu & 0 \end{pmatrix}.$$

Using the fact that $(a_\sigma)_{\sigma\in\Gamma}$ is a 1-cocycle, we deduce that

$$a_\sigma = \begin{cases} F_{\mathsf{sep}}^\times & \text{if } \sigma \in \Gamma', \\ mF_{\mathsf{sep}}^\times & \text{if } \sigma \in \sigma_0\Gamma', \\ m\sigma_0(m)F_{\mathsf{sep}}^\times & \text{if } \sigma \in \sigma_0^2\Gamma' \end{cases}$$

for all $\sigma \in \Gamma$, and $m\sigma_0(m)\sigma_0^2(m) \in F_{\mathsf{sep}}^\times$. We may assume that $\lambda = 0$: indeed put

$$c := \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \sigma_0(\lambda) \end{pmatrix}$$

then

$$\sigma_0(c)mc^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \sigma_0^2(\lambda)\mu & 0 \end{pmatrix}.$$

Since $m\sigma_0(m)\sigma_0^2(m) \in F_{\mathsf{sep}}^\times$ we have $\lambda \in F^\times$. Now suppose that the 1-cocycles $(a_{1,\sigma})_{\sigma\in\Gamma}$ and $(a_{2,\sigma})_{\sigma\in\Gamma}$ are equivalent where

$$a_{i,\sigma} = \begin{cases} F_{\mathsf{sep}}^\times & \text{if } \sigma \in \Gamma', \\ m_iF_{\mathsf{sep}}^\times & \text{if } \sigma \in \sigma_0\Gamma', \\ m_i^2F_{\mathsf{sep}}^\times & \text{if } \sigma \in \sigma_0^2\Gamma' \end{cases}$$

and

$$m_i = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \lambda_i & 0 \end{pmatrix},$$

with $\lambda_i \in F^\times$. Then we can prove by straightforward computations that

$$\lambda_2 \in \mathsf{N}_{L/F}(L^\times)\lambda_1.$$

Classification

Let A be the matrix algebra $\mathsf{M}_3(F)$ and V the subspace of A spanned by

$$u = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; v = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; w = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We shall describe the isomorphism classes of F-cubic pairs which are isomorphic to $(A, V)_{F_{\mathsf{sep}}}$ over F_{sep}. We know by Theorem 4.1.2 that they are in correspondence with the elements of $\mathsf{H}^1\big(F, \mathsf{Aut}(A, V)\big)$. Let α be a non-trivial cocycle with values in $\mathsf{Aut}(A, V)$, then by the previous subsection there exist a non-trivial Galois $\mathbb{Z}/3$-algebra (L, ρ) and a scalar $\mu \in F^\times$ such that $[\alpha] = [a_\sigma]$ where

$$a_\sigma = \begin{cases} F_{\mathsf{sep}}^\times & \text{if } \sigma|_L = \mathsf{id}_L, \\ mF_{\mathsf{sep}}^\times & \text{if } \sigma|_L = \rho, \\ m^2 F_{\mathsf{sep}}^\times & \text{if } \sigma|_L = \rho^2 \end{cases}$$

for all $\sigma \in \Gamma$, and

$$m = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & \mu & 0 \end{pmatrix}.$$

The F-cubic pair corresponding to $[\alpha]$ is the pair (A', V') where

$$\begin{aligned} A' &= \{\xi \in A_L \mid m\rho(\xi)m^{-1} = \xi\}, \\ V' &= \{\xi \in V_L \mid m\rho(\xi)m^{-1} = \xi\}. \end{aligned}$$

Suppose that F contains a primitive cube root of unity. Let $\theta \in L$ be such that $\theta^3 = d \in F$ and $\rho(\theta) = \omega\theta$. Put

$$\begin{aligned} \xi_0 &:= u + v + \mu w, \\ \eta_0 &:= \theta u + \omega\theta v + \omega^2\mu\theta w, \\ \zeta_0 &:= \theta^2 u + \omega^2\theta^2 v + \omega\mu\theta^2 w \end{aligned}$$

then ξ_0, η_0, ζ_0 are linearly independent matrices of V' such that $\xi_0^3 = \mu$, $\eta_0^3 = d\mu$ and $\xi_0\eta_0 = \omega^2\eta_0\xi_0$. Therefore V' is the subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$

spanned by ξ_0, η_0, ζ_0 and A' is the symbol algebra $(d\mu, \mu)_{\omega,F}$ generated by ξ_0 and η_0. We have

$$\zeta_0 = \frac{\omega^2}{\mu}\xi_0^2\eta_0^2$$

and f is semi-diagonal:

$$f = a\xi_0^{\star 3} + b\eta_0^{\star 3} + c\zeta_0^{\star 3} - 3\lambda\xi_0^\star\eta_0^\star\zeta_0^\star$$

for some $a, b, c, \lambda \in F$ which satisfy the relation

$$abc = \lambda^3$$

and where $(\xi_0^\star, \eta_0^\star, \zeta_0^\star)$ is the dual basis of (ξ_0, η_0, ζ_0).

Theorem 5.10.2 *Suppose F contains a primitive cube root of unity. Then, up to F-isomorphism, the F-cubic pairs with a triangle as associated cubic curve are the pairs*

$$\big((a,b)_{\omega,F}, \mathsf{span}_F\langle\xi_0, \eta_0, \xi_0^2\eta_0^2\rangle\big)$$

for all $a, b \in F^\times$, where ξ_0 and η_0 are generators of the symbol algebra such that $\xi_0^3 = a$, $\eta_0^3 = b$ and $\xi_0\eta_0 = \omega\eta_0\xi_0$. The cubic forms associated to these cubic pairs are semi-diagonal.

Now suppose that F does not contain a primitive cube root of unity and F is infinite. Let $\theta \in L$ be such that its minimal polynomial over F is equal to $x^3 - 3x + \lambda$ for some $\lambda \in F \setminus \{2 - 2\}$. Put $\theta' = \rho'(\theta)$ and $\theta'' = \rho^2(\theta)$. Then we may choose a square root x_0 of $(4 - \lambda^2)/3$ in F and a cube root ϕ of $(\lambda + (\omega - \omega^2)x_0)/2$ in F_{sep} such that

$$\begin{aligned}
\theta &= -\phi - \phi^{-1}, \\
\theta' &= -\omega\phi - \omega^2\phi^{-1} = \frac{-\theta + \delta}{2}, \\
\theta'' &= -\omega^2\phi - \omega\phi^{-1} = \frac{-\theta - \delta}{2}
\end{aligned}$$

where $\delta = x_0^{-1}(2\theta^2 + \lambda\theta - 4)$. Put

$$\begin{aligned}
\xi_0 &:= u + v + \mu w, \\
\eta_0 &:= \theta u + \frac{1}{2}(-\theta + \delta)v + \frac{1}{2}(-\theta - \delta)\mu w, \\
\zeta_0 &:= \delta u + \frac{1}{2}(-3\theta - \delta)v + \frac{1}{2}(3\theta - \delta)\mu w
\end{aligned}$$

then ξ_0, η_0, ζ_0 are linearly independent vectors of V'. Put

$$\eta_1 := -\frac{1}{2}\eta_0 + \frac{\omega - \omega^2}{6}\zeta_0 = \phi u + \omega\phi v + \omega^2\phi\mu w$$

then $\xi_0^3 = \mu$, $\eta_1^3 = \phi^3\mu$ and $\xi_0\eta_1 = \omega^2\eta_1\xi_0$. Therefore $A' \otimes_F F(\omega)$ is the symbol algebra $(\phi^3\mu, \mu)_{\omega, F(\omega)}$ generated by ξ_0 and η_1. Put $\eta_2 = \mu^{-1}\xi_0^2\eta_1$, then $\eta_2^3 = \phi^3$ and $\eta_2\overline{\eta_2} = 1$. Hence the subfield $L' := F(-\eta_2 - \eta_2^{-1})$ of A' is a Galois extension of degree 3 over F with Galois group generated by τ where $\tau(-\eta_2 - \eta_2^{-1}) = -\omega\eta_2 - \omega^2\eta_2^{-1}$. Since $\xi_0\eta_2 = \omega^2\eta_2\xi_0$ we have

$$\xi_0(-\eta_2 - \eta_2^{-1}) = \tau^2(-\eta_2 - \eta_2^{-1})\xi_0.$$

Thus A' is the cyclic algebra $(\mu, L'/F, \tau^2)$ generated by ξ_0 and L'. We observe that the mapping $-\eta_2 - \eta_2^{-1} \mapsto \theta$ defines an isomorphism between (L', τ) and (L, ρ). Furthermore $\eta_0 = \xi_0\eta_3$ and $\zeta_0 = \xi_0\big(\eta_3 + 2\tau(\eta_3)\big)$ where $\eta_3 = -\eta_2 - \eta_2^{-1}$. In particular $V = \xi_0 L'$. By Lemma 3.2.3 we know that the cubic form associated to (A', V') is a semi-trace form since $\{f_{A',V'}(\xi) = 0\}$ is a triangle. More precisely we have

$$f_{A',V'}(\xi_0\xi) = \xi_0^3 \mathsf{N}_{L'/F}(\xi)$$

for all $\xi \in L'$. We can also write

$$f(\xi) = \mathsf{Tr}_{K/F}\big(a\Theta(\xi)^3\big) - 3\mu\mathsf{N}_{K/F}\big(\Theta(\xi)\big)$$

where $K = F \times F(\omega)$, $\Theta\colon V \to K$ is the F-vector space isomorphism defined by

$$\Theta(x\xi_0 + y\eta_0 + z\zeta_0) = \big(x, -y - (\omega - \omega^2)z\big)$$

and $a = \mu(1, \phi^3)$. Note that the elements a and μ satisfy the relation $\mathsf{N}_{K/F}(a) = \mu^3$. However, the cubic form $f_{A',V'}$ is not semi-diagonal. Indeed, if $f_{A',V'}$ is semi-diagonal, then the lines of the cubic curve $\{f_{A',V'} = 0\}$ are defined over F and also their intersection points; this contradicts the assumption that F does not contain a primitive cube root of unity.

We note that V is spanned by

$$\begin{aligned}
\xi_0 &:= u + v + \mu w,\\
\eta_0 &:= \theta u + \frac{1}{2}(-\theta + \delta)v + \frac{1}{2}(-\theta - \delta)\mu w,\\
\zeta_0 &:= \delta u + \frac{1}{2}(-3\theta - \delta)v + \frac{1}{2}(3\theta - \delta)\mu w
\end{aligned}$$

for $\theta = -2$ and $\delta = 0$. The matrix $\eta_2 = \mu^{-1}\xi_0^2\eta_1$ where

$$\eta_1 := -\frac{1}{2}\eta_0 + \frac{\omega - \omega^2}{6}\zeta_0$$

is such that $\eta_2^3 = \phi^3$ for $\phi = 1$. Put

$$\eta_3 := -\eta_2 - \eta_2^{-1}, \ \ \eta_3' := -\omega^2\eta_2 - \omega\eta_2^{-1},$$

then $\eta_0 = \xi_0\eta_3$ and $\zeta_0 = \xi_0(\eta_3 + 2\eta_3')$. Put $L' := F \oplus F\eta_3 \oplus F\eta_3'$ and let ρ be the F-algebra automorphism of L' defined by $\rho(\eta_3) = \eta_3'$ and $\rho(\eta_3') = -\eta_3 - \eta_3'$. There exists an F-algebra isomorphism $\Psi\colon L \to F^3$ such that

$$\rho\big(\Psi^{-1}(1,0,0)\big) = \Psi^{-1}(0,1,0) \quad \text{and} \quad \rho\big(\Psi^{-1}(0,1,0)\big) = \Psi^{-1}(0,0,1).$$

It is also true that $A = \bigoplus_{i=0}^{2} L'\xi_0^i$ where $\xi_0\xi = \rho(\xi)\xi_0$ for all $\xi \in L'$. Note that

$$f_{A,V}(\xi) = \mathsf{Tr}_{(F\times F(\omega))/F}\big(\alpha\Theta(\xi)^3\big) - 3\mu\mathsf{N}_{(F\times F(\omega))/F}\big(\Theta(\xi)\big)$$

where $\Theta\colon V \to F \times F(\omega)$ is defined by

$$\Theta(x\xi_0 + y\eta_0 + z\zeta_0) = \big(x, -y - (\omega - \omega^2)z\big),$$

$\alpha = \mu(1, \phi^3)$ and $\mathsf{N}_{(F\times F(\omega))/F}(\alpha) = \mu^3$. But $f_{A,V}$ is also semi-diagonal:

$$f_{A,V} = u^\star v^\star w^\star$$

where $(u^\star, v^\star, w^\star)$ is the dual basis of (u, v, w).

Theorem 5.10.3 *Suppose that F is infinite and does not contain a primitive cube root of unity. Up to F-isomorphism, the F-cubic pairs with a triangle as associated cubic curve, are either*

$$\big(\mathsf{M}_3(F), \mathsf{span}_F\langle u, v, w\rangle\big)$$

where

$$u = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ v = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ w = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

or the pairs

$$\big((\mu, L/F, \rho), \xi_0 L\big)$$

for all non trivial isomorphism classes $[(L,\rho)]$ of Galois $\mathbb{Z}/3$-algebras and for all $\mu\mathsf{N}_{L/F}(L^\times) \in F^\times/\mathsf{N}_{L/F}(L^\times)$, where ξ_0 and $L = F(\theta)$ generate the cyclic algebra such that $\xi_0^3 = \mu$, $\theta^3 - 3\theta \in F$ and $\xi_0\theta = \rho(\theta)\xi_0$. The associated cubic forms are semi-diagonal.

Conclusion

We shall summarize our results on the classification of cubic pairs.

First we deal with the non-singular cubic pairs over the field F and we assume that F contains a primitive cube root of unity. By Theorems 4.4.1, 4.4.5 and 4.5.1, if (A, V) is a non-singular cubic pair over F then there exist $a, b \in F^\times$ and $\alpha \in F$ such that (A, V) is F-isomorphic to

$$\big((a,b)_{\omega,F}, \mathsf{span}_F\langle \xi_0, \eta_0, \xi_0\eta_0^2 + \alpha\xi_0^2\eta_0^2\rangle\big)$$

where ξ_0, η_0 are generators of the symbol algebra $(a,b)_{\omega,F}$ such that $\xi_0^3 = a$, $\eta_0^3 = b$ and $\xi_0\eta_0 = \omega\eta_0\xi_0$.

Conversely, let $a, b \in F^\times$ and $\alpha \in F$. Put $A := (a,b)_{\omega,F}$ and

$$V := \mathsf{span}_F\langle \xi_0, \eta_0, \xi_0\eta_0^2 + \alpha\xi_0^2\eta_0^2\rangle$$

where ξ_0, η_0 are generators of the symbol algebra A such that

$$\xi_0^3 = a, \quad \eta_0^3 = b, \quad \xi_0\eta_0 = \omega\eta_0\xi_0.$$

One can check that (A, V) is a cubic pair over F. The associated cubic form $f_{A,V}$ is semi-diagonal:

$$f_{A,V}(x\xi_0 + y\eta_0 + z\zeta_0) = ax^3 + by^3 + (ab^2 + \alpha^3a^2b^2)z^3 - 3(\omega^2\alpha ab)xyz.$$

Observe that

$$ab(ab^2 + \alpha^3a^2b^2) - (\omega^2\alpha ab)^3 = a^2b^3 \neq 0,$$

thus by Lemma 3.1.2 the cubic form $f_{A,V}$ is non-singular if and only if $ab^2 + \alpha^3a^2b^2 \neq 0$, i.e. $\alpha^3 \neq -a^{-1}$.

Lemma. *With the notation as above and $\alpha^3 \neq -a^{-1}$, the cubic pair (A, V) is of the second kind if and only if $\alpha = 0$.*

Proof: Suppose that $\alpha = 0$ then

$$(A,V)_{F_{\mathsf{sep}}} \cong \big(\mathsf{M}_3(F_{\mathsf{sep}}), \mathsf{span}_{F_{\mathsf{sep}}}\langle \xi_1, \eta_1, \xi_1\eta_1^2\rangle\big)$$

with

$$\xi_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & 0 & 0 \end{pmatrix} \quad \text{and} \quad \eta_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}.$$

Let $\theta \in F_{\mathsf{sep}}$ be a cube root of a and put

$$m := \begin{pmatrix} \omega^2\theta^2 & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

then

$$m\xi_1 m^{-1} = \omega\theta \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix} \quad \text{and} \quad m\eta_1 m^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}.$$

So $\mathsf{span}_{F_{\mathsf{sep}}}\langle \xi_1, \eta_1, \xi_1\eta_1^2\rangle$ is conjugate to the subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$ spanned by

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix}, \begin{pmatrix} 0 & \omega & 0 \\ 0 & 0 & \omega^2 \\ 1 & 0 & 0 \end{pmatrix}.$$

Using the proof of Theorem 4.3.9 we deduce that the automorphism group of $(A,V)_{F_{\mathsf{sep}}}$ is isomorphic to $\mathbb{Z}/3 \times \mathbb{Z}/3$; therefore (A,V) is a cubic pair of the second kind.

Conversely, suppose that (A,V) is a cubic pair of the second kind. By Theorem 4.3.9 the subspace V is conjugate to $\mathsf{span}_{F_{\mathsf{sep}}}\langle \xi_0, \eta_0, \xi_0\eta_0^2\rangle$. So there exist an $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$ and $\lambda_i, \mu_i, \nu_i \in F$ not all zero such that

$$\begin{aligned} m\xi_0 m^{-1} &= \lambda_1\xi_0 + \mu_1\eta_0 + \nu_1(\xi_0\eta_0^2 + \alpha\xi_0^2\eta_0^2), \\ m\eta_0 m^{-1} &= \lambda_2\xi_0 + \mu_2\eta_0 + \nu_2(\xi_0\eta_0^2 + \alpha\xi_0^2\eta_0^2), \\ m\xi_0\eta_0^2 m^{-1} &= \lambda_3\xi_0 + \mu_3\eta_0 + \nu_3(\xi_0\eta_0^2 + \alpha\xi_0^2\eta_0^2). \end{aligned}$$

Since the coefficient of $\xi_0^2\eta_0$ in $m\xi_0\eta_0 m^{-1} - \omega m\eta_0\xi_0 m^{-1}$ is equal to zero we have $\nu_1\nu_2 = 0$. Similarly we can prove that $\nu_1\nu_3 = 0$ and $\nu_2\nu_3 = 0$. We cannot have $\nu_1 = \nu_3 = 0$, since otherwise

$$m\xi_0(\xi_0\eta_0^2)m^{-1} = \omega^2 m(\xi_0\eta_0^2)\xi_0 m^{-1}$$

implies $m\xi_0 m^{-1} = \mu_1\eta_0$ and $m\xi_0\eta_0^2 m^{-1} = \lambda_3\xi_0$, and then

$$m\xi_0(\xi_0\eta_0^2)^2 m^{-1} \neq \omega ab\big(\lambda_2\xi_0 + \mu_2\eta_0 + \nu_2(\xi_0\eta_0^2 + \alpha\xi_0^2\eta_0^2)\big).$$

Similarly we cannot have $\nu_2 = \nu_3 = 0$, thus $\nu_1 = \nu_2 = 0$. Because

$$m\xi_0\eta_0 m^{-1} = \omega m\eta_0\xi_0 m^{-1}$$

we have $m\xi_0 m^{-1} = \lambda_1\xi_0$ and $m\eta_0 m^{-1} = \mu_2\eta_0$. Then

$$m\xi_0\eta_0^2 m^{-1} = \lambda_3\xi_0 + \mu_3\eta_0 + \nu_3(\xi_0\eta_0^2 + \alpha\xi_0^2\eta_0^2)$$

implies $\alpha = 0$. $\square$

We obtain the following theorem:

Theorem I *Suppose that F contains a primitive cube root of unity. Up to F-isomorphism, the non-singular cubic pairs over F are the pairs*

$$\big((a,b)_{\omega,F}, \mathsf{span}_F\langle \xi_0, \eta_0, \xi_0\eta_0^2 + \alpha\xi_0^2\eta_0^2\rangle\big)$$

for all $a, b \in F^\times$ and $\alpha \in F$ with $\alpha^3 \neq -a^{-1}$, where ξ_0, η_0 are generators of the symbol algebra such that $\xi_0^3 = a$, $\eta_0^3 = b$ and $\xi_0\eta_0 = \omega\eta_0\xi_0$. Such a cubic pair is of the second kind if and only if $\alpha = 0$. The associated cubic form is always semi-diagonal and it is diagonal if the pair is of the second kind:

$$\big(x\xi_0 + y\eta_0 + z(\xi_0\eta_0^2 + \alpha\xi_0^2\eta_0^2)\big)^3 = ax^3 + by^3 + cz^3 - 3\lambda xyz$$

where $c = ab^2 + \alpha^3 a^2 b^2$, $\lambda = \omega^2\alpha ab$ and $a^{-2}(abc - \lambda^3) = b^3$ is a non-zero cube in F.

Now we suppose that F does not contain a primitive cube root of unity and is infinite. By the remarks preceding Theorems 4.4.2, 4.4.6 and 4.5.2, if (A, V) is a non-singular cubic pair over F, then there exist a Galois $\mathbb{Z}/3$-algebra (L, ρ) and $a, \alpha, \beta \in F$, $a \neq 0$ such that

$$(A, V) \cong \Big(\bigoplus_{i=0}^{2} Le^i, \mathsf{span}_F\langle \xi_0, \eta_0, \zeta_0\rangle\Big)$$

with $e\xi = \rho(\xi)e$ for all $\xi \in L$, $e^3 = a$ and

$$\xi_0 = e, \quad \eta_0 = (\alpha + \beta e + e^2)t, \quad \zeta_0 = (\alpha + \beta e + e^2)\rho(t)$$

where $1, t, \rho(t)$ span L and

$$(x - t)\big(x - \rho(t)\big)\big(x - \rho^2(t)\big) = x^3 - 3x + \lambda$$

for some $\lambda \in F$.

Conversely, let (L,ρ) be a Galois $\mathbb{Z}/3$-algebra and $a,\alpha,\beta \in F$ such that $a \neq 0$. Let $t \in L$ be such that $1, t, \rho(t)$ span L and

$$(x-t)\big(x-\rho(t)\big)\big(x-\rho^2(t)\big) = x^3 - 3x + \lambda$$

for some $\lambda \in F$. We may name $\theta, \theta', \theta''$ the roots of $x^3 - 3x + \lambda$ in F_{sep} so that there exists an F-algebra isomorphism $\Psi\colon L \to F \oplus F\tilde{t} \oplus F\tilde{t'}$ with $\Psi(t) = \tilde{t}$, and $\Psi\big(\rho(t)\big) = \tilde{t'}$ where

$$\tilde{t} = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \theta' & 0 \\ 0 & 0 & \theta'' \end{pmatrix}, \quad \tilde{t'} = \begin{pmatrix} \theta' & 0 & 0 \\ 0 & \theta'' & 0 \\ 0 & 0 & \theta \end{pmatrix}.$$

Thus we may assume that $L = F \oplus Ft \oplus F\rho(t)$ with $t = \tilde{t}$ and $\rho(t) = \tilde{t'}$. Put

$$e := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & 0 & 0 \end{pmatrix}$$

then $e^3 = a$ and $e\xi = \rho(\xi)e$ for all $\xi \in L$. Put

$$A := \bigoplus_{i=0}^{2} Le^i \quad \text{and} \quad V := \mathsf{span}_F\langle \xi_0, \eta_0, \zeta_0 \rangle$$

where

$$\xi_0 = e, \ \ \eta_0 = (\alpha + \beta e + e^2)t, \ \ \zeta_0 = (\alpha + \beta e + e^2)\big(t + 2\rho(t)\big).$$

Then (A,V) is a cubic pair if and only if $\beta = a^{-1}\alpha^2$. Suppose that $\beta = a^{-1}\alpha^2$. There exist a square root $x_0 \in F$ of $(4-\lambda^2)/3$ and a cube root ϕ of $(\lambda + (\omega - \omega^2)x_0)/2$ such that

$$\theta = -\phi - \phi^{-1}, \ \theta' = -\omega\phi - \omega^2\phi^{-1}, \ \theta'' = -\omega^2\phi - \omega\phi^{-1}.$$

Put

$$\eta_1 := -\frac{1}{2}\eta_0 + \frac{\omega - \omega^2}{6}\zeta_0, \ \ \zeta_1 := -\frac{1}{2}\eta_0 + \frac{\omega^2 - \omega}{6}\zeta_0.$$

We note that, if $\alpha^3 = a^2$ then $\eta_1^2 = 0$ and (A,V) is singular. Suppose that $\alpha^3 \neq a^2$ then

$$(x\xi_0 + y\eta_1 + z\zeta_1)^3 = ax^3 + b'y^3 + c'z^3 - 3\mu xyz$$

where $b' = \phi^3 a^{-2}(\alpha^3 - a^2)^2$, $c' = \phi^{-3}a^{-2}(\alpha^3 - a^2)^2$, $\mu = a^{-1}\alpha(\alpha^3 - a^2)$. Since $\alpha^3 \neq a^2$ we have $b', c' \neq 0$ and $ab'c' \neq \mu^3$, so $f_{A,V}$ is non-singular. Observe that $\xi_0\eta_1 = \omega\eta_1\xi_0$ and $\xi_0\zeta_1 = \omega^2\zeta_1\xi_0$, thus

$$\zeta_1 = \alpha_{02}\eta_1^2 + \alpha_{12}\xi_0\eta_1^2 + \alpha_{22}\xi_0^2\eta_1^2$$

with $\alpha_{02} = 0$ since $\mathsf{tr}(\eta_1\zeta_1) = 0$. By the lemma above, the pair (A, V) is of the second kind if and only if $\eta_1\zeta_1 = \omega^2\zeta_1\eta_1$, i.e. $\alpha = 0$.

By the remarks preceding Theorems 4.4.2, 4.4.6 and 4.5.2 we know that $f_{A,V}$ is semi-trace but non semi-diagonal. Moreover we may choose the cubic étale F-algebra to be $F \times F(\omega)$ when we write $f_{A,V}$ as a semi-trace form: for $\xi = x\xi_0 + y\eta_0 + z\zeta_0 \in V$,

$$\begin{aligned} f_{A,V}(\xi) &= \Big(x\xi_0 + \big(-y - (\omega - \omega^2)z\big)\eta_1 + \big(-y + (\omega - \omega^2)z\big)\zeta_1\Big)^3 \\ &= \mathsf{Tr}_{(F\times F(\omega))/F}\big(\gamma\Theta(\xi)^3\big) - 3\mu\mathsf{N}_{(F\times F(\omega))/F}\big(\Theta(\xi)\big) \end{aligned}$$

where $\Theta(\xi) = \big(x, -y - (\omega - \omega^2)z\big)$, $\gamma = (a, \phi^3 a^{-2}(\alpha^3 - a^2)^2)$ and

$$\frac{\mathsf{N}_{(F\times F(\omega))/F}(\gamma) - \mu^3}{a^2} = \big(a^{-1}(a^2 - \alpha^3)\big)^3 \in F^{\times 3}.$$

Altogether we proved:

Theorem II *Suppose that F does not contain a primitive cube root of unity and is infinite. Up to F-isomorphism, the non-singular cubic pairs over F are the pairs*

$$\Big(\bigoplus_{i=0}^{2} Le^i, \mathsf{span}_F\langle\xi_0, \eta_0, \zeta_0\rangle\Big)$$

for all Galois $\mathbb{Z}/3$-algebras (L, ρ) and $a, \alpha \in F$ such that $a \neq 0$ and $\alpha^3 \neq a^2$, where $e^3 = a$, $e\xi = \rho(\xi)e$ for all $\xi \in L$,

$$\xi_0 = e, \;\; \eta_0 = (\alpha + a^{-1}\alpha^2 e + e^2)t, \;\; \zeta_0 = (\alpha + a^{-1}\alpha^2 e + e^2)\big(t + 2\rho(t)\big)$$

and $t \in L$ is such that $1, t, \rho(t)$ span L and

$$(x - t)\big(x - \rho(t)\big)\big(x - \rho^2(t)\big) = x^3 - 3x + \lambda$$

for some $\lambda \in F$. Such a cubic pair is of the second kind if and only if $\alpha = 0$. The associated cubic form f is semi-trace, and we may choose the cubic étale algebra over F to be $F \times F(\omega)$:

$$f(\xi) = \mathsf{Tr}_{(F\times F(\omega))/F}\big(\gamma\Theta(\xi)^3\big) - 3\mu\mathsf{N}_{(F\times F(\omega))/F}\big(\Theta(\xi)\big)$$

with the notation from above. However the cubic form is not semi-diagonal.

Now we deal with the singular cubic pairs over F. By Proposition 5.1.2 we know that, if (A,V) is a singular cubic pair over F such that the curve $\{f_{A,V}(\xi)=0\}$ is not a triangle, then $A \cong \mathsf{M}_3(F)$. Therefore the classification of these cubic pairs was done by matrix computations in Chapter 5. In the next statement we draw some particularly interesting conclusions.

Theorem III *There is no cubic pair over F such that the associated cubic curve is three concurrent lines or a conic plus tangent. There exists at least one cubic pair over F such that the associated cubic curve is cuspidal if and only if F contains a primitive cube root of unity. There always exists at least one cubic pair over F such that the associated cubic curve is the zero curve, a triple line, a double line plus simple line, a conic plus chord or a nodal curve.*

Finally we treat the cubic pairs with a triangle as associated cubic curve. By Theorem 5.10.2 we have:

Theorem IV *Suppose that F contains a primitive cube root of unity. Then, up to F-isomorphism, the F-cubic pairs with a triangle as associated cubic curve are the pairs*

$$\big((a,b)_{\omega,F}, \mathsf{span}_F\langle \xi_0, \eta_0, \xi_0^2\eta_0^2\rangle\big)$$

for all $a,b \in F^\times$, where ξ_0 and η_0 are generators of the symbol algebra such that $\xi_0^3=a$, $\eta_0^3=b$ and $\xi_0\eta_0=\omega\eta_0\xi_0$. The associated cubic form is semi-diagonal:

$$(x\xi_0+y\eta_0+z\xi_0^2\eta_0^2)^3 = ax^3+by^3+cz^3-3\lambda xyz$$

where $c=a^2b^2$, $\lambda=\omega^2ab$ and $abc=\lambda^3$.

Suppose that F does not contain a primitive cube root of unity and is infinite. By the remarks preceding Theorem 5.10.3, if (A,V) is a cubic pair over F such that $\{f_{A,V}(\xi)=0\}$ is a triangle, then there exist a Galois $\mathbb{Z}/3$-algebra (L,ρ) and $a\in F^\times$ such that

$$(A,V)\cong\Big(\bigoplus_{i=0}^{2} Le^i, \mathsf{span}_F\langle e, et, e\rho(t)\rangle\Big)$$

where $e^3=a$, $e\xi=\rho(\xi)e$ for all $\xi\in L$, and $t\in L$ is such that $1,t,\rho(t)$ span L and $(x-t)\big(x-\rho(t)\big)\big(x-\rho^2(t)\big)=x^3-3x+\lambda$ for some $\lambda\in F$.

Conversely, let (L, ρ) be a Galois $\mathbb{Z}/3$-algebra and $a \in F^\times$. Let $t \in L$ be such that $1, t, \rho(t)$ span L and

$$(x-t)\big(x-\rho(t)\big)\big(x-\rho^2(t)\big) = x^3 - 3x + \lambda$$

for some $\lambda \in F$. We may assume that

$$t = \begin{pmatrix} \theta & 0 & 0 \\ 0 & \theta' & 0 \\ 0 & 0 & \theta'' \end{pmatrix} \quad \text{and} \quad \rho(t) = \begin{pmatrix} \theta' & 0 & 0 \\ 0 & \theta'' & 0 \\ 0 & 0 & \theta \end{pmatrix}$$

where $\theta, \theta', \theta''$ are the roots of $x^3 - 3x + \lambda$. Put

$$e := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ a & 0 & 0 \end{pmatrix},$$

$A := \bigoplus_{i=0}^2 Le^i$ and $V = \mathsf{span}_F\langle e, et, e\rho(t)\rangle$. Then the cubic curve associated to $f_{A,V}$ is a triangle: for $\xi = xe + yet + ze\rho(t) \in V$,

$$\begin{aligned} f_{A,V}(\xi) &= a\mathsf{N}_{L/F}\big(x + yt + z\rho(t)\big) \\ &= a(x + y\theta + z\theta')(x + y\theta' + z\theta'')(x + y\theta'' + z\theta). \end{aligned}$$

We may also choose the cubic étale algebra to be $F \times F(\omega)$ when we write $f_{A,V}$ as a semi-trace form. Indeed, put

$$\eta_1 := -\frac{1}{2}et + \frac{\omega - \omega^2}{6}e\big(t + 2\rho(t)\big), \quad \zeta_1 := -\frac{1}{2}et + \frac{\omega^2 - \omega}{6}e\big(t + 2\rho(t)\big),$$

and let $x_0 \in F$ be a square root of $(4 - \lambda^2)/3$ and $\phi \in F_{\mathsf{sep}}$ a cube root of $\big(\lambda + (\omega - \omega^2)x_0\big)/2$ such that

$$\theta = -\phi - \phi^{-1}, \;\; \theta' = -\omega\phi - \omega^2\phi^{-1}, \;\; \theta'' = -\omega^2\phi - \omega\phi^{-1}.$$

Then for all $\xi = xe + yet + ze\big(t + 2\rho(t)\big) \in V$

$$\begin{aligned} f_{A,V}(\xi) &= \Big(x\xi_0 + \big(-y - (\omega - \omega^2)z\big)\eta_1 + \big(-y + (\omega - \omega^2)z\big)\zeta_1\Big)^3 \\ &= ax^3 + a\phi^3y^3 + a\phi^{-3}z^3 - 3axyz \\ &= \mathsf{Tr}_{(F \times F(\omega))/F}\big(\alpha\Theta(\xi)^3\big) - 3\mu\mathsf{N}_{(F \times F(\omega))/F}\big(\Theta(\xi)\big) \end{aligned}$$

where $\Theta(\xi) = \big(x, -y - (\omega - \omega^2)z\big)$, $\alpha = (a, a\phi^3)$, $\mu = a$ and

$$\mathsf{N}_{(F \times F(\omega))/F}(\alpha) = \mu^3.$$

Theorem V *Suppose that F is infinite and does not contain a primitive cube root of unity. Up to F-isomorphism, the F-cubic pairs with a triangle as associated cubic curve, are the pairs*

$$\Big(\bigoplus_{i=0}^{2} Le^i, \mathsf{span}_F\langle e, et, e\rho(t)\rangle\Big)$$

for all Galois $\mathbb{Z}/3$-algebras (L,ρ) and $a \in F^\times$, where $e^3 = a$, $e\xi = \rho(\xi)e$ for all $\xi \in L$, and $t \in L$ is such that $1, t, \rho(t)$ span L and

$$(x-t)\big(x-\rho(t)\big)\big(x-\rho^2(t)\big) = x^3 - 3x + \lambda$$

for some $\lambda \in F$. The associated cubic form f is semi-trace, and we may choose the cubic étale F-algebra to be $F \times F(\omega)$:

$$f(\xi) = \mathsf{Tr}_{(F\times F(\omega))/F}\big(\alpha\Theta(\xi)^3\big) - 3\mu\mathsf{N}_{(F\times F(\omega))/F}\big(\Theta(\xi)\big)$$

with notation as above.

Taken together, Theorems I–V above lead to the following result on division algebras:

Theorem VI *Let (A, V) be a cubic pair over F such that A is a division algebra. If F contains a primitive cube root of unity then the associated cubic form $f_{A,V}$ is semi-diagonal. If F is infinite and does not contain a primitive cube root of unity then $f_{A,V}$ is semi-trace, and we may choose the cubic étale algebra over F to be $F \times F(\omega)$.*

By this theorem, the results of Haile and Tignol which we mentioned in the introduction can be improved as follows:

Theorem VII *Suppose that (A, V) is a F-cubic pair where F contains a primitive cube root of unity and A is a division algebra, and let $(\varphi_1, \varphi_2, \varphi_3)$ be a basis of $V^\star$, $a_1, a_2, a_3, \lambda \in F$ such that*

$$f = a_1\varphi_1^3 + a_2\varphi_2^3 + a_3\varphi_3^3 - 3\lambda\varphi_1\varphi_2\varphi_3,$$

Then either $a_1a_2a_3 = \lambda^3$ or there exists one and only one $i \in \{1,2,3\}$ such that $(a_1a_2a_3 - \lambda^3)a_i^{-2}$ is a non-zero cube in F; in the first case necessarily

$$A \cong (a_1, a_2)_{\omega^{\pm 1},F} \cong (a_1, a_3)_{\omega^{\pm 1},F} \cong (a_2, a_3)_{\omega^{\pm 1},F};$$

in the second case necessarily $A \cong (a_i, a_j)$ for all $j \in \{1,2,3\}$, $i \neq j$.

In particular, we deduce the following theorem:

Theorem VIII *Let A, A' be division algebras of degree 3 over F. If V, V' are such that (A, V) and (A', V') are F-cubic pairs and $f_{A,V}$ is equivalent to $f_{A',V'}$, then the algebras A and A' are either isomorphic or anti-isomorphic.*

Proof : Let A^{op} denote the opposite algebra of A. By Theorem VII we have either $A \otimes_F F(\omega) \cong A' \otimes_F F(\omega)$ or $A^{\mathsf{op}} \otimes_F F(\omega) \cong A' \otimes_F F(\omega)$. In other words, the field $F(\omega)$ is a splitting field of either $A^{\mathsf{op}} \otimes_F A'$ or $A \otimes_F A'$. Since the algebras $A^{\mathsf{op}} \otimes_F A'$ and $A \otimes_F A'$ are degree 9 central simple F-algebras, and the degree of the field extension $F(\omega)/F$ divides 2, we deduce that either $A \cong A'$ or $A^{\mathsf{op}} \cong A'$. $\square$

Appendix

Some heavy computations in Chapter 4 were not done by hand: an Apple computer equipped with Wolfram's Mathematica software was of great help. We explain the use of Mathematica on several examples.

A.1 Some of Mathematica's commands

In this thesis we used Mathematica as a powerful calculator that can deal with matrices, polynomials, etc. The following commands were particularly useful:

- **Clear[a, b, c, . . .]** initializes the variables $a, b, c, \ldots$;
- **a = 1** associates to the variable a the value 1;
- **;** executes a computation without showing it;
- **{{1, 2, 3}, {4, 5, 6}}** represents the matrix 2×3 with $(1, 2, 3)$ on the first line and $(4, 5, 6)$ on the second one;
- **I** represents a complex number with square equal to -1;
- **m[[i, j]]** gives the element on row i and column j in the matrix m;
- **Det[m]** gives the determinant of the matrix m;
- **Tr[m]** gives the trace of the matrix m;
- **m. n** gives the multiplication of two matrices m and n;
- **Inverse[m]** gives the inverse of the matrix m;
- **IdentityMatrix[n]** gives the identity matrix of order n;

- **D[f, x, y]** gives the partial derivative $\frac{\partial^2 f}{\partial x \partial y}$;
- **Simplify[a]** gives a in a "more simplified way"
- **a /. b − > c** gives a replacing the variable b by the value c;
- **Factor[p]** factorizes the polynomial p;
- **Solve[{a == 0, b == 0}, {x, y, z}]** gives the solutions (x, y, z) of the system of equations $a = 0$, $b = 0$.

A.2 Conjugation of cubic subspaces

In the proof of Theorem 4.2.7 we state the existence of a matrix which conjugates two special subspaces of $\mathsf{M}_3(F_{\mathsf{sep}})$. Recall that a special subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$ is a cubic subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$ which is spanned by $u, v, w_i(\alpha)$ for some $i = 1, 2, 3$ and some $\alpha \in F_{\mathsf{sep}}$, where $u, v, w_i(\alpha)$ are the matrices introduced on page 53. We observed earlier that, given a special subspace V there are exactly 27 elements $mF_{\mathsf{sep}}^{\times} \in \mathsf{PGL}_3(F_{\mathsf{sep}})$ such that mVm^{-1} is also special. These elements $mF_{\mathsf{sep}}^{\times}$ are completely determined by their action on the flexes and the harmonic points of the cubic curve $\{f_V(\xi) = 0\}$.

Let us, as an example of our use of Mathematica, compute one of these elements. Thereto we must first find the flexes of $\{f_{V_\alpha}(\xi) = 0\}$ where $V_\alpha = \mathsf{span}_{F_{\mathsf{sep}}}\langle u, v, w_1(\alpha)\rangle$.

Let p be a flex of $\{f_{V_\alpha}(\xi) = 0\}$. Since p is a point of $\{f_{V_\alpha}(\xi) = 0\}$, we may write $p = \big(au + bv + cw_1(\alpha)\big)F_{\mathsf{sep}}$ with

$$a^2 b = \alpha^3(9\alpha - 1)c^3 + b^2 c + \frac{1}{4}(24\alpha^2 - 12\alpha + 1)bc^2.$$

Indeed,

In[1]:= $\mathbf{Clear[\boldsymbol{\alpha}]}$

In[2]:= $\mathbf{u = \{\{0,1,0\},\{0,0,1\},\{0,0,0\}\};}$
$\mathbf{v = \{\{0,0,0\},\{1,0,0\},\{1,-1,0\}\};}$
$\mathbf{w = \{\{\boldsymbol{\alpha}, \frac{-1}{2}, 1\},\{3\boldsymbol{\alpha}^2, -2\boldsymbol{\alpha}, \frac{1}{2}\},\{0,-3\boldsymbol{\alpha}^2,\boldsymbol{\alpha}\}\};}$

In[5]:= $\mathbf{Clear[x, y, z]}$

In[6]:= $\mathbf{f = Det[x\ u + y\ v + z\ w];}$

In[7]:= $\mathbf{Simplify[}$
$\quad\mathbf{f-}$
$\quad\quad\mathbf{(x^2y-}$
$\quad\quad\quad\mathbf{(\boldsymbol{\alpha}^3(9\boldsymbol{\alpha}-1)z^3 + y^2z + \frac{1}{4}(24\boldsymbol{\alpha}^2 - 12\boldsymbol{\alpha} + 1)yz^2))]}$

Out[7]= 0

If $c = 0$, then $p = uF_{\mathsf{sep}}$ because $a^2b = 0$ and

$$-8b^3 + 2(24\alpha^2 - 12\alpha + 1)a^2b = 0$$

as verified by:

In[8]:= $\mathbf{h = Det[\{\{D[f, x, x], D[f, x, y], D[f, x, z]\},}$
$\quad\mathbf{\{D[f, x, y], D[f, y, y], D[f, y, z]\},}$
$\quad\mathbf{\{D[f, x, z], D[f, y, z], D[f, z, z]\}\}];}$

In[9]:= $\mathbf{z = 0;}$

In[10]:= $\mathbf{Simplify[h - (-8y^3 + 2(24\boldsymbol{\alpha}^2 - 12\boldsymbol{\alpha} + 1)x^2y)]}$

Out[10]= 0

Now if $c = 1$, then either $b = \alpha(9\alpha - 1)$ or

$$b^3 + \alpha(9\alpha - 1)b^2 + \alpha^2(3\alpha - 1)(9\alpha - 1)b + 3\alpha^5(9\alpha - 1)$$

as shown by:

In[11]:= $\mathbf{z = 1;}$

In[12]:= $\mathbf{r =}$
$\quad\mathbf{y\ h\ /.}$
$\quad\mathbf{x^2 \rightarrow \left(\boldsymbol{\alpha}^3(9\boldsymbol{\alpha} - 1) + y^2 + \left(6\boldsymbol{\alpha}^2 - 3\boldsymbol{\alpha} + \frac{1}{4}\right)y\right)}$
$\quad\mathbf{y^{-1};}$

In[13]:= $\mathbf{Factor[r]}$

Out[13]= $-8(\mathrm{y} + \alpha - 9\alpha^2)(\mathrm{y}^3 - \mathrm{y}^2\alpha + \mathrm{y}\alpha^2 +$
$\quad 9\mathrm{y}^2\alpha^2 - 12\mathrm{y}\alpha^3 + 27\mathrm{y}\alpha^4 - 3\alpha^5 + 27\alpha^6)$

In[14]:= $\mathbf{Simplify[}$
$\quad\mathbf{r - (-8)(y - \boldsymbol{\alpha}(9\boldsymbol{\alpha} - 1))}$
$\quad\mathbf{(y^3 + \boldsymbol{\alpha}(9\boldsymbol{\alpha} - 1)y^2 + \boldsymbol{\alpha}^2(3\boldsymbol{\alpha} - 1)(9\boldsymbol{\alpha} - 1)y +}$
$\quad\mathbf{3\boldsymbol{\alpha}^5(9\boldsymbol{\alpha} - 1))]}$

Out[14]= 0

We shall find the roots of the polynomial

$$s = y^3 + \alpha(9\alpha - 1)y^2 + \alpha^2(3\alpha - 1)(9\alpha - 1)y + 3\alpha^5(9\alpha - 1)$$

using Cardano's method:

In[15]:= $\mathbf{s = y^3 + \boldsymbol{\alpha}(9\boldsymbol{\alpha} - 1)y^2 + \boldsymbol{\alpha}^2(3\boldsymbol{\alpha} - 1)(9\boldsymbol{\alpha} - 1)y +}$
$\quad\mathbf{3\boldsymbol{\alpha}^5(9\boldsymbol{\alpha} - 1);}$

In[16]:= $\mathbf{a = \boldsymbol{\alpha}(9\boldsymbol{\alpha} - 1);}$
$\mathbf{b = \boldsymbol{\alpha}^2(3\boldsymbol{\alpha} - 1)(9\boldsymbol{\alpha} - 1);}$
$\mathbf{c = 3\boldsymbol{\alpha}^5(9\boldsymbol{\alpha} - 1);}$

In[19]:= $\mathbf{y = t - \frac{a}{3};}$

In[20]:= $\mathbf{Simplify[}$
$\quad\mathbf{s-}$

$$\left(\mathbf{t^3} + \tfrac{2}{3}(1 - 9\alpha)\alpha^2\mathbf{t} + \tfrac{1}{27}\alpha^3(9\alpha - 1)(72\alpha - 7)\right)]$$

Out[20]= 0

In[21]:= $\mathbf{p} = \frac{2}{3}\alpha^2(1 - 9\alpha);$
$\mathbf{q} = \frac{1}{27}\alpha^3(9\alpha - 1)(72\alpha - 7);$

In[23]:= $\mathbf{Simplify}[\frac{\mathbf{q}^2}{4} + \frac{\mathbf{p}^3}{27} - \left(\frac{\alpha^3}{6}(8\alpha - 1)(9\alpha - 1)\right)^2]$

Out[23]= 0

In[24]:= $\mathbf{Simplify}[\frac{\mathbf{q}}{2} - \frac{\alpha^3}{6}(8\alpha - 1)(9\alpha - 1) - \left(\frac{-\alpha}{3}\right)^3(1 - 9\alpha)]$

Out[24]= 0

In[25]:= $\theta = (1 - 9\alpha)^{1/3};$

In[26]:= $\mathbf{y} = \frac{\mathbf{p}}{3}\left(\frac{-\alpha}{3}\theta\right)^{-1} - \left(\frac{-1}{3}\theta\right) - \frac{\mathbf{a}}{3};$

In[27]:= $\mathbf{Simplify}[\mathbf{y} - \frac{\alpha}{3}(-2\theta^2 + \theta + 1 - 9\alpha)]$

Out[27]= 0

So the roots of s are those $x = \frac{\alpha}{3}(-2\theta^2 + \theta + 1 - 9\alpha)$ where θ is a cube root of $1 - 9\alpha$. Since

In[28]:= $\omega = \frac{1}{2}(-1 + \mathbf{I}\sqrt{3});$

In[29]:= $\mathbf{Simplify}$
$[\left(\mathbf{y}^2 + \alpha^3(9\alpha - 1) + \mathbf{y}\left(6\alpha^2 - 3\alpha + \frac{1}{4}\right)\right)\mathbf{y}^{-1} -$
$\left(\frac{\omega - \omega^2}{18}(-4\theta^2 + 2\theta - 1)\right)^2]$

Out[29]= 0

In[30]:= $\mathbf{y} = \alpha(9\alpha - 1);$

In[31]:= $\mathbf{Simplify}[$
$\left(\mathbf{y}^2 + \alpha^3(9\alpha - 1) + \mathbf{y}\left(6\alpha^2 - 3\alpha + \frac{1}{4}\right)\right)\mathbf{y}^{-1} -$
$\left(\frac{1}{2}(8\alpha - 1)\right)^2]$

Out[31]= 0

the nine flexes of $\{f_{V_\alpha}(\xi) = 0\}$ are

$$\begin{array}{ccc}
uF_{\mathsf{sep}} & (a'u + b'v + w)F_{\mathsf{sep}} & (-a'u + b'v + w)F_{\mathsf{sep}} \\
(a_1u + b_1v + w)F_{\mathsf{sep}} & (a_2u + b_2v + w)F_{\mathsf{sep}} & (a_3u + b_3v + w)F_{\mathsf{sep}} \\
(-a_1u + b_1v + w)F_{\mathsf{sep}} & (-a_3u + b_3v + w)F_{\mathsf{sep}} & (-a_2u + b_2v + w)F_{\mathsf{sep}}
\end{array}$$

where

$$\begin{aligned}
a' &= \frac{1}{2}(8\alpha - 1), \\
b' &= \alpha(9\alpha - 1), \\
a_1 &= \frac{\omega - \omega^2}{18}(-4\theta^2 + 2\theta - 1), \\
b_1 &= \frac{\alpha}{3}(-2\theta^2 + \theta + 1 - 9\alpha),
\end{aligned}$$

$a_i = \sigma^i(a_1)$, $b_i = \sigma^i(b_1)$ and $w = w_1(\alpha)$, the automorphism σ of $F(\theta, \omega)$ being defined by $\sigma(\omega) = \omega$ and $\sigma(\theta) = \omega\theta$.

Now that we have described all the flexes of $\{f_{V_\alpha}(\xi) = 0\}$, for an arbitrary flex, we shall give a harmonic point for each of them. Let $\big(au + bv + w_1(\alpha)\big)F_{\mathsf{sep}}$ be a flex of $\{f_{V_\alpha}(\xi) = 0\}$. Put $u_2 = au + bv + w_1(\alpha)$ and $u_3 = -au + bv + w_1(\alpha)$. Then $u_3 F_{\mathsf{sep}}$ is a flex and the points uF_{sep}, $u_2 F_{\mathsf{sep}}$, $u_3 F_{\mathsf{sep}}$ are collinear. By Proposition 2.3.5 there is a harmonic point of $u_2 F_{\mathsf{sep}}$ on the line passing through vF_{sep} and $u_3 F_{\mathsf{sep}}$. Thus there exists a unique $c \in F_{\mathsf{sep}}$ such that $v_2 F_{\mathsf{sep}} = \big(-au + cv + w_1(\alpha)\big)F_{\mathsf{sep}}$ is a harmonic point of $u_2 F_{\mathsf{sep}}$, namely

$$c = \frac{\alpha^3(1-9\alpha)b + b^3}{\alpha^3(1-9\alpha) + 4a^2 b + b^2}.$$

Indeed $\det(v_2) = 0$, $\mathsf{tr}(u_2 v_2^2) = 0$ and

```
In[32]:=  Clear[a, b, c]
In[33]:=  u2 = a u + b v + w;
          v2 = −a u + c v + w;
In[35]:=  r = Tr[u2. v2. v2];
          Simplify[Det[v2]]
Out[36]=  −c^2 + α^3 − 9α^4 + c(− 1/4 + a^2 + 3α − 6α^2)
In[37]:=  s = r /. c^2 → α^3 − 9α^4 + c(− 1/4 + a^2 + 3α − 6α^2);
In[38]:=  t =
           s /.
            α^2 → −1/6(− a^2 + α^3(9α − 1)b^−1 + b − 3α + 1/4);
In[39]:=  Simplify[
           b t−
            (α^3(1 − 9α)b + b^3 − c(α^3(1 − 9α) + 4a^2b + b^2))]
Out[39]=  0
```

In the preceding computations we may replace α^2 by

$$-\frac{1}{6}\Big(-a^2 + \alpha^3(9\alpha - 1)b^{-1} + b - 3\alpha + \frac{1}{4}\Big)$$

in s because $a^2 b + \alpha^3(1 - 9\alpha) - b^2 - (6\alpha^2 - 3\alpha + 1/4)b = 0$. We have $\alpha^3(1-9\alpha) + 4a^2 b + b^2 \neq 0$ since otherwise $\alpha^3(1-9\alpha)b + b^3 = 0$ and $a = 0$ which is impossible because $u_2 F_{\mathsf{sep}}$ is not a point on the harmonic polar at uF_{sep}.

To find all the matrices which conjugate V_α into another special subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$, we search for any flex $\tilde{u}F_{\mathsf{sep}}$ of $\{f_{V_\alpha}(\xi) = 0\}$ and for any harmonic point $\tilde{v}F_{\mathsf{sep}}$ of $\tilde{u}F_{\mathsf{sep}}$, the unique $mF_{\mathsf{sep}}^{\times} \in \mathsf{PGL}_3(F_{\mathsf{sep}})$ such that

$$m\tilde{u}m^{-1}F_{\mathsf{sep}} = uF_{\mathsf{sep}} \quad \text{and} \quad m\tilde{v}m^{-1}F_{\mathsf{sep}} = vF_{\mathsf{sep}}.$$

We shall give the example where the flex $\tilde{u}F_{\mathsf{sep}}$ is $\big(a'u + b'v + w_1(\alpha)\big)F_{\mathsf{sep}}$ and the harmonic point $\tilde{v}F_{\mathsf{sep}}$ is $\big(-a'u + c'v + w_1(\alpha)\big)F_{\mathsf{sep}}$ with

$$c' = \frac{\alpha^3(1-9\alpha)b' + b'^3}{\alpha^3(1-9\alpha) + 4a'^2b' + b'^2}:$$

In[40]:= $\mathbf{a = \frac{1}{2}(8\alpha - 1);}$
$\mathbf{b = \alpha(9\alpha - 1);}$
$\mathbf{c = \frac{\alpha^3(1-9\alpha)b+b^3}{\alpha^3(1-9\alpha)+4a^2b+b^2};}$

In[43]:= **Clear[m11, m12, m13, m21, m22, m23, m31, m32, m33, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$]**

In[44]:= **m = {{m11, m12, m13}, {m21, m22, m23}, {m31, m32, m33}};**

In[45]:= **Solve[{(m. u2 − $\boldsymbol{\lambda}$u. m)[[1, 1]] == 0,
(m. u2 − $\boldsymbol{\lambda}$u. m)[[1, 2]] == 0,
(m. u2 − $\boldsymbol{\lambda}$u. m)[[1, 3]] == 0,
(m. u2 − $\boldsymbol{\lambda}$u. m)[[2, 1]] == 0,
(m. u2 − $\boldsymbol{\lambda}$u. m)[[2, 2]] == 0,
(m. u2 − $\boldsymbol{\lambda}$u. m)[[2, 3]] == 0,
(m. u2 − $\boldsymbol{\lambda}$u. m)[[3, 1]] == 0,
(m. u2 − $\boldsymbol{\lambda}$u. m)[[3, 2]] == 0,
(m. u2 − $\boldsymbol{\lambda}$u. m)[[3, 3]] == 0,
(m. v2 − $\boldsymbol{\mu}$v. m)[[1, 1]] == 0,
(m. v2 − $\boldsymbol{\mu}$v. m)[[1, 2]] == 0,
(m. v2 − $\boldsymbol{\mu}$v. m)[[1, 3]] == 0,
(m. v2 − $\boldsymbol{\mu}$v. m)[[2, 1]] == 0,
(m. v2 − $\boldsymbol{\mu}$v. m)[[2, 2]] == 0,
(m. v2 − $\boldsymbol{\mu}$v. m)[[2, 3]] == 0,
(m. v2 − $\boldsymbol{\mu}$v. m)[[3, 1]] == 0,
(m. v2 − $\boldsymbol{\mu}$v. m)[[3, 2]] == 0,
(m. v2 − $\boldsymbol{\mu}$v. m)[[3, 3]] == 0},
{m12, m13, m21, m22, m23, m31, m32, m33, $\boldsymbol{\lambda}$, $\boldsymbol{\mu}$}]**

Out[47]= $\Big\{\Big\{\mathrm{m31} \to 3\mathrm{m11}\alpha,$
$\lambda \to 1 - 8\alpha,\ \mathrm{m21} \to \mathrm{m11},\ \mathrm{m32} \to -\mathrm{m11},$
$\mu \to \alpha - 8\alpha^2,\ \mathrm{m12} \to 0,\ \mathrm{m33} \to \mathrm{m11},$
$\mathrm{m22} \to -2\mathrm{m11},\ \mathrm{m13} \to -\frac{\mathrm{m11}}{\alpha},\ \mathrm{m23} \to 0\Big\}\Big\}$

So the matrix

$$m = \begin{pmatrix} \alpha & 0 & -1 \\ \alpha & -2\alpha & 0 \\ 3\alpha^2 & -\alpha & \alpha \end{pmatrix}$$

is such that

$$m\tilde{u}m^{-1}F = uF \quad \text{and} \quad m\tilde{v}m^{-1}F = vF.$$

We determine $mV_\alpha m^{-1}$ computing $mw_1(\alpha)m^{-1}$:

In[48]:= $\mathbf{m = \{\{\alpha, 0, -1\}, \{\alpha, -2\alpha, 0\}, \{3\alpha^2, -\alpha, \alpha\}\};}$

In[49]:= $\mathbf{Simplify[m.\,w.\,Inverse[m] - \left(\frac{1}{4}(1-6\alpha)u - \frac{\alpha(6\alpha-1)(9\alpha-1)}{2(8\alpha-1)}v + \frac{10\alpha-1}{2(8\alpha-1)}w\right)]}$

Out[49]= $\{\{0,0,0\},\{0,0,0\},\{0,0,0\}\}]$

So $mV_\alpha m^{-1} = V_\alpha$ and $mF_{\mathsf{sep}}^\times \in \mathsf{Aut}\big(\mathsf{M}_3(F_{\mathsf{sep}}), V_\alpha\big)(F_{\mathsf{sep}})$.

We use the same method to find an invertible matrix such that

$$\begin{cases} m\big(a_1u + b_1v + w_1(\alpha)\big)m^{-1}F_{\mathsf{sep}} = uF_{\mathsf{sep}}, \\ m\big(-a_1u + c_1v + w_1(\alpha)\big)m^{-1}F_{\mathsf{sep}} = vF_{\mathsf{sep}} \end{cases}$$

with

$$c_1 = \frac{\alpha^3(1-9\alpha)b_1 + b_1^3}{\alpha^3(1-9\alpha) + 4a_1^2b_1 + b_1^2}.$$

But the computations take much more time and the solution of these equations given by Mathematica is too complicated. To find the solution we helped Mathematica to simplify the computations. We shall only give a matrix and check that it is an invertible matrix that satisfies these equations.

First we note that the Hessian point of $u_2F_{\mathsf{sep}} = \big(au+bv+w_1(\alpha)\big)F_{\mathsf{sep}}$ is equal to $w_2F_{\mathsf{sep}} = \big(du + ev + gw_1(\alpha)\big)F_{\mathsf{sep}}$ with

$$d = -ab^{-1}e, \quad e = 3\alpha^3(1-9\alpha) + b^2, \quad g = \alpha^3(1-9\alpha)b^{-1} - b.$$

Indeed, we have $\mathsf{tr}(u_2w_2\xi + w_2v_2\xi) = 0$ for all $\xi \in V_\alpha$ and

In[50]:= $\mathbf{Clear[a, b, d, e, g]}$

In[51]:= $\mathbf{w2 = du + ev + gw;}$

In[52]:= $\mathbf{Simplify[Tr[(u2.\,w2 + w2.\,u2).\,u]]}$

Out[52]= $2(\mathrm{bd} + \mathrm{ae})$

In[53]:= $\mathbf{Simplify[Tr[(u2.\,w2 + w2.\,u2).\,v] + 2b^{-1}\left((a^2+b)e + \left(b^2 + \left(6\alpha^2 - 3\alpha + \frac{1}{4}\right)b\right)g\right)]}$

Out[53]= 0

In[54]:= $\mathbf{Simplify[Tr[(u2.\,w2 + w2.\,u2).\,w] +}$

$$\mathbf{2\Big(\Big(b + 6\alpha^2 - 3\alpha + \tfrac{1}{4}\Big)e +}$$
$$\mathbf{\Big(3\alpha^3(9\alpha - 1) + (6\alpha^2 - 3\alpha + \tfrac{1}{4}\Big)b\Big)g\Big)]}$$

Out[54]= 0

Since

$$6\alpha^2 - 3\alpha + \frac{1}{4} = \alpha^3(1 - 9\alpha) + a^2 b - b^2$$

we have

$$\begin{aligned} (a^2 + b)e + (\alpha^3(1 - 9\alpha) + a^2 b)g &= 0, \\ \big(\alpha^3(1 - 9\alpha)b^{-1} + a^2\big)e + \big(2\alpha^3(9\alpha - 1) - b^2\big)g &= 0 \end{aligned}$$

and we obtain

$$\big(\alpha^3(1 - 9\alpha)b^{-1} - b\big)e + \big(3\alpha^3(9\alpha - 1) - b^2\big)g = 0.$$

Now we give a matrix which is a solution of the equations:

In[55]:= $\mathbf{a = \frac{\omega - \omega^2}{18}(-4\theta^2 + 2\theta - 1);}$
$\mathbf{b = \frac{\alpha}{3}(-2\theta^2 + \theta + 1 - 9\alpha);}$

In[57]:= $\mathbf{g = \alpha^3(1 - 9\alpha)b^{-1} - b;}$
$\mathbf{e = 3\alpha^3(1 - 9\alpha) + b^2;}$
$\mathbf{d = -ab^{-1}(3\alpha^3(1 - 9\alpha) + b^2);}$

In[60]:= $\mathbf{m11 = 1;}$
$\mathbf{m12 = \frac{1}{3\alpha}(\omega\theta^2 + \omega^2\theta + 1 - 9\alpha);}$
$\mathbf{m13 = \frac{1 - \omega^2}{3\alpha}(\theta^2 + 9\alpha - 1);}$
$\mathbf{m21 = \frac{\omega - 1}{3}(\theta - 1);}$
$\mathbf{m22 = \omega^2\theta;}$
$\mathbf{m23 = \frac{1}{3\alpha}(-\omega^2\theta^2 + (9\alpha - 1)\theta + \omega(9\alpha - 1));}$
$\mathbf{m31 = \frac{1}{3}(-\omega^2\theta^2 - \theta + 3(\omega - \omega^2)\alpha - \omega);}$
$\mathbf{m32 = \frac{1}{\omega - \omega^2}(\theta^2 + 9\alpha - 1);}$
$\mathbf{m33 = -\omega^2\theta^2 + \omega(9\alpha - 1);}$

In[69]:= $\mathbf{m = \{\{m11, m12, m13\}, \{m21, m22, m23\},}$
$\mathbf{\{m31, m32, m33\}\};}$

In[70]:= $\mathbf{Simplify[}$
$\mathbf{Det[m]}$
$$\mathbf{\Big(\frac{-\omega + 4(\omega - 1)\alpha}{3(8\alpha - 1)^2(9\alpha - 1)}\theta^2 - \frac{\omega + 2(1 - 4\alpha)\alpha}{3(8\alpha - 1)^2(9\alpha - 1)}\theta -}$$
$$\mathbf{\frac{\omega + 3(1 - 3\omega)\alpha - 24\alpha^2}{3(8\alpha - 1)^2(9\alpha - 1)}\Big)]}$$

Out[70]= 1

In[71]:= $\mathbf{\lambda = \frac{1}{3(\omega - \omega^2)}(2\theta^2 - \theta + 36\alpha - 4);}$

In[72]:= $\mathbf{\mu = \frac{1}{9}\Big(\frac{6\alpha - 1}{1 - 9\alpha}\theta^2 - \theta + 12\alpha - 1\Big);}$

In[73]:= $\mathbf{\nu = \frac{2}{9}(9\alpha - 1)}$

$$(-(-6\alpha-1)\theta^2-(9\alpha-1)\theta+(9\alpha-1)(12\alpha-1));$$

In[74]:= $\beta=\frac{-\omega^2\alpha}{9\alpha-1}$;

In[75]:= $\mathbf{w}'=\{\{\beta,\frac{1}{2}(\omega^2-1)\beta-1),1\},$
$\{0,\omega\beta,\frac{1}{2}((\omega^2-1)\beta+1)\},\{0,0,\omega^2\beta\}\};$

In[76]:= **Simplify[Inverse[m]. u2. m − λ u]**

Out[76]= $\{\{0,0,0\},\{0,0,0\},\{0,0,0\}\}$

In[77]:= **Simplify[Inverse[m]. v2. m − μ v]**

Out[77]= $\{\{0,0,0\},\{0,0,0\},\{0,0,0\}\}$

In[78]:= **Simplify[Inverse[m]. w2. m − ν w′]**

Out[78]= $\{\{0,0,0\},\{0,0,0\},\{0,0,0\}\}$

A.3 Description of cubic pairs

In the classification of non-singular cubic pairs, we gave an explicit representative for each F-isomorphism class of F-cubic pairs. We shall explain how we found this representative for the classification of cubic pairs of the first kind.

We put $A:=\mathsf{M}_3(F)$ and $V:=\mathsf{span}_F\langle u,v,w_1(\alpha)\rangle$ with notation as on page 53. For a non-trivial Galois $\mathbb{Z}/3$-algebra (L,ρ) we want to describe (A',V') where

$$\begin{aligned} A' &= \{\xi\in A_L \mid m\rho(\xi)m^{-1}=\xi\},\\ V' &= \{\xi\in V_L \mid m\rho(\xi)m^{-1}=\xi\}. \end{aligned}$$

Suppose that F contains a primitive cube root of unity. Let $\theta\in L$ be such that $\theta^3=d\in F$ and $\rho(\theta)=\omega\theta$. To describe V' it is sufficient to find the eigenvectors of the endomorphism $\hat{m}\colon V\to V\colon \xi\mapsto m\xi m^{-1}$: if $\xi_0\in V$ is a eigenvector of $\hat{m}$ with eigenvalue ω^i then $\theta^{2-i}\xi_0\in V'$. The eigenvectors of $\hat{m}$ are

$$\begin{aligned} &\alpha(1-6\alpha)v+w_1(\alpha),\\ &\frac{1}{2}(\omega^2-\omega)(8\alpha-1)u+\alpha(9\alpha-1)+w_1(\alpha),\\ &\frac{1}{2}(\omega-\omega^2)(8\alpha-1)u+\alpha(9\alpha-1)+w_1(\alpha), \end{aligned}$$

with eigenvalues $1,\omega^2$ and ω respectively:

In[1]:= **Clear[α]**

In[2]:= $\mathbf{u}=\{\{0,1,0\},\{0,0,1\},\{0,0,0\}\};$
$\mathbf{v}=\{\{0,0,0\},\{1,0,0\},\{1,-1,0\}\};$
$\mathbf{w}=\{\{\alpha,-\frac{1}{2},1\},\{3\alpha^2,-2\alpha,\frac{1}{2}\},\{0,-3\alpha^2,\alpha\}\};$

In[5]:= $\mathbf{m}=\{\{\alpha,0,-1\},\{\alpha,-2\alpha,0\},\{3\alpha^2,-\alpha,\alpha\}\};$

In[6]:= $\boldsymbol{\omega = \frac{1}{2}(-1 + \mathrm{I}\sqrt{3});}$

In[7]:= **Clear[x, y, z]**

In[8]:= $\boldsymbol{\xi = \mathrm{x\ u + y\ v + z\ w};}$

In[9]:= **Solve[{(m. ξ. Inverse[m] − ξ)[[1, 1]] == 0,**
(m. ξ. Inverse[m] − ξ)[[1, 2]] == 0,
(m. ξ. Inverse[m] − ξ)[[1, 3]] == 0,
(m. ξ. Inverse[m] − ξ)[[2, 1]] == 0,
(m. ξ. Inverse[m] − ξ)[[2, 2]] == 0,
(m. ξ. Inverse[m] − ξ)[[2, 3]] == 0,
(m. ξ. Inverse[m] − ξ)[[3, 1]] == 0,
(m. ξ. Inverse[m] − ξ)[[3, 2]] == 0,
(m. ξ. Inverse[m] − ξ)[[3, 3]] == 0}, {x, y}]

Out[9]= $\left\{\left\{\mathrm{x} \to 0, \mathrm{y} \to \frac{1}{2}(\mathrm{z}\alpha - 6\mathrm{z}\alpha^2)\right\}\right\}$

In[10]:= **Solve[{(m. ξ. Inverse[m] − ω²ξ)[[1, 1]] == 0,**
(m. ξ. Inverse[m] − ω²ξ)[[1, 2]] == 0,
(m. ξ. Inverse[m] − ω²ξ)[[1, 3]] == 0,
(m. ξ. Inverse[m] − ω²ξ)[[2, 1]] == 0,
(m. ξ. Inverse[m] − ω²ξ)[[2, 2]] == 0,
(m. ξ. Inverse[m] − ω²ξ)[[2, 3]] == 0,
(m. ξ. Inverse[m] − ω²ξ)[[3, 1]] == 0,
(m. ξ. Inverse[m] − ω²ξ)[[3, 2]] == 0,
(m. ξ. Inverse[m] − ω²ξ)[[3, 3]] == 0}, {x, y}]

Out[10]= $\left\{\left\{\mathrm{x} \to -\frac{1}{2}\mathrm{i}(-\sqrt{3}\mathrm{z} + 8\sqrt{3}\mathrm{z}\alpha), \mathrm{y} \to \alpha(-\mathrm{z} + 9\mathrm{z}\alpha)\right\}\right\}$

In[11]:= **Simplify[**
$\boldsymbol{-\frac{1}{2}\mathrm{I}(-\sqrt{3}\mathrm{z} + 8\sqrt{3}\mathrm{z}\alpha) - \frac{1}{2}(\omega^2 - \omega)(8\alpha - 1)\mathrm{z}}$**]**

Out[11]= 0

In[12]:= **Solve[{(m. ξ. Inverse[m] − ωξ)[[1, 1]] == 0,**
(m. ξ. Inverse[m] − ωξ)[[1, 2]] == 0,
(m. ξ. Inverse[m] − ωξ)[[1, 3]] == 0,
(m. ξ. Inverse[m] − ωξ)[[2, 1]] == 0,
(m. ξ. Inverse[m] − ωξ)[[2, 2]] == 0,
(m. ξ. Inverse[m] − ωξ)[[2, 3]] == 0,
(m. ξ. Inverse[m] − ωξ)[[3, 1]] == 0,
(m. ξ. Inverse[m] − ωξ)[[3, 2]] == 0,
(m. ξ. Inverse[m] − ωξ)[[3, 3]] == 0}, {x, y}]

Out[12]= $\left\{\left\{\mathrm{x} \to \frac{1}{2}\mathrm{i}(-\sqrt{3}\mathrm{z} + 8\sqrt{3}\mathrm{z}\alpha), \mathrm{y} \to \alpha(-\mathrm{z} + 9\mathrm{z}\alpha)\right\}\right\}$

In[13]:= **Simplify[**
$\boldsymbol{\frac{1}{2}\mathrm{I}(-\sqrt{3}\mathrm{z} + 8\sqrt{3}\mathrm{z}\alpha) - \frac{1}{2}(\omega - \omega^2)(8\alpha - 1)\mathrm{z}}$**]**

Out[13]= 0

Thus V' is the F-vector subspace of $\mathsf{M}_3(F_{\mathsf{sep}})$ spanned by

$$\begin{aligned}
\xi_0 &= \alpha(6\alpha-1)v - 2w_1(\alpha),\\
\eta_0 &= \frac{1}{2}(\omega-\omega^2)(8\alpha-1)\theta u + \alpha(1-9\alpha)\theta v - \theta w_1(\alpha),\\
\zeta_0 &= \frac{1}{2}(\omega^2-\omega)(8\alpha-1)\theta^2 u + \alpha(1-9\alpha)\theta^2 v - \theta^2 w_1(\alpha).
\end{aligned}$$

Since we have $\omega \in F$, the algebra A' is a symbol algebra. We note that $m^2 \in A'$ and $m^2\eta_0 = \omega\eta_0 m^2$. But $m^6 \in F^\times$ because $m \in \mathsf{GL}_3(F_{\mathsf{sep}})$ and $mF_{\mathsf{sep}}^\times$ has order 3; also $\eta_0^3 \in F^\times$:

In[14]:= $\mathbf{Clear[d]}$
In[15]:= $\boldsymbol{\theta = \mathrm{d}^{1/3};}$
In[16]:= $\boldsymbol{\eta 0 = \frac{1}{2}(\omega-\omega^2)(8\alpha-1)\theta \mathrm{u} + \alpha(1-9\alpha)\theta \mathrm{v} - \theta \mathrm{w};}$
In[17]:= $\mathbf{Simplify[Det[\boldsymbol{\eta} 0]] - d\boldsymbol{\alpha}(8\boldsymbol{\alpha}-1)^2(9\boldsymbol{\alpha}-1)]}$
Out[17]= 0

So A' is the symbol F-algebra generated by m^2 and η_0. We have

In[18]:= $\boldsymbol{\xi 0 = \alpha(6\alpha-1)\mathrm{v} - 2\mathrm{w};}$
$\boldsymbol{\zeta 0 = \frac{1}{2}(\omega^2-\omega)(8\alpha-1)\theta^2 \mathrm{u} + \alpha(1-9\alpha)\theta^2 \mathrm{v} - \theta^2 \mathrm{w};}$
In[19]:= $\mathbf{Simplify[\boldsymbol{\xi} 0 - \boldsymbol{\alpha}^{-1} m.m]}$
Out[19]= $\{\{0,0,0\},\{0,0,0\},\{0,0,0\}\}$
In[20]:= $\mathbf{Simplify[Det[\boldsymbol{\xi} 0] - \boldsymbol{\alpha}(8\boldsymbol{\alpha}-1)^2]}$
Out[20]= 0
In[22]:= $\mathbf{Simplify[}$
$\boldsymbol{\zeta 0 - \Big(\frac{3\omega^2}{(8\alpha-1)(9\alpha-1)}\xi 0.\eta 0.\eta 0 - \frac{\omega(6\alpha-1)}{\alpha(8\alpha-1)^2(9\alpha-1)}\xi 0.\xi 0.\eta 0.\eta 0\Big)}\mathbf{]}$
Out[22]= $\{\{0,0,0\},\{0,0,0\},\{0,0,0\}\}$

Hence A' is the symbol F-algebra $\big(\alpha(8\alpha-1)^2, d\alpha(8\alpha-1)^2(9\alpha-1)\big)_{\omega,F}$ generated by ξ_0 and η_0, and

$$\zeta_0 = \frac{3\omega^2}{(8\alpha-1)(9\alpha-1)}\xi_0\eta_0^2 - \frac{\omega(6\alpha-1)}{\alpha(8\alpha-1)^2(9\alpha-1)}\xi_0^2\eta_0^2.$$

The cubic form $f_{A',V'}$ is semi-diagonal: we have

$$f_{A',V'} = a\xi_0^{\star 3} + b\eta_0^{\star 3} + c\zeta_0^{\star 3} - 3\lambda\xi_0^\star\eta_0^\star\zeta_0^\star$$

where $(\xi_0^\star, \eta_0^\star, \zeta_0^\star)$ is the dual basis of (ξ_0, η_0, ζ_0), and a, b, c, λ satisfies

$$\frac{abc-\lambda^3}{a^2} \in F^{\times 3}.$$

Indeed:

In[23]:= $\mathbf{f = Det[x.\,\xi0 + y.\,\eta0 + z.\,\zeta0];}$

In[24]:= $\mathbf{a = \alpha(8\alpha - 1)^2;}$

$\mathbf{b = d\alpha(8\alpha - 1)^2(9\alpha - 1);}$

$\mathbf{c = d^2\alpha(8\alpha - 1)^2(9\alpha - 1);}$

$\mathbf{\lambda = d\alpha(8\alpha - 1)^2(1 - 6\alpha);}$

In[28]:= $\mathbf{Simplify[f - (ax^3 + by^3 + cz^3 - 3\lambda xyz)]}$

Out[28]= 0

In[29]:= $\mathbf{Simplify[\frac{abc-\lambda^3}{a^2} - (3d\alpha(8\alpha - 1))^3]}$

Out[29]= 0

Now we suppose that F does not contain a primitive cube root of unity and F is infinite. Let $\theta \in L$ be such that its minimal polynomial over F is equal to $x^3 - 3x + \lambda$ for some $\lambda \in F$. Then there exists a square root x_0 of $(4 - \lambda^2)/3$ in F such that

$$\begin{aligned} \theta &= -\phi - \phi^{-1}, \\ \rho(\theta) &= -\omega\phi - \omega^2\phi^{-1} = \frac{-\theta + \delta}{2}, \\ \rho^2(\theta) &= -\omega^2\phi - \omega\phi^{-1} = \frac{-\theta - \delta}{2} \end{aligned}$$

where $\delta = x_0^{-1}(2\theta^2 + \lambda\theta - 4)$ and ϕ is a cube root of $\big(\lambda + (\omega - \omega^2)x_0\big)/2$ in F_{sep}. Thus we have

$$\rho(\delta) = \frac{-3\theta - \theta}{2}.$$

To find V' we solve the equation $m\rho(\xi)m^{-1} = \xi$ where

$$\xi = (\lambda_0 + \lambda_1\theta + \lambda_2\delta)u + (\mu_0 + \mu_1\theta + \mu_2\delta)v + (\nu_0 + \nu_1\theta + \nu_2\delta)w_1(\alpha):$$

In[30]:= $\mathbf{Clear[\theta, \delta]}$

In[31]:= $\mathbf{a1 = -\frac{1}{2};}$

$\mathbf{a2 = \frac{\alpha(9\alpha-1)}{8\alpha-1};}$

$\mathbf{a3 = \frac{1}{8\alpha-1};}$

In[34]:= $\mathbf{Simplify[m.\,u.\,Inverse[m] - (a1\ u + a2\ v + a3\ w)]}$

Out[34]= $\{\{0,0,0\},\{0,0,0\},\{0,0,0\}\}$

In[35]:= $\mathbf{b1 = -\frac{1}{2\alpha};}$

$\mathbf{b2 = \frac{\alpha}{1-8\alpha};}$

$\mathbf{b3 = \frac{1}{\alpha(1-8\alpha)};}$

In[38]:= $\mathbf{Simplify[m.\,v.\,Inverse[m] - (b1\ u + b2\ v + b3\ w)]}$

Out[38]= $\{\{0,0,0\},\{0,0,0\},\{0,0,0\}\}$

In[39]:= $\mathbf{c1 = \frac{1}{4}(1 - 6\alpha);}$

$\mathbf{c2 = \frac{\alpha(6\alpha-1)(9\alpha-1)}{2(1-8\alpha)};}$

$\mathbf{c3 = \frac{10\alpha-1}{2(8\alpha-1)};}$

In[42]:= $\mathbf{Simplify[m.w.Inverse[m] - (c1\ u + c2\ v + c3\ w)]}$

Out[42]= $\{\{0,0,0\},\{0,0,0\},\{0,0,0\}\}$

In[43]:= $\mathbf{Solve\Big[\Big\{a1\ \lambda0 + b1\ \mu0 + c1\ \nu0 == \lambda0,}$
$\mathbf{a1\Big(\lambda1\frac{-1}{2} + \lambda2\frac{-3}{2}\Big) + b1\Big(\mu1\frac{-1}{2} + \mu2\frac{-3}{2}\Big) +}$
$\mathbf{c1\Big(\nu1\frac{-1}{2} + \nu2\frac{-3}{2}\Big) == \lambda1,}$
$\mathbf{a1\Big(\lambda1\frac{1}{2} + \lambda2\frac{-1}{2}\Big) + b1\Big(\mu1\frac{1}{2} + \mu2\frac{-1}{2}\Big) +}$
$\mathbf{c1\Big(\nu1\frac{1}{2} + \nu2\frac{-1}{2}\Big) == \lambda2,}$
$\mathbf{a2\ \lambda0 + b2\ \mu0 + c2\ \nu0 == \mu0,}$
$\mathbf{a2\Big(\lambda1\frac{-1}{2} + \lambda2\frac{-3}{2}\Big) + b2\Big(\mu1\frac{-1}{2} + \mu2\frac{-3}{2}\Big) +}$
$\mathbf{c2\Big(\nu1\frac{-1}{2} + \nu2\frac{-3}{2}\Big) == \mu1,}$
$\mathbf{a2\Big(\lambda1\frac{1}{2} + \lambda2\frac{-1}{2}\Big) + b2\Big(\mu1\frac{1}{2} + \mu2\frac{-1}{2}\Big) +}$
$\mathbf{c2\Big(\nu1\frac{1}{2} + \nu2\frac{-1}{2}\Big) == \mu2,}$
$\mathbf{a3\ \lambda0 + b3\ \mu0 + c3\ \nu0 == \nu0,}$
$\mathbf{a3\Big(\lambda1\frac{-1}{2} + \lambda2\frac{-3}{2}\Big) + b3\Big(\mu1\frac{-1}{2} + \mu2\frac{-3}{2}\Big) +}$
$\mathbf{c3\Big(\nu1\frac{-1}{2} + \nu2\frac{-3}{2}\Big) == \nu1,}$
$\mathbf{a3\Big(\lambda1\frac{1}{2} + \lambda2\frac{-1}{2}\Big) + b3\Big(\mu1\frac{1}{2} + \mu2\frac{-1}{2}\Big) +}$
$\mathbf{c3\Big(\nu1\frac{1}{2} + \nu2\frac{-1}{2}\Big) == \nu2\Big\},}$
$\mathbf{\{\lambda0, \lambda1, \lambda2, \mu0, \mu1, \mu2\}\Big]}$

Out[43]= $\Big\{\Big\{\lambda0 \to 0, \mu0 \to \frac{1}{2}(\nu0\ \alpha - 6\nu0\ \alpha^2),$
$\lambda1 \to \frac{3}{2}(-\nu2 + 8\nu2\ \alpha), \lambda2 \to \frac{1}{2}(\nu1 - 8\nu1\ \alpha),$
$\mu1 \to -\nu1\ \alpha + 9\nu1\ \alpha^2, \mu2 \to -\nu2\ \alpha + 9\nu2\ \alpha^2\Big\}\Big\}$

Using the fact that

$$\big(u, \theta u, \delta u, v, \theta v, \delta v, w_1(\alpha), \theta w_1(\alpha), \delta w_1(\alpha)\big)$$

is a basis of V_L we obtain that the vectors

$$\begin{aligned}
\xi_0 &= \alpha(6\alpha-1)v + w_1(\alpha),\\
\eta_0 &= \frac{1}{2}(1-8\alpha)\delta u + \alpha(9\alpha-1)\theta v + \theta w_1(\alpha),\\
\zeta_0 &= \frac{3}{2}(8\alpha-1)\theta u + \alpha(9\alpha-1)\delta v + \delta w_1(\alpha)
\end{aligned}$$

span V'. Using the case where F contains a primitive cube root of unity we know that $A'_{F(\omega)}$ is the cyclic algebra

$$\big(\alpha(8\alpha-1)^2, \phi^3\alpha(8\alpha-1)^2(9\alpha-1)\big)_{\omega, F(\omega)}$$

generated by ξ_0 and η_1 where

$$\eta_1 = \frac{1}{2}\eta_0 + \frac{\omega^2-\omega}{6}\zeta_0 = \frac{1}{2}(\omega-\omega^2)(8\alpha-1)\phi u + \alpha(1-9\alpha)\phi v - \phi w_1(\alpha)$$

(since $\phi = -\frac{1}{2}\theta + \frac{\omega-\omega^2}{6}\delta$). To describe A' we shall find a subfield of A' which is Galois extension of degree 3 over F. Thereto we search a matrix in $A'_{F(\omega)}$ with its cube equal to ϕ^3. Since

$$\overline{\eta_1} = \frac{1}{2}(\omega^2-\omega)(8\alpha-1)\phi^{-1}u + \alpha(1-9\alpha)\phi^{-1}v - \phi^{-1}w_1(\alpha),$$

using the case where F contains ω, we have

$$\phi^3\overline{\eta_1} = \frac{3\omega^2}{(8\alpha-1)(9\alpha-1)}\xi_0\eta_1^2 - \frac{\omega(6\alpha-1)}{\alpha(8\alpha-1)^2(9\alpha-1)}\xi_0^2\eta_1^2$$

and $\overline{\eta_1}^3 = \eta_1^3$. Because $\xi_0\eta_1 = \omega\eta_1\xi_0$ and $F(\xi_0)$ is a field we deduce that

$$\eta_2 := \frac{3}{(8\alpha-1)(9\alpha-1)}\xi_0\eta_1 - \frac{6\alpha-1}{\alpha(8\alpha-1)^2(9\alpha-1)}\xi_0^2\eta_1$$

is such that $\eta_2^3 = \phi^3$. We have $\eta_2\overline{\eta_2} = 1$:

In[44]:= $\mathbf{Clear[\lambda]}$

In[45]:= $\mathbf{x0 = \left(\frac{4-\lambda^2}{3}\right)^{1/2};}$

$\boldsymbol{\phi = \left(\frac{\lambda+(\omega-\omega^2)x0}{2}\right)^{1/3};}$

$\boldsymbol{\theta = -\phi-\phi^{-1};}$

$\boldsymbol{\delta = x0^{-1}(2\theta^2+\lambda\theta-4);}$

In[49]:= $\boldsymbol{\eta 0 = \frac{1}{2}(1-8\alpha)\delta u + \alpha(9\alpha-1)\theta v + \theta w;}$

$\boldsymbol{\zeta_0 = \frac{3}{2}(8\alpha-1)\theta u + \alpha(9\alpha-1)\delta v + \delta w;}$

In[51]:= $\boldsymbol{\eta_1 = \frac{1}{2}\eta 0 + \frac{\omega^2-\omega}{6}\zeta 0;}$

$\boldsymbol{\overline{\eta 1} = \frac{1}{2}\eta 0 + \frac{\omega-\omega^2}{6}\zeta 0;}$

In[53]:= $\boldsymbol{\eta 2 = \frac{3}{(8\alpha-1)(9\alpha-1)}\xi 0.\eta 1 - \frac{6\alpha-1}{\alpha(8\alpha-1)^2(9\alpha-1)}\xi 0.\xi 0.\eta 1;}$

$\boldsymbol{\overline{\eta 2} = \frac{3}{(8\alpha-1)(9\alpha-1)}\xi 0.\overline{\eta 1} - \frac{6\alpha-1}{\alpha(8\alpha-1)^2(9\alpha-1)}\xi 0.\xi 0.\overline{\eta 1};}$

In[55]:= $\mathbf{Simplify[\boldsymbol{\eta}2.\overline{\boldsymbol{\eta}2}]}$

Out[55]= $\{\{1,0,0\},\{0,1,0\},\{0,0,1\}\}$

Therefore $F(\eta_3)$ is contained in A' where $\eta_3 = -\eta_2 - \eta_2^{-1}$ and it is a Galois field extension of degree 3 over F. We shall write η_0 and ζ_0 in function of ξ_0 and η_3. We have

$$\eta_1 = \Big(\frac{3\alpha(6\alpha-1)(8\alpha-1)}{9\alpha-1} + \frac{(6\alpha-1)^2}{9\alpha-1}\xi_0 + \frac{9\alpha}{9\alpha-1}\xi_0^2\Big)\eta_2$$

because

In[56]:= $\mathbf{Simplify}[$
$$\left(\frac{3\alpha(6\alpha-1)(8\alpha-1)}{9\alpha-1}\mathbf{IdentityMatrix}[3]+\frac{(6\alpha-1)^2}{9\alpha-1}\boldsymbol{\xi}\mathbf{0}+\frac{9\alpha}{9\alpha-1}\boldsymbol{\xi}\mathbf{0}.\boldsymbol{\xi}\mathbf{0}\right).\left(\frac{3}{(8\alpha-1)(9\alpha-1)}\boldsymbol{\xi}\mathbf{0}-\frac{6\alpha-1}{\alpha(8\alpha-1)^2(9\alpha-1)}\boldsymbol{\xi}\mathbf{0}.\boldsymbol{\xi}\mathbf{0}\right)]$$

Out[56]= $\{\{1,0,0\},\{0,1,0\},\{0,0,1\}\}$

Hence

$$\begin{cases} \eta_0 = \frac{1}{1-9\alpha}\big(3\alpha(6\alpha-1)(8\alpha-1)+(6\alpha-1)^2\xi_0+9\alpha\xi_0^2\big)\eta_3, \\ \zeta_0 = \frac{1}{1-9\alpha}\big(3\alpha(6\alpha-1)(8\alpha-1)+(6\alpha-1)^2\xi_0+9\alpha\xi_0^2\big)\big(\eta_3+2\tau(\eta_3)\big). \end{cases}$$

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Index of notation

A	a central simple algebra of degree 3 over F
A°	the subspace of reduced trace zero elements of A
(A, V)	a cubic pair
$\mathsf{Aut}(A, V)$	the group scheme of automorphisms of (A, V)
F	a field of characteristic neither 2 nor 3
$\overline{F}$	an algebraic closure of F
F_{sep}	the separable closure of F in $\overline{F}$
$\{f(\xi) = 0\}$	the projective curve associated to f
$\{f(\xi) = 0\}_L$	the set of the L-points of $\{f(\xi) = 0\}$
$f_{A,V}$	the cubic form associated to (A, V)
f_V	the cubic form associated to V
Γ	the absolute Galois group of F
H_f	the Hessian curve of f
$\mathsf{N}_{K/F}$	the norm form of the F-algebra K
ω	a primitive cube root of unity in F_{sep}
$\mathbb{P}(V)$	the projective space associated to V
$\langle p, q\rangle$	the line passing through p and q
q_A	the trace quadratic form of A
$\mathsf{S}^d(V^\star)$	the d-th symmetric power of $V^\star$
$\mathsf{span}_F\langle\xi_1, \ldots \xi_r\rangle$	the F-vector space spanned by $\xi_1, \ldots, \xi_r$
t_f	the symmetric trilinear form associated to f
t_V	the symmetric trilinear form associated to f_V
$\mathsf{Tr}_{K/F}$	the trace form of the F-algebra K
Trd_A	the reduced trace of A
V	a 3-dimensional vector space over F
V_R	$V \otimes_F R$
$\overline{V}$	$V_{\overline{F}}$
V_{sep}	$V_{F_{\mathsf{sep}}}$
$V^\star$	the dual space of V
ξ_{ij}	the element on row i and column j in ξ

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