Controller validation for a validated model set†

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Abstract

This paper focuses on the validation of a controller designed from a model validated in an ellipsoidal uncertainty set. A controller is said to be validated if it stabilizes all models in this uncertainty set. This set is embedded in a coprime factor uncertainty set in order to use the results of mainstream robust control theory such as the Vinnicombe gap between plants and the related stability theorems.

1 Introduction

Considered problem. In this paper, we focus on controller validation in the case where a model is validated in a certain uncertainty region which contains the true system. By validation of a controller, we mean the verification that this controller stabilizes all the plants in the uncertainty region. For this purpose, we will take the following assumptions:

1. a model G of a true system G_0 is given;
2. an uncertainty region composed of ellipsoids at each frequency is deduced from the recent model validation theory developed in [8, 11]. This uncertainty region contains the model and the unknown true system with a certain probability level. The case of several uncertainty regions is also considered.
3. a control law C has been designed from the model G, using usual design methods such as LQG, model reference control, H_∞ and so on.

In addition, the proposed approach must easily be used for the design of a new control law, when the avalaible controller C is not satisfactory.

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Related work. The concept of model validation was extended to the concept of controller validation in [13]. In their approach, a model of the system is not avalaible, in contrast to our problem. In fact, our contribution pertains to the so-called "identification for control" approach, which has been intensively investigated recently (see [7, 6, 5, 2] and references therein). In this framework, the underlying problem is to identify a model such that if this model is used for control design, the obtained control law has an achieved performance on the true system that is close to the designed performance. There are also strong connections with uncertainty model identification [9] and robust control [17], since a possible approach is to model the actual system by an uncertainty region to which the true system belongs.

Proposed approach. The paper uses the model validation results[1] presented in [8] (closed-loop method) and [11] (open-loop method). Both methods consist in constructing a dynamic uncertainty region U which is composed of ellipsoids, at each frequency, and to which the true system belongs with a certain probability, say 0.95. A model is then said to be validated if it lies in this validated uncertainty region U.

In this paper, we propose a possible link between the ellipsoidal uncertainty region U resulting from these validation methods and the tools of mainstream robust control theory (see e.g. [17, 12, 15, 18] and references therein). These tools allow one to use some powerful robust stability theorems and controller design methods such as the loop shaping design approach [12]. Robust control theory focuses on particular uncertainty regions (for instance, the so-called additive, (inverse) multiplicative or coprime factor uncertainties), which do not correspond to our uncertainty region U made up of ellipsoids at each frequency. In order to apply the standard robust control results, we propose to embed our uncertainty region U into a standard uncertainty region. A similar approach can be found in [9] where, however, only additive uncertainty sets were considered. In our approach, we consider coprime factor uncertainty sets. Such sets present several advantages. First, a coprime factor uncertainty gives a satisfactory representation of the uncertainty both in high frequencies and in low frequencies (see [16, section 2.3.5]). Second, such sets can also be defined, via the Vinnicombe gap between two plants, as the set of plants whose Vinnicombe distance to a nominal one is less than a given number. A third reason is that some powerful stability and controller design results were developed for this kind of uncertainty set (see [12, 15, 16]).

In order to perform this embedding, we develop computationally attractive tests which boil down to a convex optimization problem involving Linear Matrix Inequality constraints [1].

An approach avoiding the embedding process can also be developed. Nevertheless, such an approach can only be used to validate a controller with respect to a model uncertainty set; it cannot be extended to the design of a robust control law. One of the main advantages of our approach is that it remains practically interesting also when considering control law design. Even if in this paper we heavily focus on the analysis problem, it is part of our continuing investigation of identification and control design interaction.

[1]As these results are given for the SISO case, we will restrict this presentation to SISO systems.

Paper outline. In the second section, the frequency domain uncertainty region U will be related to the validation theory results. In the third section, the coprime factor uncertainty that will be used in order to embed the uncertainty region U will be presented. This coprime factor uncertainty will be linked to the Vinnicombe distance and a new quantity, the worst case Vinnicombe distance, will be defined in this same section. In the fourth section, different stability theorems related to the coprime factor uncertainties described in the third section will be presented and illustrated by an example. Finally, in the last section, some conclusions will be given.

2 Preliminaries on model validation

Both model validation approaches (open-loop method [11] and closed-loop method [8]) consist in determining a dynamic uncertainty region U to which the true system belongs with a certain probability level, say 0.95[2]. This uncertainty region U is deduced from the covariance matrix of the parameters of the estimated error model $\widetilde{G}$, which is an estimate of $\Delta G = G - G_0$, and from the properties of the Gaussian probability density function [8].

The dynamic uncertainty region U is composed of ellipsoids at each frequency around the frequency response of $G + \widetilde{G}$. A model G is said to be validated if G lies in the uncertainty region. Note that whether or not a model is validated depends on the uncertainty region. Indeed, for every validation data set, a new U is constructed and G may or may not lie in the corresponding uncertainty region U. In other words, if G is not validated for a particular U, it may well be validated for another data set (and another U). In this paper, we will always consider a model G that is validated in a certain region U. In the sequel, we will talk about "validated uncertainty region" for such U.

To say that a model G is validated in an uncertainty region U tells us that the frequency responses of G and the true system G_0 (modulo a certain probability level) lie in U. This paper proposes a way to use this information in order to guarantee that a controller C stabilizes all the plants in this validated uncertainty region U (and hence, also the true system G_0).

3 The worst case Vinnicombe distance

As explained in the introduction, an uncertainty region composed of ellipsoids that contains G_0 is not one that fits the standard tools of mainstream robust control theory. Indeed, a validated uncertainty region is neither an additive nor (inverse) multiplicative, nor a coprime perturbation of G, ... whilst robust control theory provides theorems about the stabilization of the true system G_0, but with the assumption that G_0 lies in one of those particular uncertainty regions. In order to link the uncertainty region composed of ellipsoids at each frequency to standard robust control theory, G_0 must be included in one of those particular uncertainty regions. This can be done by embedding the validated region U in one of those

[2]The probability level depends on the number of standard deviations chosen to construct those ellipsoids [8]. For a Gaussian distribution, two standard deviations correspond to 95 %

particular uncertainty sets. In this paper, we have opted for the coprime uncertainty set described in [16], for the reasons given in the introduction.

3.1 Coprime factor uncertainty and the Vinnicombe distance

Since G will be used as our nominal model for control design, the embedding coprime factor uncertainty regions that we now construct will be centered at the model G. The coprime factor uncertainty set of size ξ described in [16] is the set of all plants G_{in} which can be written as a perturbation of the normalized coprime factor description $[N\ D]$ of the model G, with a perturbation $\Delta = \begin{bmatrix} \Delta_N \\ \Delta_D \end{bmatrix} \in L_\infty$ such that $\| \Delta \|_\infty \leq \xi$:

$$\mathcal{G}(G,\xi) = \left\{ G_{in} \mid G_{in} = \frac{N+\Delta_N}{D+\Delta_D}, \quad \| \Delta \|_\infty \leq \xi \quad \text{and} \quad \eta(G_{in}) = wno(D+\Delta_D) \right\}. \tag{1}$$

Here $\eta(G)$ denotes the number of open right half plane poles of G and $wno(G)$ denotes the winding number about the origin of $G(s)$ as s follows the standard Nyquist D-contour indented into the right half plane around any imaginary axis poles and zeros of $G(s)$.

An alternative expression, easier to handle than definition (1), was proposed in [16]:

$$\mathcal{G}(G,\xi) = \{ G_{in} \mid \delta_\nu(G, G_{in}) \leq \xi \}. \tag{2}$$

Expression (2) shows that the coprime factor uncertainty set is a ball of systems, centered on the model G and whose radius is equal to ξ. In this expression, a new quantity, the Vinnicombe distance $\delta_\nu(G, G_{in})$, is used. In the SISO case, this distance is defined as follows:

$$\delta_\nu(G, G_{in}) = \begin{cases} \max_\omega \kappa(G(j\omega), G_{in}(j\omega)) = \max_\omega \frac{|G-G_{in}|}{\sqrt{1+|G|^2}\sqrt{1+|G_{in}|^2}} & \textit{if (4) is satisfied} \\ 1 & \textit{otherwise} \end{cases} \tag{3}$$

The condition to be fulfilled in order to have $\delta_\nu(G, G_{in}) < 1$ is :

$$\begin{aligned} (1+G^* G_{in})(j\omega) \neq 0 \quad \text{for all } \omega \text{ and} \\ wno(1+G^* G_{in}) + \eta(G_{in}) - \tilde{\eta}(G) = 0. \end{aligned} \tag{4}$$

where $G^*(s) = G(-s)$ and $\tilde{\eta}(G)$ denotes the number of closed right half plane poles of G.

If these last two conditions are satisfied, then the distance between two plants has a simple frequency domain interpretation (in the SISO case). Indeed, the quantity $\kappa(G(j\omega), G_{in}(j\omega))$ is the chordal distance between the projections of $G(j\omega)$ and $G_{in}(j\omega)$ onto the Riemann sphere [15]. The distance $\delta_\nu(G, G_{in})$ between G and G_{in} is therefore, according to (3), the supremum of these chordal distances over all frequencies.

These definitions also hold in discrete time via the use of the bilinear transform $s = (z-1)/(z+1)$ [16, page 259]. In the sequel, we will use a discrete time formalism since this formalism is used in the validation methods of [8, 11].

3.2 The worst case Vinnicombe distance

Expression (2) shows that to embed a validated uncertainty region U in a coprime factor uncertainty set $\mathcal{G}(G,\xi)$, one only has to find the smallest size ξ such that $U \subset \mathcal{G}(G,\xi)$. This corresponds to finding the largest Vinnicombe distance between the model G and all plants inside U. This largest distance will be denoted the worst case Vinnicombe distance in the sequel: $\delta_{WC}(G,U)$. A first step will be to make sure that a frequency domain interpretation is sufficient to compute the worst case Vinnicombe distance. This can be guaranteed by making one weak assumption on the structure of U as proved in [16, page 137]. This weak assumption is that, given a metric space Θ and the set of all rational transfer functions $\mathcal{R}$, there exists a mapping

$$\Theta \to \mathcal{R} \quad : \quad \theta \to G_\theta,$$

continuous in the graph topology, such that all plants in U can be parametrized with a θ inside a pathwise connected closed subset of Θ.

Assuming the existence of such a mapping, the size ξ of the embedding coprime factor uncertainty set (i.e. the worst case Vinnicombe distance) can be found using the following procedure. Recall that a validated uncertainty region is composed of ellipses at each frequency centered at $G(e^{j\Omega}) + \widetilde{G}(e^{j\Omega})$ and that both G_0 and G lie in these ellipses since the uncertainty region U is assumed validated. At a particular frequency Ω, we define the worst case chordal distance $\kappa_{WC}(G(e^{j\Omega}), U(\Omega))$ as the maximum chordal distance between the projection of the point $G(e^{j\Omega})$ and the projections of all the points of the ellipsoid onto the Riemann sphere. The computation of this quantity is a convex optimization problem involving LMI constraints (see appendix A.1 for more details).

Definition of the worst case Vinnicombe distance Recalling the definition of the Vinnicombe distance, the worst case Vinnicombe distance is then the maximum over Ω of this chordal distance :

$$\delta_{WC}(G,U) = \max_{\Omega} \kappa_{WC}(G(e^{j\Omega}), U(\Omega)) \tag{5}$$

where $U(\Omega)$ denotes the ellipsoid at frequency Ω.

This definition without respect to the conditions (4) is sufficient. Indeed if there exists a plant $G_U \in U$ and a frequency Ω_0 such that $(1 + G^* G_U)(e^{j\Omega_0}) = 0$[3], then it is proved in [15] that $\kappa(G(e^{j\Omega_0}), G_U(e^{j\Omega_0})) = 1$. The worst case Vinnicombe distance (5) is therefore equal to 1 in such case. Furthermore, the assumption of the existence of a continuous mapping assures that if there exists a plant G_1 in U for which the second part of the conditions (4) is not verified, then there also exists another plant G_2 in U for which the first part fails for a given frequency (see [16]). The chordal distance between the model and this particular plant G_2 at this frequency is then equal to 1 and the worst case Vinnicombe distance (5) is therefore also equal to 1.

[3] $G^*(z) = G(1/z)$

Embedding coprime factor uncertainty set The worst case Vinnicombe distance $\delta_{WC}(G,U)$ having been defined, the coprime factor uncertainty set which embeds U can also be defined as follows:

$$\{G_{in} \mid \delta_\nu(G,G_{in}) \leq \delta_{WC}(G,U)\}. \tag{6}$$

Indeed, all the plants in the uncertainty region lie in this set since the distance between the model G and a plant in U is smaller than (or equal to) $\delta_{WC}(G,U)$.

3.3 Properties of the worst case Vinnicombe distance and of the embedding uncertainty set

Before presenting an example of this worst case Vinnicombe distance, let us give some of its properties. Since the true system G_0 lies in the validated uncertainty region U (with probability 0.95), it also lies in the coprime factor uncertainty set which embeds this validated region U. Therefore,

$$G_0 \in \{G_{in} \mid \delta_\nu(G,G_{in}) \leq \delta_{WC}(G,U)\}. \tag{7}$$

The true system also lies in another set which will be useful later.

$$G_0 \in \{G_{in} \mid \kappa(G,G_{in}) \leq \kappa_{WC}(G,U) \ \forall \ \Omega \in [0,\pi]\} \tag{8}$$

Another property is that the center of the embedding coprime factor uncertainty is the model G while $G+\widetilde{G}$ was the center of the validated uncertainty region U.

Finally, since the true system is now included in a coprime factor uncertainty set, we can apply the different tools of mainstream robust control theory in order to design a robust controller or to verify the stability of the closed-loop composed of the true system G_0 and of the controller designed from the model G. This last point is developed in section 4.

3.4 Example of the worst case Vinnicombe distance

Consider the following true system and model, respectively,

$$y_0 = G_0 u + H_0 e = \frac{z+0.5}{z^2-1.5z+0.7}u + \frac{z^2}{z^2-1.5z+0.7}e$$

$$y = Gu + He = \frac{z+0.503}{z^2-1.545z+0.73}u + \frac{z^2}{z^2-1.545z+0.73}e$$

The actual Vinnicombe distance $\delta_\nu(G,G_0)$ between G and G_0 is equal to 0.023.

For this model G, an open-loop and a closed-loop validation were achieved leading to two uncertainty regions U_{OL} and U_{CL} correponding to a probability level of 0.95 [8]. The controller chosen for closed-loop validation is a dynamic controller such that $u = r-(0.9z-0.8)/(z+0.5)y$.

The model was validated with data sets of 1000 data in open-loop and closed-loop, respectively, having the following statistics:

$$Open-loop: \quad \sigma_u^2 = 0.025 \quad and \quad \sigma_e^2 = 1 \Longrightarrow \sigma_y^2 = 8.05$$

$$Closed-loop: \quad \sigma_r^2 = 0.5 \quad and \quad \sigma_e^2 = 1 \Longrightarrow \sigma_y^2 = 3.28$$

The dynamic uncertainty regions U_{OL} and U_{CL} are represented in Figure 1 and Figure 2, respectively. Observe that G_0 and G lie inside both U_{OL} and U_{CL} for all frequencies.

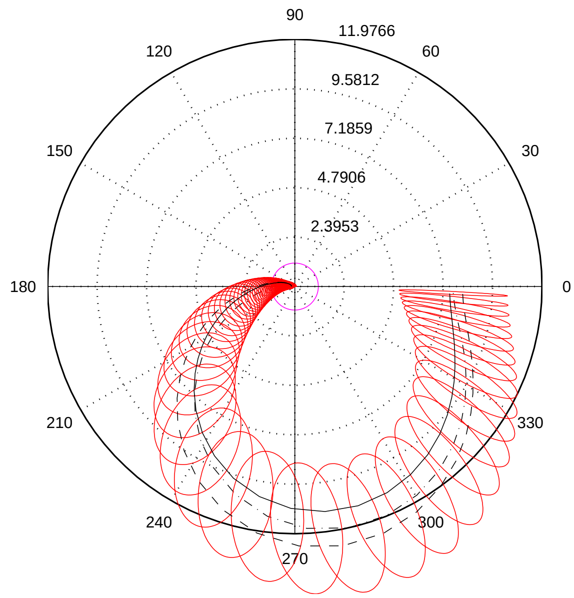

Figure 1: Nyquist plot of the open-loop validation : G (dash), $G + \tilde{G}$ (dashdot), G_0 (solid) and the uncertainty ellipsoids at each frequency

The uncertainty regions being validated, the worst case Vinnicombe distance $\delta_{WC}(G, U)$ can now be computed for the two cases.

$$Open-loop: \quad \delta_{WC}(G, U_{OL}) = 0.3318$$

$$Closed-loop: \quad \delta_{WC}(G, U_{CL}) = 0.0600$$

It can be noted that the two quantities are of course larger than the actual Vinnicombe distance between the true system G_0 and the model G which is equal to 0.023, but the important observation is that the worst case Vinnicombe distance is much smaller in closed-loop than in open-loop. Consequently, it is always better to perform more than one validation experiment and to select the validated uncertainty region that yields the smallest worst case Vinnicombe distance, and hence the tightest bound for the actual Vinnicombe distance between G and G_0. Thus, one should always choose the experimental conditions for identification and validation such that the corresponding uncertainty region U minimizes the worst case Vinnicombe distance $\delta_{WC}(G, U)$.

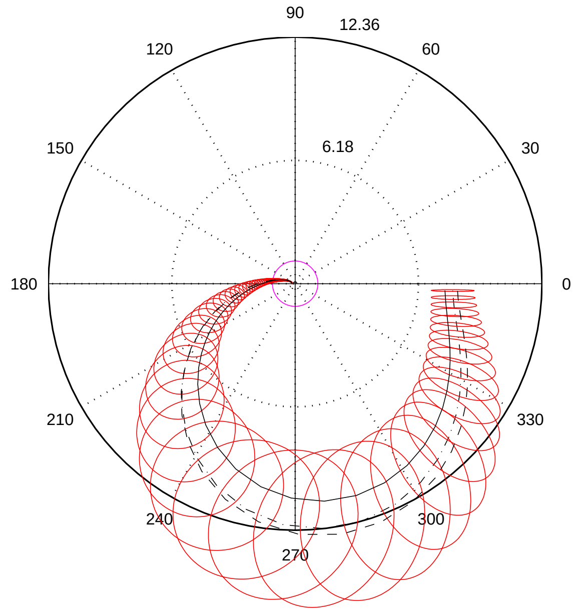

Figure 2: Nyquist plot of the closed-loop validation : G (dash), $G + \widetilde{G}$ (dashdot), G_0 (solid) and the uncertainty ellipsoids at each frequency

4 Stability

4.1 Introduction

In the previous section, the worst case Vinnicombe distance has been defined. This worst case distance allows one to embed a validated uncertainty region U in a coprime factor uncertainty region as can be seen in expression (6). This coprime factor uncertainty can now be used in the Vinnicombe stability results. That would not have been the case if an uncertainty region composed of ellipsoids at each frequency had been used. Before presenting the Vinnicombe stability results, let us first recall the problem we want to solve. From the nominal model $G(z)$, a controller $C(z)$ can be designed. This controller stabilizes G and achieves satisfactory performance with this model. However, this controller is not assured to stabilize the true system G_0. We therefore need conditions guaranteeing this robust stabilization.

4.2 Min-Max type condition

First we recall the definition of generalized stability margin for a closed-loop system made up of the feedback connection of a plant G and a controller C.

Definition:

$$b_{GC} = \begin{cases} \min_\Omega \kappa(G(e^{j\Omega}), -\frac{1}{C(e^{j\Omega})}) & if\ [C\ G]\ is\ stable \\ 0 & otherwise \end{cases} \tag{9}$$

where $\kappa(G_1, G_2)$ was defined in (3).

A main result of [15] is the following sufficient but not necessary condition to guarantee the stabilization of G_0 by C.

Proposition Consider a model G and a controller C that stabilizes G with a stability margin b_{GC}. Then C stabilizes G_0 if

$$G_0 \in \{G_{in} \;\mid\; \delta_\nu(G, G_{in}) < b_{GC}\}. \tag{10}$$

As the true system G_0 is unknown, this condition can not be verified in practice. Nevertheless in the previous section, it has been shown that, modulo a probability level of 0.95, the true system G_0 lies in the coprime factor uncertainty set $\{G_{in} \mid \delta_\nu(G, G_{in}) \leq \delta_{WC}(G, U)\}$ where U is a validated uncertainty region. Therefore, a sufficient condition to guarantee the stabilization of the true system G_0 by a controller C is to verify that :

$$\delta_{WC}(G, U) < b_{GC}. \tag{11}$$

4.3 A less conservative condition

The condition $G_0 \in \{G_{in} \mid \delta_\nu(G, G_{in}) < b_{GC}\}$ is rather conservative. Indeed, it is proved in [15] that the stabilization of the true system G_0 by the controller C is already guaranteed if

$$G_0 \in \{G_{in} \;\mid\; \kappa(G_{in}(e^{j\Omega}), G(e^{j\Omega})) < \kappa(G(e^{j\Omega}), -\frac{1}{C(e^{j\Omega})}) \;\forall\, \Omega \;\mathit{and}\; \delta_\nu(G, G_{in}) < 1\} \tag{12}$$

The classical condition $G_0 \in \{G_{in} \mid \delta_\nu(G, G_{in}) < b_{GC}\}$ is a much stronger (and thus more conservative) constraint since $\delta_\nu(G, G_{in}) = \max_\Omega \kappa(G, G_{in})$ (if $\delta_\nu(G, G_{in}) < 1$) and $b_{GC} = \min_\Omega \kappa(G(e^{j\Omega}), -\frac{1}{C(e^{j\Omega})})$.

With the worst case Vinnicombe distance, this less conservative condition can be rewritten as follows.

Main theorem: Let G be a model and let U be a validated set of models containing G and the true plant G_0. If $\delta_{WC}(G, U) < 1$, then the stabilization of the true system G_0 by C is guaranteed if

$$\kappa_{WC}(G(e^{j\Omega}), U(\Omega)) < \kappa(G(e^{j\Omega}), -\frac{1}{C(e^{j\Omega})}) \quad \forall\, \Omega \in [0, \pi] \tag{13}$$

Proof: According to expression (8), G_0 lies in $\{G_{in} \mid \kappa(G, G_{in}) \leq \kappa_{WC}(G, U) \;\forall\, \Omega\}$. If expression (13) holds, one can thus write

$$G_0 \in \{G_{in} \;\mid\; \kappa(G_{in}(e^{j\Omega}), G(e^{j\Omega})) < \kappa(G(e^{j\Omega}), -\frac{1}{C(e^{j\Omega})}) \;\forall\, \Omega\}$$

In order to verify expression (12), we still have to verify that the distance between G and G_0 is smaller than one. This is indeed the case since the worst case Vinnicombe distance $\delta_{WC}(G, U)$ between the model G and the plants in the uncertainty region U which contains G_0 is smaller than one. □

4.4 Usefulness of having several validated uncertainty regions

If two validated uncertainty regions U_1 and U_2 are available, then two embedding coprime factor uncertainty sets are also available (i.e. two worst case Vinnicombe distances are available). The robust stability can therefore be tested for these two embedding uncertainty sets and it may happen that the stabilization of G_0 by C is established only for one of these two worst case Vinnicombe distances and not for the other (for instance, $\delta_{WC}(G, U_1) < b_{GC} < \delta_{WC}(G, U_2)$), as we shall illustrate in the next subsection.

Furthermore, two uncertainty regions U_1 and U_2 having been validated, a new validated uncertainty region can be constructed from these two, namely their intersection $U_{inter} = U_1 \cap U_2$. Indeed, since G lies in both U_1 and U_2, it also lies in U_{inter}. The true system G_0 lies in U_{inter} with probability $(0.95)^2 = 0.9025$ since it lies in U_1 with probability 0.95 and in U_2 with probability 0.95.

The advantage of using the intersection U_{inter} as uncertainty region is that we have:

$$\kappa_{WC}(G(e^{j\Omega}), U_{inter}(\Omega)) \leq min(\kappa_{WC}(G(e^{j\Omega}), U_1(\Omega)), \kappa_{WC}(G(e^{j\Omega}), U_2(\Omega))) \tag{14}$$

$$\delta_{WC}(G, U_{inter}) \leq min(\delta_{WC}(G, U_1), \delta_{WC}(G, U_2)) \tag{15}$$

It is then possible that the stabilization by a controller C is established for all plants in U_{inter}, but not separately for U_1 or U_2. Conversely, if the stability is proved for U_1 or U_2, then it is also proved for U_{inter}. However, the use of U_{inter} also has a drawback : the probability that G_0 lies in U_{inter} is only 0.9025 instead of 0.95 with U_1 and U_2.

The computation of the worst case chordal distance $\kappa_{WC}(G(e^{j\Omega}), U_{inter}(\Omega))$ at a given frequency for a region U_{inter} that is the intersection of two ellipsoidal regions, is explained in appendix A.2.

4.5 Sequel of the example

In order to illustrate the results of this section, we take the same true system G_0 and model G as in Section 3.4. For this choice of systems, we consider the following controller C which stabilizes the model G:

$$u = r - C(z)y = r - \frac{z - 0.6}{z + 0.7}y$$

In practice, the true system is unknown. But, in this case, the stabilization of G_0 by the controller can easily be tested. Indeed, the stability margin b_{GC} is equal to 0.0347 which is larger than the actual Vinnicombe distance between the true system and the model (=0.023).

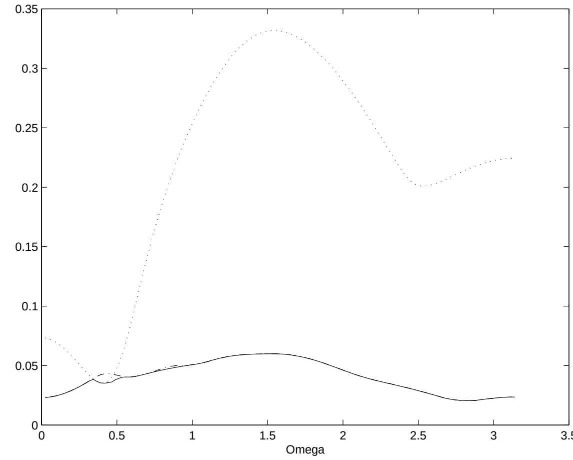

Figure 3: Frequency by frequency comparison of $\kappa_{WC}(G(e^{j\Omega}), U_{OL}(\Omega))$ (dotted), $\kappa_{WC}(G(e^{j\Omega}), U_{CL}(\Omega))$ (dashdot), and $\kappa_{WC}(G(e^{j\Omega}), U_{inter}(\Omega))$ (solid). $\kappa_{WC}(G(e^{j\Omega}), U_{CL}(\Omega))$ and $\kappa_{WC}(G(e^{j\Omega}), U_{inter}(\Omega))$ coincide at almost every frequency

Consequently, C stabilizes G_0 according to (10).

In practice, the only information we have about G_0 is that it lies in the validated uncertainty regions U_{OL} and U_{CL} with probability level 0.95. These uncertainty regions have both been embedded in a coprime factor uncertainty set and the corresponding worst case Vinnicombe distances $\delta_{WC}(G, U_{OL})$ and $\delta_{WC}(G, U_{CL})$ were equal to 0.3318 and 0.06, respectively. A third validated uncertainty region can be deduced from these two, namely $U_{inter} = U_1 \cap U_2$. The worst case Vinnicombe distance for U_{inter} is also equal to 0.06. Figure 3 represents the worst case chordal distance at each frequency for U_{OL}, U_{CL} and U_{inter}. It can be seen in this figure that $\kappa_{WC}(G(e^{j\Omega}), U_{inter}(\Omega))$ is a lower bound for the minimum of the chordal distances related to U_{OL} and U_{CL} at each frequency.

The stabilization of G_0 by the controller C can only be guaranteed by verifying expression (11) or expression (13). It is to be recalled that if expression (13) is satisfied, then expression (11) is also verified. Therefore, the less conservative condition will always be prefered. We show the results with the Min-Max type condition to underline the extreme conservatism of this condition.

Verification of the Min-Max type condition (11): In this example, the stabilization of G_0 by C can not be deduced from the condition (11), neither with the uncertainty regions U_{OL} or U_{CL}, nor with their intersection. The corresponding worst case Vinnicombe distances are all larger than b_{GC}.

$$b_{GC} = 0.0347 < \delta_{WC}(G, U_{CL}) = \delta_{WC}(G, U_{inter}) = 0.06 < \delta_{WC}(G, U_{OL}) = 0.3318$$

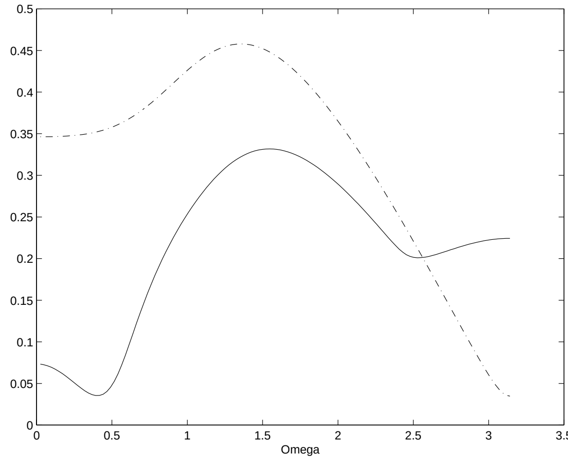

Figure 4: Frequency by frequency comparison of the worst case chordal distance $\kappa_{WC}(G(e^{j\Omega}), U_{OL}(\Omega))$ (solid) and $\kappa(G(e^{j\Omega}), -\frac{1}{C(e^{j\Omega})})$ (dashdot)

Verification of the less conservative condition (13): Figure 4 illustrates the use of U_{OL}. It shows that, even though this condition is less conservative, the worst case chordal distances $\kappa_{WC}(G(e^{j\Omega}), U_{OL}(\Omega))$ corresponding to U_{OL} are not smaller than $\kappa(G(e^{j\Omega}), -\frac{1}{C(e^{j\Omega})})$ at each frequency. The stabilization of G_0 by C is therefore not guaranteed if the validated region U_{OL} is used. But the stabilization is guaranteed with the worst case chordal distances corresponding to U_{CL} as can be seen in Figure 5. According to (14), this will also be true with the worst case chordal distances corresponding to U_{inter}.

Some comments are to be made about this example. As the stabilization is only guaranteed for the less conservative condition applied to U_{CL} (and therefore also U_{inter}), this example once more shows:

- the conservatism of the Min-Max type condition with respect to the frequency by frequency condition since the stabilization is not guaranteed with the first condition but is guaranteed with the other one,
- the usefulness of making several validation experiments since the stabilization is not established with U_{OL} but is established with U_{CL}.

5 Conclusions

In this paper, the robust stability problem of guaranteeing the stabilization of a true system by a controller designed from a model has been developed in the case where a validated uncertainty region, composed of ellipsoids at each frequency, is available. In order to use robust control theory tools, this validated uncertainty region has been embedded in a coprime factor uncertainty set. A new quantity, the worst case Vinnicombe distance, has also been defined; it corresponds to the size of the coprime factor uncertainty set. This new quantity

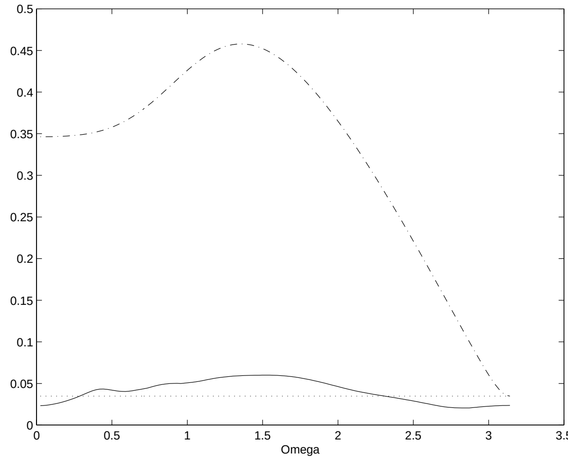

Figure 5: Frequency by frequency comparison of the worst case chordal distance $\kappa_{WC}(G(e^{j\Omega}), U_{CL}(\Omega))$ (solid) and $\kappa(G(e^{j\Omega}), -\frac{1}{C(e^{j\Omega})})$ (dashdot). The horizontal dotted line is b_{GC}

has been used in the robust stability theorems developed by Vinnicombe in [15]. An example shows that a pointwise use of the classical Vinnicombe theorem can significantly reduce the conservatism of this sufficient but not necessary condition. One example also shows the usefulness of having more than one validated uncertainty region. Indeed, with several uncertainty regions, a better estimate of the actual Vinnicombe distance between the true system and the model can be obtained and, furthermore, a new and "smaller" uncertainty region can be constructed by considering the intersection of these uncertainty regions. A last remark is that the results of this paper are only true modulo some probability level since the presence of the true system in a validated uncertainty region can only be guaranteed at a certain probability level.

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A LMI approach for the computation of the worst case chordal distance at a given frequency

A.1 Case of one ellipsoid

The problem of determining the worst case chordal distance $\kappa_{WC}(G(e^{j\Omega}), U(\Omega))$ for a given frequency can be expressed as follows : for a given frequency Ω, find a bound on the chordal distance κ between a given point $G(e^{j\Omega})$ and any point y belonging to an ellipsoid in the complex plane. For the sake of clarity, let us denote $x = G(e^{j\Omega})$.

For a given y, the distance $\kappa(x, y)$ is given by:

$$\frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}}.$$

In the sequel, the complex vectors and matrices are decomposed in their real part, referred to with the subscript R and their imaginary part, referred to with the subscript I. The ellipsoid is defined as:

$$\mathcal{E}_Q = \left\{ y \mid \begin{bmatrix} y_R - c_R \\ y_I - c_I \end{bmatrix}^T Q \begin{bmatrix} y_R - c_R \\ y_I - c_I \end{bmatrix} \leq \alpha \right\}.$$

where c is the ellipsoid center (i.e. $G(e^{j\Omega}) + \widetilde{G}(e^{j\Omega})$) and α describes the number of standard deviations chosen to construct the ellipsoids (i.e. $\alpha = 2$ for a probability level of 95%).

Thus the previous problem boils down to find the smallest γ such that:

$$\left(\frac{|x-y|}{\sqrt{1+|x|^2}\sqrt{1+|y|^2}} \right)^2 \leq \gamma \tag{16}$$

for all y such that:

$$\begin{bmatrix} y_R - c_R \\ y_I - c_I \end{bmatrix}^T \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12} & Q_{22} \end{bmatrix} \begin{bmatrix} y_R - c_R \\ y_I - c_I \end{bmatrix} \leq \alpha. \tag{17}$$

Note that condition (16) can be straightforwardly rewritten as:

$$\begin{bmatrix} y \\ 1 \end{bmatrix}^* \begin{bmatrix} 1-\gamma(1+x^*x) & -x \\ -x^* & x^*x - \gamma(1+x^*x) \end{bmatrix} \begin{bmatrix} y \\ 1 \end{bmatrix} \leq 0 \tag{18}$$

that is,

$$\overbrace{\begin{bmatrix} y_R \\ y_I \\ 1 \end{bmatrix}^T \begin{bmatrix} 1-\gamma(1+x^*x) & 0 & -x_R \\ 0 & 1-\gamma(1+x^*x) & -x_I \\ -x_R & -x_I & x^*x-\gamma(1+x^*x) \end{bmatrix} \begin{bmatrix} y_R \\ y_I \\ 1 \end{bmatrix}}^{\sigma_0(y_R, y_I)} \leq 0. \tag{19}$$

Let us introduce:

$$\sigma_1(y_R, y_I) = \begin{bmatrix} y_R \\ y_I \\ 1 \end{bmatrix}^T \left[\begin{array}{c|c|c} Q_{11} & Q_{12} & -(Q_{11}c_R + Q_{12}c_I) \\ \hline Q_{12} & Q_{22} & -(Q_{12}c_R + Q_{22}c_I) \\ \hline -(Q_{11}c_R + Q_{12}c_I) & -(Q_{12}c_R + Q_{22}c_I) & \begin{bmatrix} c_R \\ c_I \end{bmatrix}^T Q \begin{bmatrix} c_R \\ c_I \end{bmatrix} - \alpha \end{array}\right] \begin{bmatrix} y_R \\ y_I \\ 1 \end{bmatrix}.$$

The problem boils down to find (the smallest) γ such that $\sigma_0(y_R, y_I) \leq 0$ for all (y_R, y_I) for which $\sigma_1(y_R, y_I) \leq 0$. By the $\mathcal{S}$ procedure [10, 1], this problem is equivalent to finding (the smallest) γ and a positive scalar τ such that $\sigma_0(y_R, y_I) - \tau\sigma_1(y_R, y_I) \leq 0$, for all $(y_R, y_I) \in \mathbf{R}^2$, that is:

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{over} & \gamma,\ \tau \\ \text{subject to} & \tau \geq 0 \text{ and} \end{array}$$

$$\left[\begin{array}{c|c|c} 1 - \gamma(1 + x^*x) - \tau Q_{11} & -\tau Q_{12} & -x_R + \tau(Q_{11}c_R + Q_{12}c_I) \\ \hline -\tau Q_{12} & 1 - \gamma(1 + x^*x) - \tau Q_{22} & -x_I + \tau(Q_{12}c_R + Q_{22}c_I) \\ \hline -x_R + \tau(Q_{11}c_R + Q_{12}c_I) & -x_I + \tau(Q_{12}c_R + Q_{22}c_I) & \begin{array}{l} x^*x - \gamma(1 + x^*x) \cdots \\ \cdots - \tau\left(\begin{bmatrix} c_R \\ c_I \end{bmatrix}^T Q \begin{bmatrix} c_R \\ c_I \end{bmatrix} - \alpha\right) \end{array} \end{array}\right] \leq 0.$$

There are many efficient algorithms for solving such LMI optimization problem, see e.g. [4]. The LMI problem can be solved using the free ware code SP [14] and its Matlab/Scilab interface `LMITOOL` [3], or the available commercial Matlab Toolbox, LMI Control Toolbox [4].

It is to be noted that the corresponding worst case chordal distance $\kappa_{WC}(G(e^{j\Omega}), U(\Omega))$ is equal to $\sqrt{\gamma}$.

A.2 Case of the intersection of N ellipsoids

The previous results readily extend to the case of N ellipsoids:

$$\mathcal{E}_{Q^i} = \left\{ y \mid \begin{bmatrix} y_R - c_R^i \\ y_I - c_I^i \end{bmatrix}^T Q^i \begin{bmatrix} y_R - c_R^i \\ y_I - c_I^i \end{bmatrix} \leq \alpha \right\} \qquad i = 1, \cdots, N.$$

Note that, in the case $N > 1$, the $\mathcal{S}$ procedure condition is no longer necessary. We obtain the following LMI formulation.

$$\begin{array}{ll} \text{minimize} & \gamma \\ \text{over} & \gamma,\ \tau_i,\ \ i = 1, \cdots, N \\ \text{subject to} & \tau_i \geq 0,\ \ i = 1, \cdots, N \text{ and} \end{array}$$

$$\begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} \\ \phi_{12} & \phi_{22} & \phi_{23} \\ \phi_{13} & \phi_{23} & \phi_{33} \end{bmatrix} \leq 0 \quad \text{with}$$

$$\phi_{11} = 1 - \gamma(1 + x^* x) - \sum_{i=1}^{i=N} \tau_i {Q^i}_{11},$$

$$\phi_{12} = - \sum_{i=1}^{i=N} \tau_i {Q^i}_{12},$$

$$\phi_{13} = -x_R + \sum_{i=1}^{i=N} \tau_i ({Q^i}_{11} c_R^i + {Q^i}_{12} c_I^i),$$

$$\phi_{22} = 1 - \gamma(1 + x^* x) - \sum_{i=1}^{i=N} \tau_i {Q^i}_{22},$$

$$\phi_{23} = -x_I + \sum_{i=1}^{i=N} \tau_i ({Q^i}_{12} c_R^i + {Q^i}_{22} c_I^i),$$

$$\phi_{33} = x^* x - \gamma(1 + x^* x) - \sum_{i=1}^{i=N} \tau_i \left(\begin{bmatrix} c_R^i \\ c_I^i \end{bmatrix}^T Q^i \begin{bmatrix} c_R^i \\ c_I^i \end{bmatrix} - \alpha \right).$$

As, in the case of $N > 1$, the $\mathcal{S}$ procedure condition is no more necessary, the result of this procedure is only an upper bound of the worst case chordal distance at the considered frequency. However, by using a gridding of the points inside the intersection of the ellipsoids and by computing the chordal distance for those points, a lower bound of the searched quantity can also be found. This way, a tighter estimate of the worst case chordal distance can be found. In our particular example of section 4.5, these lower and upper bounds are nearly identical.