# Unbiased estimation of the Hessian for Iterative Feedback Tuning (IFT)

Gabriel SOLARI* and Michel GEVERS*
Center for Systems Engineering and Applied Mechanics (CESAME)
Université Catholique de Louvain, Bâtiment Euler
B-1348 Louvain-la-Neuve, Belgium
{solari,gevers}@csam.ucl.ac.be

***Abstract*— Iterative Feedback Tuning (IFT) is a data-based method for the optimal tuning of a low order controller. The tuning of the controller parameters is performed iteratively, using a generalized Robbins-Monro type gradient descent scheme. An update step of the controller parameters is performed at each iteration on the basis of data obtained partly during normal operating conditions and partly from some special experiments. These data come from the closed loop system with the current controller. This paper presents a simple improvement to the IFT scheme: it is shown that one can compute an unbiased estimate of the Hessian on the basis of additional experiments on the closed loop system.**

## I. INTRODUCTION

Iterative Feedback Tuning (IFT) is a model-free data-based method for the optimal tuning of the parameters of a controller of given structure, typically a restricted complexity controller [3]. The method is simple and has proved very successful in a wide range of applications. A number of extensions and applications of IFT have been proposed: see the survey [2] and the special issue [4]. The control performance objective used in IFT is a quadratic performance criterion, which is minimized by a stochastic gradient descent scheme of Robbins-Monro type. Thus, the iterative parameter update rule of IFT is given by

$$\rho_{n+1} = \rho_n + \gamma_n R_n^{-1} \text{est}\left[\frac{\partial J}{\partial \rho}(\rho_n)\right], \qquad (1)$$

where $\rho_n$ is the controller parameter vector at iteration $n$, est $\left[\frac{\partial J}{\partial \rho}(\rho_n)\right]$ is an estimate of the gradient of the performance criterion $J$, $\gamma_n$ is the step size, and $R_n$ is a positive definite matrix.

The key feature of IFT is that an unbiased estimate of the gradient $\frac{\partial J}{\partial \rho}$, evaluated at the current controller parameter values, is computed using a filtered set of data obtained from both normal operation conditions and from a special experiment performed on the plant at each iteration. As for the matrix $R_n$, a simple choice is to take the identity, but this may lead to slow convergence. An optimal choice in the vicinity of the optimal $\rho^*$ is to take the Hessian of the criterion. A typical choice is to take a Gauss-Newton approximation of the Hessian. In the implementation of the IFT algorithm, a biased estimate of the Gauss-Newton direction can be computed from the signals that have been generated for the computation of the gradient. This choice has been recommended in [3], and has been commonly adopted since then.

The contribution of this paper is to show that, with two additional special experiments at each iteration of the IFT algorithm, one can construct an unbiased estimate of the Hessian, evaluated at the present controller parameter vector, directly from data collected on the closed loop system. The idea of computing an unbiased estimate of the Hessian for the IFT iterations has already been proposed in [1], however at the cost of two identification steps.

*This paper presents research results of the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian Federal Science Policy Office. The scientific responsibility rests with its authors.

In this paper, we show how one can compute an unbiased estimate of the Hessian for a simple version of the IFT algorithm, namely where the control performance objective is to perform disturbance rejection only, and where no penalty is added on the control energy. This keeps our presentation simple and focused on the main ideas. While the extension to an IFT criterion that incorporates a tracking objective may not be straightforward, the extension to the case of an added penalty on the control is almost trivial.

## II. IFT FOR DISTURBANCE REJECTION

We consider that the task is to optimize the controller parameters of a linear time-invariant closed loop system driven by stochastic disturbances, and described by

$$y_t(\rho) = \frac{1}{1+G(q)C(q,\rho)} v_t = S(q,\rho) v_t$$

where $G(q)$ is a linear time-invariant operator, $C(q,\rho)$ is a linear time-invariant transfer function parameterized by some parameter vector $\rho \in \mathbb{R}^{n_\rho}$, and $v_t$ is a zero mean weakly stationary noise. The transfer function $S(q,\rho)$ from $v_t$ to $y_t(\rho)$ is called the sensitivity function. For the purposes of implementing the special experiments of IFT, we consider the block diagram shown in the figure where $r_t$ is a reference signal that is set to zero or a constant under normal operating conditions. The transfer function from $r_t$ to $y_t(\rho)$ is called the complementary sensitivity function $T(q,\rho)$. For ease of notation we omit the $q$ argument from now on.

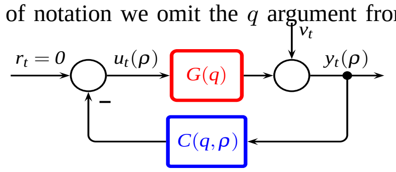


***Fig. 1: System under normal operating conditions***

The goal is to minimize the quadratic criterion

$$J(\rho) = \frac{1}{2N} E\left[\sum_{t=1}^{N} y_t^2(\rho)\right]$$

by choice of the controller parameter vector $\rho$, where $E[.]$ denotes expectation with respect to the disturbance $v_t$. The optimal controller parameter $\rho$ is defined by $\rho^* = \arg\min_\rho \ J(\rho)$.

The IFT method is based on a stochastic gradient descent scheme of Robbins-Monro type: see (1). In the IFT scheme, the gradient estimate is constructed from batches of data of length $N$, obtained on the closed loop system [3]. The exact expressions of the gradient and the Hessian of the cost function are as follows.

$$\begin{aligned}\frac{\partial J}{\partial \rho}(\rho) &= \frac{1}{N} E\left[\sum_{t=1}^{N} y_t(\rho)\frac{\partial y_t}{\partial \rho}(\rho)\right] \\ \frac{\partial^2 J}{\partial \rho^2}(\rho) &= \frac{1}{N} E\left[\sum_{t=1}^{N}\left(y_t(\rho)\frac{\partial^2 y_t}{\partial \rho^2}(\rho) + \frac{\partial y_t}{\partial \rho}(\rho)\frac{\partial y_t}{\partial \rho}(\rho)^T\right)\right] \end{aligned} \qquad (2)$$

The gradient and second derivative of $y_t(\rho)$ are given by:

$$\frac{\partial y_t}{\partial \rho}(\rho) = -T(\rho)S(\rho)\frac{\partial C}{\partial \rho}(\rho)v_t$$
$$\frac{\partial^2 y_t}{\partial \rho^2}(\rho) = -T(\rho)S(\rho)\frac{\partial^2 C}{\partial \rho^2}(\rho)v_t + 2T^2(\rho)S(\rho)\frac{\partial C}{\partial \rho}(\rho)\frac{\partial C}{\partial \rho}(\rho)^T v_t$$

Here $\frac{\partial C}{\partial \rho}(\rho)$, $\frac{\partial^2 C}{\partial \rho^2}(\rho)$ are known functions of $\rho$; however, $T(\rho)$ and $S(\rho)$ are unknown because they depend on the unknown $G(\rho)$.

It was shown in [3] that one can compute an unbiased estimate of the gradient $\frac{\partial J}{\partial \rho}$ by storing a batch of $N$ output data $y_t^1(\rho)$ collected under normal operating conditions, and by then applying these $N$ data at the reference input $r$ during a second (special) experiment. The signals collected during these two periods of length $N$ are indexed by 1 and 2. Thus we get:

$$y_t^1(\rho) = S(\rho)v_t^1 \quad \text{with } r_t^1 = 0$$
$$y_t^2(\rho) = T(\rho)S(\rho)v_t^1 + S(\rho)v_t^2 \quad \text{with} \quad r_t^2 = y_t^1(\rho)$$

From the collected signals, we can construct a noisy, but unbiased, estimate of the gradient of $y_t$ w.r.t. $\rho$, denoted $\text{est}_1$:

$$\text{est}_1\left[\frac{\partial y_t}{\partial \rho}(\rho)\right] = -\frac{\partial C}{\partial \rho}(\rho)y_t^2(\rho) = \frac{\partial y_t}{\partial \rho} - S(\rho)\frac{\partial C}{\partial \rho}(\rho)v_t^2$$

## III. UNBIASED HESSIAN ESTIMATOR FOR IFT

We now show that, with two additional special experiments, we can construct an unbiased estimate of the Hessian. We index the signals with 3 and 4.

$$y_t^3(\rho) = T(\rho)S(\rho)v_t^1 + S(\rho)v_t^3 \quad \text{with } r_t^3 = y_t^1(\rho)$$
$$y_t^4(\rho) = T^2(\rho)S(\rho)v_t^1 + T(\rho)S(\rho)v_t^3 + S(\rho)v_t^4 \quad \text{with } r_t^4 = y_t^3(\rho)$$

With these additional experiments we can construct an estimate of the second derivative and an extra estimation of the gradient:

$$\begin{aligned}\text{est}\left[\frac{\partial^2 y_t}{\partial \rho^2}(\rho)\right] &= 2\frac{\partial C}{\partial \rho}(\rho)\frac{\partial C}{\partial \rho}(\rho)^T y_t^4(\rho) - \frac{\partial^2 C}{\partial \rho^2}(\rho)y_t^2(\rho)\\ &= \frac{\partial^2 y_t}{\partial \rho^2} + 2\left(T(\rho)S(\rho)\frac{\partial C}{\partial \rho}(\rho)\frac{\partial C}{\partial \rho}(\rho)^T v_t^3\right.\\ &\quad \left.+ S(\rho)\frac{\partial C}{\partial \rho}(\rho)\frac{\partial C}{\partial \rho}(\rho)^T v_t^4\right) - S(\rho)\frac{\partial^2 C}{\partial \rho^2}(\rho)v_t^2\\ \text{est}_2\left[\frac{\partial y_t}{\partial \rho}(\rho)\right] &= -\frac{\partial C}{\partial \rho}(\rho)y_t^3(\rho) = \frac{\partial y_t}{\partial \rho} - S(\rho)\frac{\partial C}{\partial \rho}(\rho)v_t^3\end{aligned}$$

The disturbances $v_t^i$ during two different experiments are assumed to be mutually independent because $N$ is large compared to the correlation time of these disturbances. We then have

$$\mathbf{E}\left\{\text{est}\left[\frac{\partial^2 y_t}{\partial \rho^2}(\rho)\right]y_t^1(\rho)\right\} = \mathbf{E}\left[\frac{\partial^2 y_t}{\partial \rho^2}(\rho)y_t(\rho)\right] \quad (3)$$
$$\mathbf{E}\left\{\text{est}_1\left[\frac{\partial y_t}{\partial \rho}(\rho)\right]\text{est}_2\left[\frac{\partial y_t}{\partial \rho}(\rho)\right]^T\right\} = \mathbf{E}\left[\frac{\partial y_t}{\partial \rho}(\rho)\frac{\partial y_t^T}{\partial \rho}(\rho)\right] \quad (4)$$

Combining these properties, we can thus construct an unbiased estimate of the Hessian, $\frac{\partial^2 J}{\partial \rho^2}(\rho)$, described in (2).

## IV. ILLUSTRATION BY MONTE CARLO SIMULATIONS

We evaluate the benefits of our new computation of $R_n$ in the IFT algorithm on the following system:

$$G(q) = \frac{0.2826q^{-3} + 0.5067q^{-4}}{1 - 1.418q^{-1} + 1.589q^{-2} - 1.316q^{-3} + 0.8864q^{-4}}$$
$$H(q) = \frac{1}{1 - 1.418q^{-1} + 1.589q^{-2} - 1.316q^{-3} + 0.8864q^{-4}}$$
$$\text{with } v_t = H(q)e_t \text{ and } \sigma_e^2 = 1 \quad (\sigma_v^2 = 12.875).$$

For a disturbance rejection problem ($r_t = 0$) with a cost function $J(\rho) = \frac{1}{2}\mathbf{E}\left[y_t(\rho)^2\right]$, and a controller of the form $C(\rho, q) = \frac{\rho_1 + \rho_2 q^{-1}}{1 + \rho_3 q^{-1} + \rho_4 q^{-2}}$, one local minimum is at $\rho^* = [0.08204 \ -0.55654 \ 0.0075 \ 0.15019]$, yielding $J(\rho^*) = 3.3719$.

We have compared three possible choices for $R_n$ in (1):

$$R^1 = \frac{1}{N}\sum_{t=1}^{N}\left(\text{est}_1\left[\frac{\partial y_t}{\partial \rho}(\rho)\right]\text{est}_1\left[\frac{\partial y_t}{\partial \rho}(\rho)\right]^T\right)$$
$$R^2 = \frac{1}{N}\sum_{t=1}^{N}\left(\text{est}\left[\frac{\partial^2 y_t}{\partial \rho^2}(\rho)\right]y_t^1(\rho) + \text{est}_1\left[\frac{\partial y_t}{\partial \rho}(\rho)\right]\text{est}_2\left[\frac{\partial y_t}{\partial \rho}(\rho)\right]^T\right)$$
$$R^3 = \frac{1}{N}\sum_{t=1}^{N}\left(\text{est}_1\left[\frac{\partial y_t}{\partial \rho}(\rho)\right]\text{est}_2\left[\frac{\partial y_t}{\partial \rho}(\rho)\right]^T\right)$$

$R^1$ is the classical choice made in the IFT literature, $R^2$ is the unbiased estimator of the Hessian developed in this paper, while $R^3$ is the same as $R^2$ but leaving aside the second derivative. To compare the quality of these three estimates of the Hessian, we have performed $M = 500$ Monte Carlo runs of length $N = 4000$ each on the system $[G \ H]$ with the controller $C(\rho^*)$ in the loop, in order to generate the signals $y_t^i(\rho^*), i = 1, \ldots, 4$, and hence $M$ estimates $R_m^i, m = 1, \ldots, M$ of $R^i$ for $i = 1, 2, 3$. The following table shows a measure of the experimental bias and mean square error, for all three estimates:

$$b_{R^i} = \left\|\frac{1}{M}\sum_{m=1}^{M}R_m^i - \mathbf{H}\right\|_F, \quad MSE_{R^i} = \frac{1}{M}\sum_{m=1}^{M}\left\|R_m^i - \mathbf{H}\right\|_F^2$$

| | $R^1$ | $R^2$ | $R^3$ |
|---|---|---|---|
| $b_{R^i}$ | 6.4451 | 0.096 | 0.8966 |
| $MSE_{R^i}$ | 35.7496 | 1.9676 | 1.1644 |

***TABLE I: Bias and mean square error of the three estimates of*** $R$

These simulations confirm that the new estimates of the Hessian proposed in this paper, based on additional special experiments, yield significantly improved estimates with a much reduced bias error. Even the estimate $R^3$, which requires only one additional experiment, has a significantly smaller bias error than the traditionally used estimate. We have performed tests on actual IFT iterations, in which the estimate $R^1$ is replaced by $R^3$ near the optimum; this leads to a significant reduction in the number of iterations required to reach the convergence point.

## V. CONCLUSIONS

In this contribution we have proposed a new way of computing the controller parameter updates in the IFT controller tuning scheme. We have shown how to construct an unbiased estimate of the Hessian on the basis of either one or two additional experiments. Simulations have shown the claimed unbiasedness properties.

## REFERENCES


[1] Franky De Bruyne and Leonardo C. Kramer. Iterative Feedback Tuning with guaranteed stability. In *Proceedings of the American Control Conference*, pages 3317–3321, San Diego, California, USA, June 1999.

[2] H. Hjalmarsson. Iterative Feedback Tuning - an overview. *International Journal of Adaptive Control and Signal Processing*, 16(5):373–395, 2002.

[3] H. Hjalmarsson, M. Gevers, S. Gunnarsson, and O. Lequin. Iterative Feedback Tuning: theory and applications. *IEEE Control Systems Magazine*, 18:26–41, August 1998.

[4] Special Section on Algorithms and Applications of Iterative Feedback Tuning. *Control Engineering Practice*, 11(9):1021–1094, September 2003.