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Extending identifiability results from isolated networks to embedded networks

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***Abstract*—This paper deals with the design of Excitation and Measurement Patterns (EMPs) for the identification of dynamic networks, when the objective is to identify only a subnetwork embedded in a larger network. Recent results have shown how to construct EMPs that guarantee identifiability for a range of networks with specific graph topologies, such as trees, loops, parallel networks, or Directed Acyclic Graphs (DAGs). However, an EMP that is valid for the identification of a subnetwork taken in isolation may no longer be valid when that subnetwork is embedded in a larger network. Our main contribution is to exhibit conditions under which it does remain valid, and to propose ways to enhance such EMP when these conditions are not satisfied.**

***Index Terms*—Network Analysis and Control; Dynamic networks; Network identification.**

I. Introduction

This paper deals with the design of Excitation and Measurement Patterns (EMP) for the identification of dynamic networks. The network framework used here was introduced in [1], where signals are represented as nodes of the network which are related to other nodes through transfer functions. These networks can be interpreted as directed graphs (or digraphs) where the transfer functions, also called modules, are the edges of the graph and the node signals are the vertices.

In [1] and in subsequent contributions, it was assumed that either all nodes are excited or all nodes are measured. A breakthrough was made in [2], where the first identifiability results were obtained for networks where it is not assumed that all nodes are either excited or measured. This scenario is referred to as "partial excitation and measurement".

Since the publication of [2], the search has been for the construction of EMPs which guarantee network identifiability and that are preferably sparse. Once it was discovered that one could identify a network, or part of the network, using only a selection of excited and measured nodes, this opened up a whole new ball-game: designing valid EMPs that have desirable properties, such as small cardinality. An EMP defines which nodes are excited and which nodes are measured. It is called valid for a given network when it guarantees generic identifiability of that network. It is sparse when it is valid and the sum of the number of excited nodes and measured nodes, called cardinality of the EMP, is kept small. It is called minimal when it is valid with minimal cardinality. Precise definitions will be given in Section II.

In this paper, we focus on the synthesis of EMPs for networks with partial excitation and measurement. We briefly review the existing results so far. They are all based on the graph-theoretic framework developed in [3]. In [2], the first results were obtained for the generic identification of networks with specific graph topologies, namely trees and loops. Subsequent results for the construction of valid EMPs have been obtained for some classes of parallel networks in [4], for DAGs in [5], for isolated loops in [6], where a necessary and sufficient condition was derived leading to a minimal EMP. The construction of a valid EMP for the identification of a single module was proposed in [7]. The paper [8], while not proposing a synthesis method, presented a new set of necessary conditions for network identifiability in the context of partial excitation and measurement. For this same context, a different approach was proposed in [9], [10]; it is not based on the synthesis of a valid EMP, but on an efficient and fast algorithm that allows to check the validity of large numbers of EMPs.

A first new result of the present paper is a necessary and sufficient condition for the generic identifiability of another class of networks with a specific topology, namely Parallel Paths Networks (PPN), leading to the construction of minimal EMPs for such structures. However, constructing a valid EMP for a new class of networks with specific structure is not the main object of the present paper. Instead, we consider a novel approach to network identification, by addressing the following problem.

It is often the case that one wants to identify a subnetwork that is part of a larger network. All the results summarized above lead to the construction of an EMP that is valid for the subnetwork under consideration (whether its graph is a tree, a loop, a PPN, or any other structure) when it is treated in isolation. However, an EMP that is valid for such subnetwork considered in isolation may no longer be valid when it is embedded in a larger network, as we illustrate in Section IV.

Our main contribution will be to present a set of sufficient conditions under which an EMP that is valid for a subnetwork in isolation remains valid when that subnetwork is embedded

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in a larger network. Our theorem will also indicate how one can often complement an EMP that is valid for an isolated subnetwork so that the augmented EMP remains valid for that subnetwork when it is embedded into the larger network. Based on that, it is possible to obtain valid EMPs for the whole network by designing valid EMPs for specific subnetworks, which are easy to construct.

The paper is organized as follows. In Section II we introduce the notations and the main concepts about generic identifiability of networks used in this paper, and we recall the necessary conditions for the generic identifiability of any network. In Section III we first recall existing necessary and sufficient conditions for the identifiability of trees and loops, leading to minimal EMPs for these structures. We then present necessary and sufficient conditions for the generic identifiability of Parallel Paths Networks. In Section IV we illustrate why an EMP that is valid for a subnetwork treated in isolation may no longer be valid when that subnetwork is embedded in a larger network. Our main result is in Section V: we present two sets of conditions under which an EMP that is valid for a subnetwork treated in isolation remains valid when that subnetwork is embedded. We illustrate this result with an example, which shows how the main theorem can be used to construct a valid EMP for the whole network by combining EMPs that are valid, on the basis of the main theorem, for subnetworks that, together, compose the whole network.

II. Definitions, Notations and Preliminaries

In this section, we introduce the dynamic networks with partial excitation and measurement that we deal with in this paper. We recall the necessary conditions for generic identifiability that were derived in [2], and we also recall the concept of a valid EMP.

We consider dynamic networks composed of n nodes (or vertices) which represent internal scalar signals $\{w_k(t)\}$ for $k \in \{1, 2, \ldots, n\}$. These nodes are interconnected by discrete time transfer functions, represented by edges, which are entries of a *network matrix* $G(z)$. The dynamics of the network is given by the following equations:

$$w(t) = G(z)w(t) + Br(t), \tag{1a}$$

$$y(t) = Cw(t), \tag{1b}$$

where $w(t) \in \mathbb{R}^n$ is the node vector, $r(t) \in \mathbb{R}^m$ is the input vector, and $y(t) \in \mathbb{R}^p$ is the set of measured nodes, considered as the output vector of the network. The matrix $B \in \mathbb{Z}_2^{n\times m}$, where $\mathbb{Z}_2 \triangleq \{0, 1\}$, is a binary selection matrix with a single 1 and $n - 1$ zeros in each column; it selects which nodes are excited. Similarly, $C \in \mathbb{Z}_2^{p\times n}$ is a matrix with a single 1 and $n-1$ zeros in each row that selects which nodes are measured.

To each network matrix $G(z)$ we associate a directed graph (called digraph) $\mathcal{G}$ defined by the tuple $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of vertices and $\mathcal{E} \subseteq \mathcal{V}\times\mathcal{V}$ is the set of edges. The digraph $\mathcal{G}$ defines the topology of the network. The edges are associated to transfer functions, which define the relationships between the nodes. We adopt the terminology $j \to i$ to denote an edge (j, i). Node j is called an in-neighbor of node i, and node i is an out-neighbor of node j. There is an edge (j, i) associated with the transfer function $G_{ij}(z)$ of the network matrix only if $G_{ij}(z)$ is nonzero. In this paper, we consider only *connected* graphs, meaning that every node can be reached by at least one other node in the network. A *source* is a node with no in-neighbors, and a *sink* is a node with no out-neighbors. Nodes that are neither sources nor sinks are called *internal nodes*.

For the digraph $\mathcal{G}$ associated to the network matrix $G(z)$ we introduce the following notations.

- $\mathcal{V}$ – the set of all n nodes;
- $\mathcal{B}$ – the set of excited nodes, defined by B in (1a);
- $\mathcal{C}$ – the set of measured nodes, defined by C in (1b);
- $\mathcal{F}$ – the set of sources;
- $\mathcal{D}_F$ – the set of dources: see Definition 1 below;
- $\mathcal{S}$ – the set of sinks;
- $\mathcal{D}_S$ – the set of dinks: see Definition 1 below;

The dources and dinks are defined as follows (see [11]).

Definition 1. *A node j is called a **dource** if it has at least one out-neighbor to which all its in-neighbors have a directed edge. A node j is called a **dink** if it has at least one in-neighbor that has a directed edge to all its out-neighbors.*

Assumptions on the network matrix $G(z)$

Throughout the paper, we shall make the following assumptions on the network matrix:

- the diagonal elements are zero and all other elements are proper;
- $(I - G(z))^{-1}$ is proper and all its elements are stable.

One can represent the dynamic network in (1a)-(1b) as an input-output model as follows

$$y(t) = M(z)r(t), \text{ with } M(z) \triangleq CT(z)B. \tag{2}$$

where $T(z) \triangleq (I - G(z))^{-1}$. Observe that the matrix $T(z)$ is generically nonsingular by construction.

In analyzing the generic identifiability of the network matrix, it is assumed that the input-output model $M(z)$ is known; the identification of $M(z)$ from input-output (IO) data $\{y(t), r(t)\}$ is a standard identification problem, provided the input signal $r(t)$ is sufficiently rich. The question of generic identifiability of the network is then whether the network matrix $G(z)$ can be fully recovered from the transfer matrix $M(z)$. It is defined as follows.

Definition 2. *([3]) The network matrix $G(z)$ is generically identifiable from excitation signals applied to $\mathcal{B}$ and measurements made at $\mathcal{C}$ if, for any rational transfer matrix parametrization $G(P, z)$ consistent with the directed digraph associated with $G(z)$, there holds*

$$C[I-G(P, z)]^{-1}B = C[I-\tilde{G}(z)]^{-1}B \implies G(P, z) = \tilde{G}(z),$$

for all parameters $P \in \mathbb{R}^N$ except possibly those lying on a zero measure set in $\mathbb{R}^N$, where $\tilde{G}(z)$ is any network matrix consistent with the digraph.

The following Proposition provides necessary conditions for generic identifiability of any network; it combines Theorem III.1, Corollary III.1 from [2] with Theorem III.1 from [11].

Proposition II.1. *The network matrix $G(z)$ is generically identifiable only if $\mathcal{B},\mathcal{C} \neq \emptyset$; $\mathcal{F},\mathcal{D}_F \subset \mathcal{B}$; $\mathcal{S},\mathcal{D}_S \subset \mathcal{C}$ and $\mathcal{B} \cup \mathcal{C} = \mathcal{V}$.*

This paper deals with networks in which not all nodes are excited and not all nodes are measured. Finding conditions that guarantee generic identifiability for such networks is equivalent to constructing an EMP that guarantees identifiability. The concept of EMP and of valid EMP, which led to the concept of minimal EMP, was introduced in [12]. They are defined in the following.

Definition 3. *A pair of selection matrices B and C, with its corresponding pair of node sets $\mathcal{B}$ and $\mathcal{C}$, defines an* ***excitation and measurement pattern (EMP)****. An EMP is called* ***valid*** *for the network (1a)-(1b) if this network is generically identifiable with this EMP. Let $\nu = |\mathcal{B}| + |\mathcal{C}|$ [1] be the cardinality of an EMP. A given EMP is called* ***minimal*** *for this network if it is valid and there is no other valid EMP with smaller cardinality.*

Proposition II.1 shows that for an EMP to be valid, it must contain at least one excitation and one measurement, all sources and dources must be excited and all sinks and dinks must be measured, and every other node must be either excited or measured. From now on, we drop the arguments z and t used in (1a)-(1b) whenever there is no risk of confusion.

III. Identifiability results for specific network structures

In this section, we present generic identifiability results for some specific network structures, for the situation of partial excitation and measurement. Stated otherwise, we present valid EMPs for these network structures. We consider three types of network structures that are defined by their specific topologies, namely trees, loops and parallel paths networks. We first briefly recall recent existing identifiability results for trees and loops. We then present novel identifiability results for parallel paths networks. First, we define these three specific network structures.

A. Trees

Definition 4. *A directed tree is a weakly connected graph which has no loops even if one were to change the edges directions.*

For the identifiability of directed trees, the following necessary and sufficient conditions were derived in [2].

Theorem III.1. *A directed tree is generically identifiable if and only if $\mathcal{F} \subseteq \mathcal{B}$, $\mathcal{S} \subseteq \mathcal{C}$, $\mathcal{B} \cup \mathcal{C} = \mathcal{V}$.*

B. Loops

Definition 5. *A loop is a network consisting of a path that starts and ends at the same node. If the loop is part of a larger network, it is called an isolated loop if no other loop in the graph contains any of the nodes of the loop of interest.*

[1] $|\cdot|$ - Denotes the cardinality of a set.

In [6] the following necessary and sufficient conditions were derived for the generic identifiability of isolated loops that have at least three nodes.

Theorem III.2. *All transfer functions in an isolated loop are generically identifiable if and only if $\mathcal{B} \cup \mathcal{C} = \mathcal{V}$ and, in addition: (i) either $\mathcal{B} \cap \mathcal{C} \neq \emptyset$, or (ii) the excited nodes (and hence also the measured nodes) are not all consecutive along the loop.*

Alternative versions of necessary and sufficient conditions for the identifiability of isolated loops can be found in [6], but the formulation given in Theorem III.2 is by far the simplest; the verification can be done by visual inspection of the loop.

C. Parallel Paths Networks

We now consider a class of networks with a single source, a single sink, and n_p paths with an arbitrary number of nodes between the source and the sink. Figure 1 depicts an example of such network.

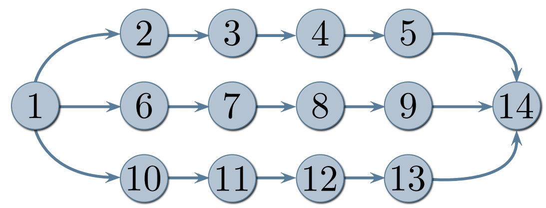

Fig. 1. An example of a parallel paths network.

Definition 6. *A* ***parallel paths network (PPN)*** *is a network composed of a single source and a single sink. There are $n_p \geq 2$ paths from the source to the sink, which are the only nodes common to each path. At most one path may have no internal nodes; every other path has at least three nodes.*

This definition extends the concept of parallel network presented in [4], where each path contained only a single node. The following theorem gives necessary and sufficient conditions for the generic identifiability of a PPN.

Theorem III.3. *Consider a parallel paths network from Definition 6 with one source $\mathcal{F} = \{1\}$ connected through $n_p \geq 2$ paths $\left(\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{n_p}\right)$ to one sink $\mathcal{S} = \{n\}$. Let $\mathcal{V}_{\mathcal{P}_j}$ be the set of nodes of path $\mathcal{P}_j$. Assume that the nodes have been labeled sequentially, such that for every path $\mathcal{P}_j$ there exists a path from k'_j to k''_j only if $k'_j < k''_j$. A parallel paths network is generically identifiable if and only if $\mathcal{F} \subset \mathcal{B}$, $\mathcal{S} \subset \mathcal{C}$, $\mathcal{B} \cup \mathcal{C} = \mathcal{V}$, and in addition there are $n_p - 1$ paths $\mathcal{P}_j$ for which there exist at least $k'_j, k''_j \in \mathcal{V}_{\mathcal{P}_j} \setminus \{1, n\}$, such that $k'_j \leq k''_j$, $k'_j \in \mathcal{B}$, $k''_j \in \mathcal{C}$.*

Proof. Necessity: The first three conditions are known to be necessary for any network, see Proposition II.1. It remains to justify the last condition. Assume that two paths do not obey the last condition. Without loss of generality, we denote them $\mathcal{P}_1$, $\mathcal{P}_2$. This implies that, for each of these two paths, there are three possibilities: 1) all *internal* nodes of a particular path are excited, 2) all *internal* nodes of a path are measured, or 3) the first l *internal* nodes of a path are measured and the remaining are excited. Assume, without loss of generality, that each path $\mathcal{P}_k$ for $k = 1, 2, \ldots, n_j$, has $p_k + 2$ nodes, where one of the p_k can possibly be zero. Hence, the two paths $\mathcal{P}_1$ and $\mathcal{P}_2$ have a total of $p_1 + p_2 + 2$ unknown edges. Now, from Lemma III.1 in [5], we know that if there is no path from node i to node j then $T_{ji} = 0$. Thus, for the two paths $\mathcal{P}_1$ and $\mathcal{P}_2$ there are $p_1 + p_2 + 1$ useful equations, because there are $p_1 + p_2$ elements $T_{ji} \neq 0$ and in addition, we have $T_{n1} \neq 0$, which is common to the paths. Therefore, there are more unknowns than available useful equations.

Sufficiency: Let us consider that path $\mathcal{P}_m$ is the only path that does not obey the last condition stated in the Theorem; hence all other paths do obey this condition. We now show that for each path $\mathcal{P}_l \neq \mathcal{P}_m$ all edges can be generically identified. Let $\mathcal{V}_{\mathcal{P}_l} = \{1, k_l^1, k_l^2, \ldots, k_l^{p_k}, n\}$, and assume that node k_l^i is excited and node k_l^j is measured, with $k_l^j \geq k_l^i$. The transfer functions corresponding with the edges of $\mathcal{P}_l$ can be identified as follows:

$$\text{if } k_l^1 \in \mathcal{B},\ G_{k_l^1,1} = T_{k_l^j,1}/T_{k_l^j,k_l^1}; \tag{3}$$

$$\text{if } k_l^1 \in \mathcal{C},\ G_{k_l^1,1} = T_{k_l^1,1}. \tag{4}$$

Consider now k_l^2. If $k_l^2 \in \mathcal{C}$ then we can recover $G_{k_l^2,k_l^1} = T_{k_l^2,1}/G_{k_l^1,1}$. Conversely, if $k_l^2 \in \mathcal{B}$, then we can recover $G_{k_l^2,k_l^1} = T_{k_l^j,1}/(T_{k_l^j,k_l^2} G_{k_l^1,1})$. We can apply the same reasoning for all nodes $k = k_l^1, k_l^2, \ldots, k_l^j$ (remember that node k_l^j is measured) and thus recover all transfer functions up to $G_{k_l^j,k_l^{j-1}}$. In a similar fashion we can recover the remaining transfer functions as:

$$\text{if } k_l^j + 1 \in \mathcal{B},\ G_{k_l^j+1,k_l^j} = T_{n,k_l^i}/(T_{k_l^j,k_l^i} T_{n,k_l^j+1}); \tag{5}$$

$$\text{if } k_l^j + 1 \in \mathcal{C},\ G_{k_l^j+1,k_l^j} = T_{k_l^j+1,k_l^i}/T_{k_l^j,k_l^i}. \tag{6}$$

Applying this reasoning to the remaining nodes $k = k_l^j + 1, k_l^j + 2, \ldots, n$ in $\mathcal{P}_l$ one can recover all transfer functions in $\mathcal{P}_l$, for all paths other than $\mathcal{P}_m$.

Now, consider the remaining path $\mathcal{P}_m$. It has $p_m + 2$ nodes corresponding to $\mathcal{V}_{\mathcal{P}_m} = \{1, k_m^1, k_m^2, \ldots, k_m^{p_m}, n\}$ and $p_m + 1$ unknown transfer functions. Since we assume that P_m does not obey the last condition of the Theorem, its EMP obeys one of the three possible scenarios previously stated. Suppose that k_m^1 is excited. It follows that we are in scenario 1, and hence all other nodes of $\mathcal{P}_m$ are excited. In this situation, we can recover all transfer functions of $\mathcal{P}_m$, except for $G_{k_m^1,1}$. Suppose now that $k_m^{p_m}$ is measured. This means that we are in scenario 2, and hence all other nodes of $\mathcal{P}_m$ are measured. In a similar way, we can recover all transfer functions of $\mathcal{P}_m$, except for $G_{n,k_m^{p_m}}$. The last scenario is where the first $\{k_m^1, k_m^2, \ldots, k_m^j\}$ are measured and the remaining $\{k_m^{j+1}, \ldots, k_m^{p_m}\}$ are excited. For this case, we can recover all modules of $\mathcal{P}_m$, except for $G_{k_m^{j+1},k_m^j}$. We conclude that, in all cases, all transfer functions in path $\mathcal{P}_m$ can be identified except one. Since all other transfer functions in the network are known, this remaining unknown transfer function can be successfully recovered from T_{n1}, which has not been used for the computation of the other transfer functions. ■

What Theorem III.3 says is that, for a PPN to be generically identifiable, all paths except one must have an excitation that precedes a measurement, in addition to the universal necessary condition that the source must be excited, the sink must be measured, and all other nodes must be either excited or measured.

IV. Isolated versus embedded

The necessary and sufficient conditions for generic identifiability of trees, loops and PPNs, respectively, allow one to construct valid and even minimal EMPs for these specific structures. However, these necessary and sufficient conditions, and the EMPs that are based on them, are valid for these structures taken in isolation. When these same structures are part of a more complex network, these EMPs may no longer be valid. To clarify this point, we will now introduce the concept of subdigraph, and specify exactly what we mean by an isolated subdigraph and an embedded subdigraph. We will then illustrate why an EMP that is valid for a subdigraph taken in isolation may no longer be valid when that subdigraph is embedded in a larger network.

Let $\mathcal{V}_A$ be a subset of nodes of $\mathcal{V}$, and let $\mathcal{G}_A(\mathcal{V}_A, \mathcal{E}_A)$ be the subdigraph of $\mathcal{G}$ defined by the subset of nodes $\mathcal{V}_A$ and all the edges $\mathcal{E}_A$ that link them, and let G_A be the associated subnetwork matrix. We extend Definition 3 to deal with subdigraphs.

Definition 7. *Consider a digraph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ and a subdigraph $\mathcal{G}_A(\mathcal{V}_A, \mathcal{E}_A)$ of $\mathcal{G}(\mathcal{V}, \mathcal{E})$. Consider an EMP that is valid for the subdigraph $\mathcal{G}_A(\mathcal{V}_A, \mathcal{E}_A)$ when all vertices and edges from $\mathcal{G}(\mathcal{V}, \mathcal{E})$ that do not belong, respectively, to $\mathcal{V}_A$ and $\mathcal{E}_A$ have been removed. This EMP is said to be **valid in isolation** for the isolated subdigraph $\mathcal{G}_A(\mathcal{V}_A, \mathcal{E}_A)$. If the same EMP is valid when $\mathcal{G}_A(\mathcal{V}_A, \mathcal{E}_A)$ is embedded into $\mathcal{G}(\mathcal{V}, \mathcal{E})$, then this EMP is said to be **embedded valid** with respect to $\mathcal{G}(\mathcal{V}, \mathcal{E})$ for the embedded $\mathcal{G}_A(\mathcal{V}_A, \mathcal{E}_A)$.*

Finding a valid, possibly minimal, EMP for the isolated graph is likely to be a much simpler problem than finding an EMP that is valid for the embedded graph. We illustrate this with the following example.

Example 1

Consider the 5-node network depicted in Figure 2, and let $\mathcal{G}(\mathcal{V}, \mathcal{E})$ be its associated digraph, with $\mathcal{V} = \{1, 2, 3, 4, 5\}$ and $\mathcal{E} = \{1 \to 2, 2 \to 3, 3 \to 4, 4 \to 1, 4 \to 5, 5 \to 3\}$.

Suppose first that the dotted red edges are not present in this network ($G_{54} = G_{35} = 0$), and that our objective is to identify the loop formed by nodes 1, 2, 3 and 4. According to Theorem III.2, there are two minimal EMPs for this isolated loop, namely: $\text{EMP}_1 : \mathcal{B}_1 = \{1, 3\}, \mathcal{C}_1 = \{2, 4\}$ and $\text{EMP}_2 : \mathcal{B}_2 = \{2, 4\}, \mathcal{C}_2 = \{1, 3\}$. However, when this isolated loop is connected to the other loop - and hence embedded in the 5-node network - as seen in Figure 2, these EMPs are no longer

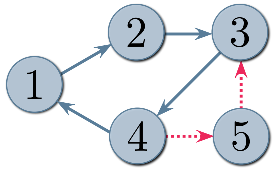

Fig. 2. A network formed by two loops: the one on the left formed by the solid blue edges, and the other on the right formed by the dotted red edges and the edge $3 \to 4$.

valid for this embedded loop, i.e. they are not embedded valid w.r.t. $\mathcal{G}$.

This example shows that a valid EMP for an isolated digraph may no longer be valid when this digraph is embedded into a more complex network. Therefore, using the theorems of Section III to produce valid EMPs for subdigraphs with specific structures may no longer yield valid EMPs when these subdigraphs are embedded in a larger digraph. In the next section we present our main result which exhibits conditions under which an EMP that is valid for an isolated digraph remains valid when it is embedded into a larger network.

V. Main result

Theorem V.1. *Consider a digraph $\mathcal{G}(\mathcal{V},\mathcal{E})$ and its associated network matrix G. Let $\mathcal{V}_A$ be a subset of connected nodes of $\mathcal{V}$, and $\mathcal{V}_B \triangleq \mathcal{V} \setminus \mathcal{V}_A$, Let $\mathcal{G}_A(\mathcal{V}_A,\mathcal{E}_A)$ be the subdigraph of $\mathcal{G}$ defined by the subset of nodes $\mathcal{V}_A$ and all edges $\mathcal{E}_A$ that link them, and let G_A be the associated subnetwork matrix. Assume that an EMP that is valid for the isolated subdigraph $\mathcal{G}_A$ has been obtained. This same EMP is embedded valid w.r.t. $\mathcal{G}$ for $\mathcal{G}_A$ if at least one of the following conditions holds.*

1) *There is no path starting in $\mathcal{V}_A$, passing through $\mathcal{V}_B$, and returning to $\mathcal{V}_A$.*
2) *All paths that leave $\mathcal{V}_A$ and return to $\mathcal{V}_A$ are known.*

Proof. Let us start by partitioning the network matrix G and the corresponding input-output matrix T according to the sets $\mathcal{V}_A$ and $\mathcal{V}_B$.

$$G = \begin{bmatrix} G_A & G_{AB} \\ G_{BA} & G_B \end{bmatrix}, T = \begin{bmatrix} T_{AA} & T_{AB} \\ T_{BA} & T_{BB} \end{bmatrix}.$$

The choice of an EMP for $\mathcal{G}_A$ corresponds to the choice of a submatrix of T_{AA}, namely the submatrix defined by the corresponding excited and measured nodes of $\mathcal{V}_A$. This EMP is valid if G_A can be uniquely reconstructed from this submatrix of T_{AA}. To examine whether an EMP that is valid for the identification of G_A, taken in isolation, remains valid when the nodes $\mathcal{V}_A$ are connected to the nodes $\mathcal{V}_B$, we use the following identity that relates T_{AA} to the other submatrices:

$$\begin{aligned} T_{AA} &= (I_A - G_A - G_{AB}(I_B - G_B)^{-1}G_{BA})^{-1} \\ &= (I_A - G_A - G_{AB}T_BG_{BA})^{-1}, \end{aligned} \tag{7}$$

where $T_B \triangleq (I - G_B)^{-1}$. All elements of $G_{AB}T_BG_{BA}$ can be written as:

$$\sum_{k_1,k_2 \in \mathcal{V}_A, j,i \in \mathcal{V}_B} G_{k_1j}T_{ji}G_{ik_2}.$$

Now, if there is no path from $\mathcal{V}_A$ passing through $\mathcal{V}_B$ and returning to $\mathcal{V}_A$, then one of the following holds: 1) $G_{k_1j} = 0$; 2) $T_{ji} = 0$; 3) $G_{ik_2} = 0$, for all $k_1, k_2 \in \mathcal{V}_A$ and $j, i \in \mathcal{V}_B$. Hence, all elements of $G_{AB}T_BG_{BA}$ are zero, which implies that

$$T_{AA} = (I_A - G_A)^{-1} \triangleq T_A \tag{8}$$

This is the expression that relates G_A to T_A in the subnetwork $\mathcal{G}_A$ taken in isolation. We have shown that the same relationship holds when that subnetwork is embedded in the complete network. Hence, the network matrix G_A can be recovered if an EMP is applied to T_{AA} that is valid for the subnetwork $\mathcal{G}_A$ taken in isolation, which proves item 1.

In order to prove item 2, we note that the expression (8) relates T_{AA} to G_A for the subnetwork $\mathcal{G}_A$ taken in isolation. If an EMP is valid for this subnetwork taken in isolation, it means that, with the submatrix of T_{AA} corresponding to this EMP, G_A can be recovered from $G_A = I_A - T_{AA}^{-1}$. When this subnetwork is embedded in the full network, this relation becomes (using a rewrite of (7)):

$$G_A = I_A - T_{AA}^{-1} - G_{AB}T_BG_{BA}. \tag{9}$$

If all paths leaving $\mathcal{V}_A$ and returning to $\mathcal{V}_A$ are known, then $G_{AB}T_BG_{BA}$ is also known. Hence, G_A can also be identified from (9), and this completes the proof. ■

Theorem V.1 is a powerful result for the situation where only part of the digraph is of interest, and where the task is to design a valid EMP for the corresponding subdigraph. But it also allows one to synthesize EMPs for the full network if a convenient decomposition of the digraph can be obtained. We illustrate this with the following Example.

Example 2

Consider the network and its digraph denoted $\mathcal{G}(\mathcal{V},\mathcal{E})$ in Figure 3, with ten nodes and twelve edges. Assume first that the edge $6 \to 3$ does not exist, i.e. disregard the dotted red edge for now. Suppose first that one only wants to identify the subdigraph, denoted $\mathcal{G}_A(\mathcal{V}_A,\mathcal{E}_A)$, that connects nodes 1 to 4, i.e. $\mathcal{V}_A = \{1, 2, 3, 4\}$. It is a PPN. According to Theorem III.3, the following EMP, denoted EMP_1, is *valid in isolation* for $\mathcal{G}_A$: $\mathcal{B}_1 = \{1, 3\}$, $\mathcal{C}_1 = \{2, 3, 4\}$; it is actually minimal. Now, consider that same subdigraph $\mathcal{G}_A(\mathcal{V}_A,\mathcal{E}_A)$ embedded in the whole digraph formed by all ten nodes shown in Figure 3, still without the edge $6 \to 3$. We observe that this subdigraph obeys condition 1 of Theorem V.1, and hence EMP_1 is *embedded valid* w.r.t $\mathcal{G}(\mathcal{V}, \mathcal{E} \setminus \{6 \to 3\})$ for $\mathcal{G}_A$.

Now consider that the dotted edge $6 \to 3$ (i.e. G_{36}) is added to the previous graph. Then condition 1 of Theorem V.1 is no longer satisfied, since there is an unknown path leaving $\mathcal{V}_A$ and returning to itself: $(4 \to 5 \to 6 \to 3)$ $\{G_{54}, G_{65}, G_{36}\}$. If these edges are known, then condition 2 of Theorem V.1 will be valid for the identification of the embedded $\mathcal{G}_A$. So, one possibility to produce an embedded valid EMP w.r.t. $\mathcal{G}$ for the identification of all edges in $\mathcal{G}_A$ is to identify also the edges in this path. This path, together with $3 \to 4$ (G_{43}),

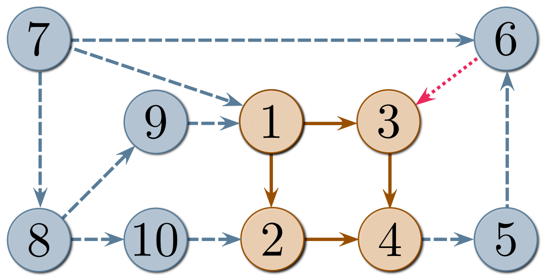

Fig. 3. A network with multiple structures. The nodes $1, 2, 3, 4$ form a PPN. The nodes $3, 4, 5, 6$ form a loop. The dashed blue edges linking nodes $1, 2, 6, 7, 8, 9, 10$ form a tree.

forms a loop: $(3, 4, 5, 6, 3)$. Consider first this loop in isolation. It then follows from Theorem III.2 that the following EMP, denoted EMP_2, allows the identification of all edges in this isolated loop: $\mathcal{B}_2 = \{3, 5\}$, $\mathcal{C}_2 = \{4, 6\}$. Now, consider this loop as embedded in the complete network $\mathcal{G}$. We observe that there is no path in $\mathcal{G}$ that leaves this loop and re-enters it. Hence, the conditions of item 1) of Theorem V.1 apply, which means that this EMP_2 is embedded valid w.r.t. $\mathcal{G}$ for this loop $(3, 4, 5, 6, 3)$.

Combining EMP_1 and EMP_2, we have EMP_3, defined as $\mathcal{B}_3 = \{1, 3, 5\}$, $\mathcal{C}_3 = \{2, 3, 4, 6\}$, which allows to identify all edges in $\mathcal{G}_A$ when it is embedded in the whole network shown in Figure 3, including node 6. This illustrates how Theorem V.1 can be used to construct a valid EMP for a subdigraph that is embedded in a more complete digraph. As a byproduct, EMP_3 also allows the identification of the transfer functions G_{54}, G_{65}, G_{36}.

Finally, consider that one wants to identify the whole network, that is, all the edges in $\mathcal{G}$. Let $\mathcal{G}_C$ be the digraph formed by the nodes $\mathcal{V}_C = \{1, 2, 8, 9, 10\}$ and by the edges $\mathcal{E}_C = \{1 \to 2, 8 \to 9, 8 \to 10, 9 \to 1, 10 \to 2\}$. Actually, $\mathcal{G}_C$ is a PPN. The EMP $\mathcal{B}_4 = \{8, 9\}$, $\mathcal{C} = \{1, 2, 10\}$, denoted EMP_4, is valid in isolation for it. The PPN $\mathcal{G}_C$ has no paths leaving and returning to it. Thus, EMP_4 is embedded valid w.r.t. $\mathcal{G}$ for $\mathcal{G}_C$ as Condition 1 from Theorem V.1 is satisfied. It remains to identify the edges $7 \to 1$, $7 \to 6$ and $7 \to 8$. Let $\mathcal{G}_D$ be the digraph formed by nodes $\mathcal{V}_D = \{1, 6, 7, 8, 10\}$ and by edges $\mathcal{E}_D = \{7 \to 1, 7 \to 6, 7 \to 8, 8 \to 10\}$. This digraph is a tree, and the EMP $\mathcal{B}_5 = \{7, 8\}$, $\mathcal{C}_5 = \{1, 6, 10\}$, denoted EMP_5, is valid in isolation for that tree. For this EMP, condition 2 from Theorem V.1 is satisfied for $\mathcal{G}_D$ since all other edges are known from the combination of EMP_3 and EMP_4.

Therefore, the combination of EMP_3, EMP_4 and EMP_5 allows the identification of the whole network. Define it as EMP_6: $\mathcal{B}_6 = \{1, 3, 5, 7, 8, 9\}$, $\mathcal{C}_6 = \{1, 2, 3, 4, 6, 10\}$. Its cardinality is 12, much smaller than the maximum 19 - the graph has only one source and no sinks - and only slightly more than the minimum 10. Notice that different combinations of valid EMPs for each subnetwork can be used to produce a range of EMPs that are valid for the identification of the whole network.

Example 2 has allowed us to illustrate the main result of Theorem V.1. It has shown how a valid, possibly minimal, EMP constructed for a subdigraph taken in isolation can be preserved or enhanced when that subdigraph is embedded in a larger digraph. At the same time, Example 2 has illustrated that, when a network is decomposed into a set of simple subdigraphs for which valid EMPs are easy to construct, the combination of these valid EMPs can lead to an EMP that is valid for the whole digraph, via the use of Theorem V.1. Our ongoing work is to develop a decomposition algorithm of a general network into subdigraphs to which this scenario can be applied.

VI. Conclusion

We have presented new results for the generic identifiability of Parallel Paths Networks. But our main contribution has been to show that an EMP that is valid for a subnetwork treated in isolation is typically no longer valid when that subnetwork is embedded in a larger network, and to present sets of sufficient conditions under which it remains valid. This result is of importance for the practical situation when only part of a network needs to be identified. With a proper decomposition of the global network, it may also lead to an efficient method for the identification of this global network.

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