C33 CDC 1987 C33

EXPONENTIAL CONVERGENCE OF A NEW ERROR SYSTEM ARISING FROM ADAPTIVE OBSERVERS

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Part of this research was supported by the Biotechnology Action Programme of the EEC

ABSTRACT

We prove the uniform asymptotic stability of a new error system, whose transition matrix contains both a regression vector and a filtered version of that vector. We show that stability is under conditions which are different from the usual SPR or small gain requirements. We show that this error system arises from the design of adaptive observers for some classes of time-varying or nonlinear systems.

1. MAIN RESULT

In this paper, we study the stability of the following error system :

$$\begin{cases} \dot{x}(t) = A(t)x(t) + B(t)\theta(t) & , x_0 \in \mathbb{R}^m \\ \dot{\theta}(t) = -\Gamma v(t)C^T x(t) & , \theta_0 \in \mathbb{R}^n \end{cases} \quad (1)$$

where $C \in \mathbb{R}^m$ and $\Gamma \in \mathbb{R}^{nxn}$ are constant with $\Gamma = \Gamma^T > 0$, $A(t) \in \mathbb{R}^{mxm}$ and $B(t) \in \mathbb{R}^{mxn}$ are time-varying, and $v(t) \in \mathbb{R}^n$ is the output of the following auxiliary filter :

$$\begin{cases} \dot{\psi}(t) = A\psi(t) + B(t) \, , & \psi_0 \in \mathbb{R}^{mxn} \\ v(t) = \psi^T(t)C \end{cases} \quad (2)$$

where $A \in \mathbb{R}^{mxm}$ is a constant matrix, and C is as above. The matrix $B(t)$ can take various forms ; a typical example is $B(t) = b\varphi^T(t)$ where $b \in \mathbb{R}^m$ is a constant vector and $\varphi(t) \in \mathbb{R}^n$ is a regression vector.

Note that the state of the auxiliary filter is a matrix, and its transition matrix is of order n. The following assumption will be made about the error system and its auxiliary filter.

A1 : $A(t) = A - \psi(t)\Gamma v(t)C^T$
Re $\lambda_i(A) < -a$, $a > 0$, $i = 1,\dots,m$

A2 : $v(t)$ is persistently exciting (PE), i.e. there exist s_0, $T > 0$ and $\beta > \alpha > 0$ such that

$$0 < \alpha I \leqslant \frac{1}{T} \int_s^{s+T} v(t)v^T(t)dt \leqslant \beta I = \|v\|_\infty^2 I \quad \forall s \geqslant s_0$$

We now prove the following result.

Theorem 1 : Under assumptions A1 and A2, the time-varying linear system (1), with B(t) and v(t) related by the auxiliary filter (2), is uniformly asymptotically stable.

Proof : Define the vector $e \triangleq x - \psi\theta$. Then the system (1) is equivalent with

$$\begin{bmatrix} \dot{e} \\ \dot{\theta} \end{bmatrix} = \begin{bmatrix} A & 0 \\ -\Gamma v(t)C^T & -\Gamma v(t)v^T(t) \end{bmatrix} \begin{bmatrix} e \\ \theta \end{bmatrix} \quad , \quad \begin{bmatrix} e_0 \\ \theta_0 \end{bmatrix} \quad (3)$$

The exponential stability follows directly from A1 and A2.

2. COMMENTS

Comment 1 : The transition matrix of (3) is given by

$$H(t,\tau) = \begin{bmatrix} \exp(-A(t-\tau)) & 0 \\ -\int_\tau^t F(t,s)\,\Gamma v(s)C^T \exp(-A(s-\tau))ds & F(t,\tau) \end{bmatrix} \quad (4)$$

where $F(t,\tau)$ is the transition matrix for the linear time-varying system $\dot{z} = -\Gamma v(t)v^T(t)z$.

Comment 2 : The rate of exponential convergence of (3) can be estimated as follows. Let α_1 be the rate of exponential convergence of $F(t,\tau)$, i.e.

$$\|F(t,\tau)\| < K_1 e^{-\alpha_1(t-\tau)} \quad \forall t,\tau, \text{ for some } K_1 > 0 \quad (5)$$

The rate α_1 and the constant K_1 are related to Γ and the bounds of the PE conditions as follows (see [1]) :

$$\alpha_1 = \frac{1}{2T} \ln \frac{1}{1-\eta} \quad \text{and } K_1 = (1-\eta)^{-\frac{T}{2}} \quad (6)$$

where

$$\eta = \frac{2T\alpha\,\lambda_{min}(\Gamma)}{[1+n\,\lambda_{max}(\Gamma)\beta T]^2} \quad (7)$$

The rate of exponential convergence of (3) is then given by

$$\|H(t,\tau)\| < K_2 e^{-\alpha_2(t-\tau)} \quad \forall t,\tau \quad \text{and some } K_2 > 0$$

where

$$\alpha_2 = \min(a,\alpha_1) \text{ if } a \neq \alpha_1$$
$$= a - \epsilon, \quad \epsilon > 0 \text{ but arbitrary if } a = \alpha_1.$$

Comment 3 : Note that there is no SPR condition and no constraint on the gain Γ ; however, see our further comment about Γ in the last section.

3. MOTIVATION

The error system (1) was originally obtained as the error system of an adaptive observer designed for the on-line estimation of parameters in nonlinear fermentation processes.

An adaptive observer, inspired by one designed by Kreisselmeier [2] (see also [3]) for linear time-invariant systems, was derived for this nonlinear system; it led to an error system which was a simplified version of (1)-(2). We later recognized that the Kreisselmeier adaptive observer [1] could be modified to handle the on-line state and parameter estimation of a class of linear time-varying systems described in scalar differential equation form. The error system for this class of observers is the system (1)-(2). We now illustrate these two points.

The following set of nonlinear equations is a model for a large class of fermentation processes :

$$\begin{cases} \dot{X} = \alpha XS - DX \\ \dot{S} = -\beta XS + D[S_{in} - S] \end{cases} \tag{8}$$

Here X and S are, respectively, the concentration of biomass and substrate in a continuous stirred tank bioreactor, D is the influent flow rate and α and β are strictly positive unknown and unmeasurable time-varying coefficients, which are nonlinear functions of X, S and other variables. All the variables and parameters in these two equations are time-varying, but the yield coefficient α/β is constant.

In a typical application, it is required to obtain on-line estimates of the state $S(t)$ and the parameters $\alpha(t)$ and $\beta(t)$ from measurements of the inputs $D(t)$ and $S_{in}(t)$, and of the state components $X(t)$. Note that, instead of identifying the nonlinear dependence of α and β on the states, these parameters will be treated as time-varying parameters, to be estimated on line. The only requirement will be the boundedness of $|\dot{\alpha}|$ and $|\dot{\beta}|$. To solve this problem, we first perform the following state transformation :

$$Z \triangleq S + \frac{\beta}{\alpha} X$$

The model (8) is then replaced by

$$\begin{cases} \dot{X} = \alpha XZ - \beta X^2 - DX \\ \dot{Z} = D[S_{in} - Z] \end{cases} \tag{9}$$

The following adaptive observer was derived in [4] :

$$\begin{cases} \dot{\hat{Z}} = D[S_{in} - \hat{Z}] \ , \quad \hat{Z}(0) \geq 0 \text{ but arbitrary} \\ \begin{bmatrix} \dot{v}_0 \\ \dot{v}_1 \\ \dot{v}_2 \\ \dot{v}_3 \end{bmatrix} = -a \begin{bmatrix} v_0 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} + \begin{bmatrix} X \\ X\hat{Z} \\ -X^2 \\ -DX \end{bmatrix} \qquad (10) \\ \hat{X} = av_0 + \hat{\alpha} v_1 + \hat{\beta} v_2 + v_3 \qquad (11) \\ \dot{\hat{\theta}} = \Gamma v(X - \hat{X}), \ \hat{\theta}_0 \text{ arbitrary} \qquad (12) \end{cases}$$

where a is a strictly positive constant scalar, $v = (v_1 \ v_2)^T$, $\hat{\theta} = (\hat{\alpha} \ \hat{\beta})^T$ and $\Gamma = \Gamma^T > 0$ but otherwise arbitrary.

Defining $\tilde{X} = X - \hat{X}$, $\tilde{\theta} = \theta - \hat{\theta}$ and $\varphi = (\hat{Z}X - X^2)^T$ yields the following third order system, after a number of manipulations :

$$\begin{bmatrix} \dot{\tilde{X}} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} -a - v^T \Gamma v & \varphi^T \\ -\Gamma v & 0 \end{bmatrix} \begin{bmatrix} \tilde{X} \\ \tilde{\theta} \end{bmatrix} + \begin{bmatrix} \alpha X \tilde{Z} \\ \dot{\theta} \end{bmatrix} \tag{13}$$

The driving terms can be shown to be bounded, due to the physical assumptions on the fermentation process, and under realistic experimental conditions the vectors $\varphi(t)$ anf $v(t)$ are PE, as demonstrated in [5]. It is easy to see that the homogeneous part of (13) is a special case of the error system (1). Its uniform asymptotic stability therefore ensures the global stability of the adaptive observer.

As a second application, consider the following linear time-varying differential equation model :

$$y^{(n)} + a_1(t)y^{(n-1)} + \ldots + a_n(t)y(t) = b_1(t)u^{(n-1)}(t) + \ldots + b_n(t)u(t) \tag{14}$$

where the $a_i(t)$ and $b_i(t)$ are in C^1 and $u(t)$, $y(t)$ in C^n.

Let $F(s) = s^n + f_1 s^{n-1} + \ldots + f_n$ be an arbitrary Hurwitz polynomial with roots $\gamma_1, \ldots, \gamma_n$ and let $\gamma = \sup_i \mathrm{Re}(\gamma_i)$.

Also, denote

$$\zeta(t) = [y^{(n-1)} \ldots y^{(1)} y \ u^{(n-1)} \ldots u^{(1)} \ u]^T$$

$$\theta(t) = [f_1 - a_1(t) \ \ldots \ f_n - a_n(t) \ b_1(t) \ \ldots \ b_n(t)]^T$$

Then (14) can be rewritten as :

$$y(t) = \frac{1}{F(s)} \{\theta^T(t)\zeta(t)\} + \mathrm{dec}(\gamma t) \tag{15}$$

where $\mathrm{dec}(\gamma t)$ will denote a time function that decays at a rate $\exp(\gamma t)$. One can then show (see [6]) that :

$$y(t) = C^T x(t) + \eta(t) + \mathrm{dec}(\gamma t) \tag{16}$$

where $C^T = [1 \ 0 \ \ldots \ 0] \in \mathbb{R}^n$,

$$\dot{x}(t) = Fx(t) + B(t)\theta(t) + \psi(t)\dot{\theta}(t) \tag{17}$$

where $F \in \mathbb{R}^{n \times n}$ is a constant matrix in observer canonical form such that $\det(sI-F) = F(s)$,

$$B(t) = \begin{bmatrix} y & & & u & & \\ & y & & & \ddots & \\ \bigcirc & & \ddots \ y & \bigcirc & & u \end{bmatrix} \in \mathbb{R}^{n \times 2n} \tag{18}$$

and $\psi(t) \in \mathbb{R}^{n \times 2n}$ is the state of the following auxiliary system :

$$\begin{cases} \dot{\psi}(t) = F\psi(t) + B(t) \\ v^T(t) = C^T \psi(t) \end{cases} \tag{19}$$

Finally, $\eta(t) = \frac{1}{F(s)}\{\theta^T(t)\zeta(t)\} - \theta^T(t)\frac{1}{F(s)}\{\zeta(t)\}$.

Using Morse's Swapping Lemma [7], one can show that, for some $K > 1$:

$$\|\eta(t)\| \leq \frac{K}{\gamma} \|\psi\|_\infty \|\dot{\theta}\|_\infty \tag{20}$$

The following adaptive observer can then be used :

$$\begin{cases} \dot{\hat{x}}(t) = F\hat{x}(t) + B(t)\hat{\theta}(t) + \psi(t)\Gamma v(t) \ [y(t) - C^T \hat{x}(t)]^T \\ \dot{\hat{\theta}}(t) = \Gamma v(t) \ [y(t) - C^T \hat{x}(t)]^T \end{cases} \tag{21}$$

The error system is

$$\begin{bmatrix}\dot{\tilde{x}}\\ \dot{\tilde{\theta}}\end{bmatrix} = \begin{bmatrix}F - \psi(t)\Gamma v(t)C^T & B(t)\\ -\Gamma v(t)C^T & 0\end{bmatrix}\begin{bmatrix}\tilde{x}\\ \tilde{\theta}\end{bmatrix} + \begin{bmatrix}\psi(\dot{\theta} - \Gamma v\eta)\\ \dot{\theta} - \Gamma v\eta\end{bmatrix} \quad (22)$$

Compare with (1). The conditions under which v(t) is PE have been studied in [8]. Under these conditions, and the assumed boundedness on $|\dot{\theta}|$, the state variables $\tilde{x}$, $\tilde{\theta}$ of (22) are bounded for all t, using Theorem 1.

4. EXTENSIONS : A ROBUSTNESS PROPERTY

We shall now demonstrate the robustness of the exponential convergence property under some type of multiplicative perturbation. Such multiplicative perturbation may, e.g., arise due to numerical round-off errors in the actual implementation of the algorithm.

Consider again the error system (1) and the auxiliary filter (2) and replace assumption A1 by :

A3 : $A(t) = A - \psi(t)\Gamma v(t)C^T + E(t)$

where the transition matrix $G(t,\tau)$ of the linear system
$\dot{z} = (A + E(t))z$ satisfies
$\|G(t,\tau)\| \leq K \exp(-d(t-\tau))$ for some $K > 1$ and $d > 0$.

Comment 4 : E(t) could arise from numerical errors in implementing $A - \psi(t)\Gamma v(t)C^T$. Note that to get a feeling for the effect of high adaptation gains on the practical implementation of the algorithm, we have to look at this type of multiplicative perturbation.

Theorem 2 : Under assumptions A2 and A3, the error system (1) is uniformly asymptotically stable if

$$\|E\|_\infty \|\Gamma\| < K_3(\|\psi\|_\infty, C, \alpha_1, d, K, K_1) \quad (23)$$

$$\text{where } K_3 = \frac{\max(|\alpha_1 - d| \min(\alpha_1,d),\ \frac{1}{2}\min^2(\alpha_1,d))}{K_1 K \|\psi\|_\infty^2 \|CC^T\|}$$

Proof : Under the transformation $e \triangleq x - \psi\theta$ and the assumption A3, system (1) is equivalent with :

$$\begin{bmatrix}\dot{e}\\ \dot{\theta}\end{bmatrix} = \begin{bmatrix}A + E(t) & E(t)\psi(t)\\ -\Gamma v(t)C^T & -\Gamma v(t)v^T(t)\end{bmatrix}\begin{bmatrix}e\\ \theta\end{bmatrix}, \quad \begin{bmatrix}e_0\\ \theta_0\end{bmatrix} \quad (24)$$

It follows that

$$\dot{\theta}(t) = -\Gamma v(t)C^T G(t,0)e_0 - \Gamma v(t)C^T \int_0^t G(t,\tau)E(\tau)\psi(\tau)\theta(\tau)d\tau - \Gamma v(t)v^T(t)\theta(t) \quad (25)$$

Hence, using $F(t,\tau)$ as defined in section 2, we get

$$\theta(t) = F(t,o)\theta_0 - \int_0^t F(t,s)\Gamma v(s) C^T G(s,o)\, e_0 ds - \int_0^t F(t,s)\Gamma v(s) C^T \int_0^s G(s,\tau)E(\tau)\psi(\tau)\theta(\tau)d\tau ds$$

Denote the last term by M(t). It can be estimated as follows :

$$\|M(t)\| \leq K_1 K \|\Gamma\| \|\psi\|_\infty^2 \|CC^T\| \|E\|_\infty \int_0^t e^{-\alpha_1(t-s)} \cdot \int_0^s e^{-d(s-\tau)}\|\theta(\tau)\|\, d\tau ds \quad (26)$$

We now consider three separate cases.

a) $\alpha_1 < d$. Then

$$\|M(t)\| \leq K_1 K \|\Gamma\| \|\psi\|_\infty^2 \|CC_\infty^T\| \|E\|_\infty \frac{1}{(d-\alpha_1)} \cdot \int_0^t e^{-\alpha_1(t-\tau)}\|\theta(\tau)\|d\tau$$

b) $\alpha_1 > d$. Then

$$\|M(t)\| \leq K_1 K \|\Gamma\| \|\psi\|_\infty^2 \|CC^T\| \|E\|_\infty \frac{1}{(\alpha_1-d)} \cdot \int_0^t e^{-d(t-\tau)}\|\theta(\tau)\|d\tau$$

c) $\alpha_1 = d$. Then

$$\|M(t)\| \leq K_1 K \|\Gamma\| \|\psi\|_\infty^2 \|CC^T\| \|\dot{E}\|_\infty \int_0^t (t-\tau)e^{-\alpha_1(t-\tau)}\|\theta(\tau)\|d\tau$$

The Bellman-Gromwall lemma then guarantees exponential stability if either

* $K_1 K \|\Gamma\| \|\psi\|_\infty^2 \|CC^T\| \|E\|_\infty \frac{1}{|\alpha_1 - d| \min(\alpha_1,d)} < 1$

  in case a) or b) or

* $K_1 K \|\Gamma\| \|\psi\|_\infty^2 \|CC^T\| \|E\|_\infty \frac{1}{me(\alpha_1 - m)} < 1$

  where $m \in (0,\alpha_1)$, in case c).

Comment 5 : Since the right hand side of inequality (23) is positive, it can always be satisfied for $\|E\|_\infty$ sufficiently small, i.e. for sufficiently small perturbations.

Comment 6 : For sufficiently small Γ and with $\Gamma = \gamma I$, we get $\eta \approx 2T\alpha\gamma$ and $\alpha_1 \approx \alpha\gamma$ (see (6) and (7)). Inequality (23) becomes :

$$\frac{KK_1 \|\psi\|_\infty^2 \|CC^T\| \|E\|_\infty}{d\alpha} < 1$$

Since $\beta = \|v\|_\infty^2 \leq \|\psi\|_\infty^2 \|CC^T\|$, this requires in particular that

$$\frac{KK_1 \|E\|_\infty \beta}{d\alpha} < 1$$

We conclude that the amount of allowable perturbation cannot be increased by decreasing the adaptation gain. However, the last inequality shows that the condition number of the PE matrix (see A2) may play a role in determining how much perturbation can be allowed. See also [9] for a discussion of the effect of this condition number.

CONCLUSION

We have presented a new error system which arises from applying a specific adaptive observer structure to a variety of nonlinear and time-varying systems. We have exhibited two such problems where this adaptive observer can be used, but we believe that it may have much wider applications. This error system has some very nice features. We have shown the uniform asymptotic stability of its homogeneous part without any requirement of a strictly positive real transfer function or of slow adaptation. We have shown the robustness of this exponential stability property under some forms of multiplicative perturbations which can arise from numerical errrors in the actual implementation of the equations. This robustness is an inherent property of this particular error system which distinguishes it from the more usual error systems, where either a SPR condition or a small gain condition is required : see e.g. [10].

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