UNIFICATION OF DISCRETE AND CONTINUOUS TIME STOCHASTIC ADAPTIVE CONTROL ALGORITHMS.

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Abstract. In this paper, we present a formulation of a continuous time stochastic adaptive control algorithm and explore the relationship with classical continuous time deterministic schemes and with discrete time stochastic adaptive control algorithms. We give a complete global stability theory and highlight the technical difficulties which arise in the continuous time stochastic case.

Keywords. Adaptive systems, stochastic systems, identification, model reference control.

INTRODUCTION

In recent years, considerable progress has been made towards the unification of many adaptive control algorithms which have been proposed over the last fifteen years. Particular emphasis has been given to the problem of unifying discrete and continuous time deterministic adaptive control schemes (see, for example, Goodwin and Mayne, 1987) and in unifying discrete deterministic and stochastic schemes (Goodwin and Sin, 1984). However, up to now, continuous time stochastic control algorithms have only been derived with restrictive assumptions, which make them appear very different from their discrete counterparts (see for example Chen, 1985, Moore, 1986 and Chen and Moore, 1987).

We believe that there is a need for a formulation of a continuous time adaptive stochastic control algorithm which is close to similar formulations in the deterministic or discrete-time cases. Indeed, as pointed out in Moore (1986), such an algorithm would give greater insight into the discrete stochastic case when fast sampling is employed and into the continuous time deterministic case under unbounded, or wide band, noise.

Hence, in this paper, we present a formulation of a continuous-time stochastic adaptive control algorithm which is closely related to the discrete time formulation given for example in Goodwin and Sin (1984). We establish global stability of the algorithm and highlight the technical difficulties which arise in the continuous time case. As in deterministic systems, a central issue is that of the relative degree of the system input-output transfer function. Up to now, the only known "solutions" treat the case of relative degree zero. The direct extension of these results to larger relative degrees seems difficult.

This paper gives a solution to the problem. This solution depends on additional properties of the parameter estimator and these depend upon using a new algorithm structure which differs from that used in discrete time.

The paper is organized as follows. Section 2 presents and discusses the continuous-time stochastic model that will be used. Section 3 discusses the predicted output based on pseudoregressions. The estimation algorithm and some of its properties are presented in section 4. An indirect adaptive control algorithm is presented in section 5 together with a proof of global stability.

THE MODEL

When dealing with continuous time stochastic processes, even the choice of an appropriate model becomes difficult. A commonly used model is the following state space innovations representation

\[ dx_t = Ax_t dt + Bu_t dt + Kd_u \]
\[ dy_t = Cx_t dt + d_w \]

(2.1a)

(2.1b)

where \( w_t \) is a Wiener process with incremental covariance \( \sigma^2 dt \), i.e. \( \mathbb{E}[w_t^2] = \sigma^2 dt \). We shall denote by \( \mathcal{F}_t \) the increasing \( \sigma \)-algebra generated by \{\( x_s; 0 \leq s \leq t \}\}. This model has been used extensively in the non-adaptive literature (see e.g. Aström, 1970) but also more recently in the adaptive case (Moore, 1986 and Gevers et al, 1987). Notice that \( \mathcal{F}_t \) is in fact the integral of the output of this model.

One of the aims of this paper is to establish links between continuous-time and discrete-time stochastic adaptive control algorithms. In the latter case, input-output (ARMAX) descriptions have been preferred, and for this reason we shall also use an input-output representation in this paper. This is a compact formal notation for the corresponding state space results as given in Gevers et al, 1987. One way to define an input output representation for (2.1) is to use the integral operator as in Chen (1985) or Chen and Moore (1987). However, the use of an integral operator makes the treatment of continuous time stochastic processes apparently different from the treatment of continuous time deterministic processes. To make the connections more apparent, we will introduce another formal representation which uses the differential operator.

Let \( \rho \Delta \frac{d}{dt} \), \( x_t \Delta \frac{dy_t}{dt} \), \( w_t \Delta \frac{dw_t}{dt} \). Then (2.1) can be rewritten as

\[ A(\rho)z_t = B(\rho)u_t + C(\rho)w_t \]

(2.2a)

where
$A(p) = p^n + a_{n-1}p^{n-1} + \cdots + a_0$ \hspace{1cm} (2.2b)

$B(p) = b_m p^m + \cdots + b_0$ \hspace{1cm} (2.2c)

$C(p) = p^n + c_{n-1}p^{n-1} + \cdots + c_0$ \hspace{1cm} (2.2d)

\( n \) is the order of the process and \( n-m+1 \) is the relative degree of the input-output transfer function. The model (2.2) contains an abuse of notation, since we "differentiate" a Winner process. However, this is purely a compact notation for the system described by (2.1). In future, unless otherwise specified, all polynomials will be polynomials in \( p \).

Let \( E(p) \) be any stable monic polynomial

$E(p) \triangleq p^n + a_{n-1}p^{n-1} + \cdots + a_0$ \hspace{1cm} (2.3)

Then (2.3a) can be rewritten as

$z_k = E^{-1}(E-A)z_k + B_i + C \omega_k$ \hspace{1cm} (2.4)

$\hat{z}_k = \hat{\phi}_T^T \psi_{\hat{\theta}} + \omega_k$ \hspace{1cm} (2.5)

$\hat{\phi}_T^T \triangleq [\phi_{n-1}^T \phi_{n-2}^T \cdots \phi_0^T]$, \hspace{1cm} (2.6)

$\hat{\theta}_i^T \triangleq [\phi_{n-1-1}^T \phi_{n-2-1}^T \cdots \phi_0^T]$ \hspace{1cm} (2.7)

Let \( \psi_{\hat{\theta}} \) be any measurable estimate of \( \theta_i \). Then we define the "prediction" \( \hat{z}_k \) of \( z_k \) by using pseudo regressions as:

$\hat{z}_k = \psi_{\hat{\theta}}^T \hat{\theta}_i$ \hspace{1cm} (2.8)

where

$\psi_{\hat{\theta}}^T \triangleq [\phi_{n-1}^T \phi_{n-2}^T \cdots \phi_0^T]$, \hspace{1cm} (2.9)

Finally we define

$\hat{\eta}_k = \hat{z}_k - \phi_{\hat{\theta}} \hat{\theta}_i$ \hspace{1cm} (2.10)

\( \hat{\eta}_k \) can be viewed as the "noise reduced" prediction error. Since there is an instantaneous feedback of white noise into the output \( \phi_{\hat{\theta}} \hat{\theta}_i \) (see 2.15), obviously the best one can hope for is to prove that \( \hat{\eta}_k \) goes to zero in some sense, but not \( \hat{\eta}_k \). It will be useful for future use to establish a relationship between \( \hat{\eta}_k \) and \( \phi_{\hat{\theta}}^T \hat{\theta}_i \). From (2.5) and (2.8) to (2.10), it follows that.

$C_{\phi_{\hat{\theta}}} \hat{\eta}_k = -\phi_{\hat{\theta}}^T \hat{\theta}_i$ \hspace{1cm} (2.11)

Comment 2.1

As pointed out above, all the expressions from (2.2) are shorthand notations that should be interpreted as integral equations - see, for example, Govers et. al. (1987) for the appropriate state space forms.

ESTIMATION ALGORITHM

Motivated by the discrete time algorithm given in Goodwin and Sin (1984), we introduce the following estimation algorithm

$\hat{\phi}_i = \frac{\psi_{\hat{\theta}}^T \psi_{\hat{\theta}}}{\psi_{\hat{\theta}}^T \phi_{\hat{\theta}}}$ \hspace{1cm} (3.1)

where \( \tau_k \) is a suitably chosen normalization constant.

In discrete time, \( \tau_k = i \) is usually defined as the cumulative sum of \( \psi_{\phi_{\hat{\theta}}}^T \psi_{\hat{\theta}} \). This ensures, amongst other things, that \( \psi_{\phi_{\hat{\theta}}}^T \psi_{\phi_{\hat{\theta}}} \) is bounded. However, if we replace the sum by an integral in continuous time then we lose this boundedness property. We therefore modify the usual definition of \( \tau_k \) as follows:

$\tau_k \triangleq \sup_{t \leq \tau} \int_0^t \psi_{\phi_{\hat{\theta}}}^T \psi_{\phi_{\hat{\theta}}} \, dt + \mathcal{C}_0 : \mathcal{C}_0 > 0$ \hspace{1cm} (3.2)

Note that this re-establishes the boundedness property as required in our subsequent analysis.

Noting that \( \psi_{\phi_{\hat{\theta}}}^T = \psi_{\phi_{\hat{\theta}}}^T \psi_{\phi_{\hat{\theta}}} \) and using the definition of \( \tau_k \) given in (2.10), we can write that \( \phi_{\hat{\theta}} \) is the solution of the following stochastic differential equation:

$\frac{d\phi_{\hat{\theta}}}{\tau_k} = \frac{\psi_{\phi_{\hat{\theta}}}^T \psi_{\phi_{\hat{\theta}}} \phi_{\hat{\theta}}}{\psi_{\phi_{\hat{\theta}}}^T \phi_{\phi_{\hat{\theta}}}} \, dt + \mathcal{C}_0 \, d\omega_k$ \hspace{1cm} (3.3)

We shall next establish properties of the estimator which hold irrespectively of the control law that will be used. (Compare with the deterministic case in Goodwin and Mayne 1987). We shall need the following assumption.

Assumption 1:

\( \phi_{\theta} \) is strictly minimum phase and the filter \( H(s) \) is chosen such that

1. \( \text{Re} \, \sigma(E(s)) \leq -\alpha \leq 0 \), \( i=1, \ldots, n \) where \( \sigma(E) \) are the roots of \( E \).

2. \( \text{Re} \, \frac{C}{\phi_{\phi_{\hat{\theta}}}} \) is input strictly passive, i.e. \( \beta > 0 \) such that

$\psi_{\phi_{\phi_{\hat{\theta}}}}^T \phi_{\phi_{\phi_{\hat{\theta}}}} \psi_{\phi_{\phi_{\hat{\theta}}}} + \int_0^T \psi_{\phi_{\phi_{\hat{\theta}}}}^T \psi_{\phi_{\phi_{\hat{\theta}}}} \, dt \, \psi_{\phi_{\phi_{\hat{\theta}}}} \psi_{\phi_{\phi_{\hat{\theta}}}} \, dt$ \hspace{1cm} (3.4)

where \( \gamma_k \) is the output of the filter \( \frac{C}{\phi_{\phi_{\hat{\theta}}}} \) driven by \( \phi_{\phi_{\hat{\theta}}} \).

Comment 3.1

Note that the filter \( E \) is crucial in satisfying the above passivity condition. In discrete time, \( E \) is often implicitly assumed to be \( \phi_{\phi_{\phi_{\hat{\theta}}}}^0 \). However, the continuous time theory indicates that this choice will be inadequate when fast sampling is employed.

Comment 3.2

Another difficulty in continuous time is that one must show that finite escape do not occur. This is straightforward in discrete time provided zero dividers are avoided. (Caines and Meyn 1985)). However, in continuous time it is necessary to show that the solutions of (3.3) possess moments of arbitrary high order for all finite \( t \). This is done in the following Lemma.
Lemma 3.1: The solutions of (3.3) satisfy
\[
K \left( \sup_{0 \leq t \leq T} \left\| \hat{\Theta}_t \right\|^{2p} \right) < (k + 3B \left( \left\| \hat{\Theta}_0 \right\|^{2p} \right))e^{KT}
\]
for any \( p > 0 \), where \( K \) depends only on \( p \) and \( T \).

**Proof** The result depends upon showing that the coefficients of (3.3) are linearly bounded. The result then follows from Lemma 3.8, page 140 of Gibson and Stoorvogel (1979). See Gevers et al. (1987).

This result implies, in particular, that \( \hat{\theta}_t \) is almost surely bounded for all finite \( t \), i.e., the solutions of (3.3), almost surely, have no finite escape time.

We can then establish the following result which gives properties of the parameter estimator:

**Theorem 3.1:** Under assumption 1, the following properties hold for the model (2.2) and the estimator (3.1), (3.2):

(i) \( \limsup_{t \to \infty} \left\| \hat{\Theta}_t \right\| < K < \infty \) a.s. \hspace{2cm} (3.4)

(ii) \( \lim_{t \to \infty} \int_0^t \frac{r_x^2}{r} \, dt \leq K_2 \) \hspace{2cm} a.s. \hspace{2cm} (3.5)

(iii) For all finite \( \Delta, \limsup_{t \to \infty} \left\| \hat{\Theta}_t \right\| = 0 \) a.s. \hspace{2cm} (3.6)

**Proof** See Appendix A.

Comment 3.3:

The above proof uses the result in equation (A.1)

It is instructive to compare expression (A.1) obtained from the Ito rule (Wong, 1971) with the corresponding discrete time result.

The discrete time form of (3.1) is (Goodwin, Ramadge, Caines (1981)):

\[
\hat{\theta}_t = \hat{\theta}_{t-1} + \frac{\hat{y}_{t-1}}{r_{t-1}} (\hat{\theta}_{t-1} - \hat{\gamma}_{t-1})
\]

Squaring both sides we obtain

\[
\tilde{\theta}_t^2 = \tilde{\theta}_{t-1}^2 + 2\Delta \left( \frac{\hat{y}_{t-1}}{r_{t-1}} \hat{\gamma}_{t-1} \right) + 2 \Delta \left( \frac{\hat{y}_{t-1}}{r_{t-1}} \hat{\gamma}_{t-1} \right) \left( \hat{\gamma}_{t-1} + \tilde{\omega}_t \right)
\]

where \( \tilde{\omega}_t \) is the deterministic part of the error and \( \tilde{\omega}_t \) is the discrete time noise. It is argued in Salgado, Middleton, Goodwin (1987) that if an appropriate pre-sample filter is used, then \( \tilde{\omega}_t \) will have variance of the order of \( \Delta^2 \).

Thus

\[
\tilde{\theta}_t^2 - \tilde{\theta}_{t-1}^2 = \Delta^2 \left( \frac{\hat{y}_{t-1}}{r_{t-1}} \right)^2 \left( \hat{\gamma}_{t-1} + \tilde{\omega}_t \right) + 2 \Delta \left( \frac{\hat{y}_{t-1}}{r_{t-1}} \hat{\gamma}_{t-1} \right) \left( \hat{\gamma}_{t-1} + \tilde{\omega}_t \right)
\]

Comparing this result with that given in (A.1), we see that, in the limit as \( \Delta \to 0 \), the first term disappears since it is of order \( \Delta^2 \), the second term becomes the first term in (3.7) and \( \frac{\Delta^2 \left( \frac{\hat{y}_{t-1}}{r_{t-1}} \right)^2 \hat{\gamma}_{t-1} \hat{\gamma}_{t-1}} {r_{t-1}} \) becomes \( \frac{\Delta^2 \hat{\gamma}_{t-1} \hat{\gamma}_{t-1}} {r_{t-1}} \) on application of the Ito rule.

We can now establish a continuous time equivalent of the discrete-time stochastic key technical lemma given in Goodwin and Sin (1984).

**Lemma 3.2 (Stochastic Key Technical Lemma):**

Suppose the estimator is such that (3.5) is satisfied, and the controller is such that the following growth condition is satisfied:

\[
\frac{r_t}{r} \leq C + K \int_0^t \eta_s^2 \, ds \quad \text{a.s.}
\]

where \( r_t, \eta_t \) are defined before and \( C, K \) are finite positive constants. Then:

(i) \( \limsup_{t \to \infty} \frac{r_t}{r} \leq K_3 < \infty \) a.s. \hspace{2cm} (3.9)

(ii) \( \lim_{t \to \infty} \int_0^t \eta_s^2 \, ds = 0 \) a.s. \hspace{2cm} (3.10)

**Proof** (1) From (3.8) and the nonnegativity of \( \eta_s^2 \) and \( r_t \), it follows that:

\[
\frac{r_t}{r} \leq C + K \int_0^t \eta_s^2 \, ds \leq C + K \int_0^t \eta_s^2 \, ds + K_3 \text{ a.s.}
\]

The result then follows from the Bellman–Gronwall lemma (see e.g. Doetsch and Vidyasagar 1975) using (3.5).

(ii) Suppose first that \( \lim_{t \to \infty} \frac{r_t}{r} = \infty \). Then, by the continuous time Konoike (Gevers et al. 1987):

\[
\lim_{t \to \infty} \int_0^t \eta_s^2 \, ds = 0 \quad \text{a.s.}
\]

Hence, using (3.9) and (3.11),

\[
\lim_{t \to \infty} \int_0^t \eta_s^2 \, ds = \lim_{t \to \infty} \int_0^t \eta_s^2 \, ds \leq \lim_{t \to \infty} \int_0^t \eta_s^2 \, ds = 0 \quad \text{a.s.}
\]

Alternatively, if \( \lim_{t \to \infty} \frac{r_t}{r} \leq K_3 \) for some \( K_3 \), then

\[
\lim_{t \to \infty} \int_0^t \eta_s^2 \, ds = \lim_{t \to \infty} \int_0^t \eta_s^2 \, ds \leq \lim_{t \to \infty} \int_0^t \eta_s^2 \, ds = 0 \quad \text{a.s.}
\]

Using (3.5).

Q.E.D.

Comment 3.3

Comparing the properties of our estimator with those proved for the continuous-time deterministic algorithm in Goodwin and Mayne (1987), we observe that we have not proved the uniform boundedness of \( \frac{r_t}{r} \) or of \( \frac{\hat{\theta}_t}{r} \); neither
have we proved $\hat{\theta} \in L^2$. The uniform boundedness of $\frac{\eta_0}{r_1^2}$ is not needed in the subsequent analysis ($\frac{\eta_0}{r_1^2} \in L^2$ suffices).

As for $\hat{\beta}$, it does not exist in our stochastic framework, but the conditions on $\hat{\beta}$ in Goodwin and Mayne (1987) are replaced by the weaker condition (3.6).

AN INDIRECT ALGORITHM

We now consider a general class of control laws in state feedback form:

$$u_t = -\hat{\phi}_1^T \hat{\beta} + y_t^*$$  \hspace{1cm} (4.1)

where $\beta$ denotes a control law parameter vector and where $y_t^*$ denotes a bounded reference signal. This gives a strictly proper controller. In the adaptive case, we replace $\beta$ by $\hat{\beta}$ which will be evaluated as a suitable function of $\theta$. This gives

$$u_t = -\hat{\phi}_1^T \hat{\beta}_t + y_t^*$$  \hspace{1cm} (4.2)

We make the following assumptions about the control law:

Assumptions

C.1: $[\beta]$ is uniformly bounded by $[\theta]$

C.2: $\beta$ has a bounded derivative w.r.t. $\theta$

C.3: Let $\hat{\beta} = [\hat{\beta}_0, \ldots, \hat{\beta}_n| \hat{\beta}_0, \ldots, \hat{\beta}_n]$ and form

$$\hat{\beta} = [\hat{\beta}_0, \ldots, \hat{\beta}_n| \hat{\beta}_0, \ldots, \hat{\beta}_n]^T$$

and

$$L_t = \hat{\beta}_{n-1} \rho^{n-2} + \ldots + \hat{\beta}_0; \quad \hat{\beta}_0 = B + L$$

$$\hat{P}_t = \hat{\beta}_0 \rho^{n-2} + \ldots + \hat{p}_0$$

$$\hat{H}_t = \hat{\beta}_{n-1} \rho^{n-2} + \ldots + \hat{h}_0$$

Then

$$\forall t, \lambda^* \hat{A} + \hat{PB} = \hat{A}(\theta)$$

where $\hat{A}(\theta)$ is continuous, has a uniform stability margin, i.e. $\lambda^* \hat{A}(\theta) \leq -\beta < 0$ $\forall t = 1, \ldots, 2n$ and $\lim_{T \to \infty} \sup_{\Delta} \| \hat{A}(\theta) - \hat{A}(\theta) \| = 0$ for some $T$.

A special case of the above class of control laws is the adaptive linear quadratic optimal controller structure with an infinite horizon. $Q$ and $P$ are then computed as follows:

Let

$$\hat{\beta} = [\hat{\beta}_0, \ldots, \hat{\beta}_n| \hat{\beta}_0, \ldots, \hat{\beta}_n]$$

and define

$$\hat{A}_1 = \hat{A}_1 \hat{\eta}_t^*; \quad \hat{c}_1 = \hat{c}_1 + \hat{f}_t^* \quad \text{i=0, \ldots, n-1}$$

$$\hat{A}(\theta) = \rho_0 + \hat{\beta}_{n-1} \rho^{n-2} + \ldots + \hat{c}_0$$

$$\hat{H}(\theta) = \hat{h}_{n-1} \rho^{n-2} + \ldots + \hat{c}_0$$

Then $\hat{Q}, \hat{P}$ satisfy an equation of the form $\hat{Q} \hat{A} + \hat{P} \hat{B} = \hat{A}(\theta)$, where $\hat{A}(\theta)$ depends upon $\hat{A}, \hat{B}$ and $\hat{C}$ and the weights used in the performance index.

In this case, assumptions C.1 to C.3 are satisfied provided (a) $\hat{\theta}_1$ is bounded, and (b) no near-pole-zero cancellations occur between $(\hat{A}_1, \hat{B}_1)$ and $(\hat{A}, \hat{C})$.

Comment 4.1

Several strategies have been described elsewhere to avoid the problem of "near-pole-zero cancellations" in the above algorithm. For example, one idea presented in Middleton et al. (1987) for the deterministic case is to consider a set of convex regions in the parameter space in which no pole-zero cancellations occur. A separate estimator is then run in each of these regions with an orthogonal projection to ensure that the estimates remain inside the corresponding convex region. If we assume that the true system lies inside at least one of these regions then it is readily shown that properties (1) to (iii) of Lemma 3.1 will hold for at least one of the estimators. If one then monitors $\int_0^T \hat{y}_t^2 \, dt$ and chooses the estimator having the least value of this function (with some hysteresis to avoid swapping between estimators infinitely often), then the essential properties of the parameter estimator are retained together with the additional feature that near-pole-zero cancellations are now impossible. The full proof of these facts follows by combining the ideas in Middleton et al. (1988) with the results in Theorem 3.1.

We now show that the regression $\hat{\phi}_1^*$ can be written as the state of a linear time-varying system driven by the prediction error $\hat{\epsilon}_t$, and the reference signal $\hat{y}_t^*$. From the model (2.1), the controller (4.2), the definition of $\hat{\phi}_1^*$ (2.9), and the definition of the errors $\eta_t$ and $\epsilon_t$, we can write

$$d\hat{\phi}_1^* = A_1 \hat{\phi}_1^* \, dt + B_1 (\eta_t \, dt + \epsilon_t \, dt) + B_2 \hat{y}_t^* \, dt$$  \hspace{1cm} (4.4)

where
\[ H_1 \tau = [1 \ 0 \ ... \ 0 \ 1 \ 0 \ ... \ 0] \quad (4.5b) \]
\[ H_2 \tau = [0 \ ... \ 0 \ 1 \ 0 \ ... \ 0] \quad (4.5c) \]

where \( H_1 \ Tau \) has 1's in the 1st and (2n+1)st positions and \( H_2 \ Tau \) has 1 in the (n+1)st position.

We can then establish that the homogeneous part of (4.4) is exponentially stable.

**Lemma 4.1:** Consider the differential equation

\[ \frac{d}{dt} \phi = \Lambda \phi \quad (4.6) \]

with \( \Lambda \) given by (4.5a). Assume that the \( a_j, b_j \) are estimated using the parameter estimator of Section 3 and that C1 to C3 holds. Then (4.6) is exponentially stable. a.s.

**Proof:** This result follows from Lemma 3 of Krisselmeier (1985) and the structure of \( \Lambda \) given in (4.5) — see Gevers et al. (1987) for details.

We then have the main result of this paper.

**Theorem 4.1**

Consider a system (2.1), the parameter estimator of Section 3 satisfying Assumption 1 and any control law of the form (4.2) satisfying C1 to C3. Then, for arbitrary finite initial conditions and an arbitrary, piecewise continuous, uniformly bounded reference input \( \gamma \), then:

(i) \( \limsup_{t \to \infty} \frac{1}{t} \int_{0}^{t} \| \psi(t) \|^2 dt < \infty \) a.s.

(ii) \( \limsup_{t \to \infty} \frac{1}{t} \int_{0}^{t} \| w(t) \|^2 dt < \infty \) a.s.

(iii) \( \limsup_{t \to \infty} \frac{1}{t} \int_{0}^{t} \| Cx(t) \|^2 dt < \infty \) a.s.

**Proof:** Outline (see Gevers et al. (1987)).

We first establish a growth condition between \( y \) and \( \phi \) using Lemma 4.1, i.e.

\[ \frac{\dot{y}}{y} \leq K_0 \tau + K_1 \int_{0}^{t} \psi(t) \tau \quad (4.7) \]

We then use the Stochastic Key Technical Lemma (Lemma 3.2) to complete the proof.

**Comment 4.3**

Note that \( Cx \) is the 'deterministic part' of the output. Thus, part (iii) of Theorem 4.2 establishes that this part of the output is sample mean square bounded. This is all that can be said about the output since it contains a Wiener process.

**CONCLUSIONS**

The paper has analyzed a class of continuous time stochastic adaptive control algorithms and has shown that, under suitable conditions, they will be globally stable. The results for the discrete time-case and differences, where they exist, have been highlighted.

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**REFERENCES**


**Appendix A:** Proof of Theorem 3.1.

(i) Starting from (3.3) and using Ito's rule (Wong 1971), we get

\[ d\mu_{\tau} (\theta_{\tau}) = 2 \theta_{\tau} ^T \psi_{\tau} (\eta_{\tau} dt + dw) + \sigma \frac{\psi_{\tau}^T}{\tau} \mu_{\tau} - 2 \frac{\psi_{\tau}^T}{\tau} \mu_{\tau} dt \]

\[ = - \frac{1}{\tau} \left( \theta_{\tau} ^T \psi_{\tau} (\eta_{\tau} dt + dw) + \frac{\psi_{\tau}^T}{\tau} \mu_{\tau} \right) \]

\[ + \frac{2}{\tau} \theta_{\tau} ^T \psi_{\tau} (\eta_{\tau} dt) + \sigma \frac{\psi_{\tau}^T}{\tau} \mu_{\tau} - 2 \frac{\psi_{\tau}^T}{\tau} \mu_{\tau} dt \]
for some $\epsilon > 0$.

Defining

$$\xi_t \triangleq -\theta_t^T \eta_t$$

and integrating (A.2) yields

$$\theta_t \triangleq \theta_0 + \int_0^t \frac{\xi_{\lambda} \cdot \lambda}{\lambda} d\lambda + 2 \int_0^t \frac{\eta_{\lambda} \cdot \lambda}{\lambda} d\lambda + 2 \int_0^t \frac{\theta_{\lambda} \cdot \lambda}{\lambda} d\lambda$$

$$\sup_{0 \leq t \leq \Delta} \| \tilde{\theta}_t \| < \epsilon$$

Consider now the integral

$$\int_0^t \frac{\| \phi_{\lambda} \|^2}{\lambda} d\lambda$$

We have

$$\int_0^t \frac{\| \phi_{\lambda} \|^2}{\lambda} d\lambda < \int_0^t \frac{\| \phi_{\lambda} \|^2}{\lambda} d\lambda + \frac{1}{\lambda_0^2} \left( \frac{\| \phi_{\lambda} \|^2}{\lambda} - \frac{\| \phi_{\lambda} \|^2}{\lambda_0^2} \right)$$

since $\| \phi_{\lambda} \|^2$ is a non-negative contribution when $\| \phi_{\lambda} \|^2 > \sup_{s < \lambda} \| \phi_{\lambda} \|^2$. We now define

$$X_t \triangleq \theta_t + 2 \int_0^t \frac{\xi_{\lambda} \cdot \lambda}{\lambda} d\lambda + 2 \int_0^t \frac{\eta_{\lambda} \cdot \lambda}{\lambda} d\lambda + 2 \int_0^t \frac{\theta_{\lambda} \cdot \lambda}{\lambda} d\lambda$$

By Lemma 3.1 and the definition of $\lambda$, all terms in $X_t$ have finite expectation. In addition, by (A.4) and Assumption 1, $X_t$ is non-negative.

Using (A.3) in (A.5) yields

$$X_t = \theta_0 + \int_0^t \frac{\theta_{\lambda} \cdot \lambda}{\lambda} d\lambda$$

Now denote by $\mathcal{F}_s$ the increasing $\sigma$-fields generated by $\omega_s$.

Then for $t \geq s$, $\mathbb{P}[X_s] = \mathbb{E}[\mathbb{E}[X_s | \mathcal{F}_s]] = \mathbb{E}[X_s] = \theta_0$.

Therefore, for $t \geq s$, $\mathbb{P}[X_t | \mathcal{F}_s] = \theta_0$ and in addition, for $t \geq s$,

$$\mathbb{E}[X_t | \mathcal{F}_s] = X_s$$

Therefore $(X_t, \mathcal{F}_s)$ is a Martingale and by the Martingale Convergence Theorem (Dobr, 1983) it follows that

$$\lim_{t \to \infty} X_t = X_s < \epsilon \text{ a.s.}$$

This proves (i) and (ii).

(b) To prove (iii), we must prove that for all $\epsilon > 0$ and $\Delta > 0$, there exists $t_0$ such that

$$\sup_{t \geq t_0} \| \tilde{\theta}_t \| < \epsilon$$

(A.10)

From (3.3) we can write

$$\| \tilde{\theta}_{t+T} - \tilde{\theta}_t \|^2 \leq 2 \left( \int_{t+T}^{t+T} \frac{\psi_{\lambda} \cdot \lambda}{\lambda} d\lambda \right)^2 + 2 \left( \int_{t+T}^{t+T} \frac{\psi_{\lambda} \cdot \lambda}{\lambda} d\lambda \right)^2$$

$$\leq 2T \left( \int_{t+T}^{t+T} \frac{\psi_{\lambda} \cdot \lambda}{\lambda} d\lambda \right)^2 + 2 \left( \int_{t+T}^{t+T} \frac{\psi_{\lambda} \cdot \lambda}{\lambda} d\lambda \right)^2$$

(A.11)

The last inequality is obtained by applying the Schwartz inequality. Consider now the terms on the R.H.S. of (A.11). From (i) it follows that, given $\epsilon > 0$, $\Delta > 0$, there exists $t_0$ such that

$$\int_{t+T}^{t+T} \frac{\psi_{\lambda} \cdot \lambda}{\lambda} d\lambda \leq \frac{T}{2} \Delta$$

for some $\delta > 0$.

Since $\| \psi_{\lambda} \|^2 \leq 1$ it follows that

$$\| \tilde{\theta}_{t+T} - \tilde{\theta}_t \|^2 \leq \frac{4T}{2} \Delta^2$$

(A.12)

Next denote $S_t \triangleq \int_0^t \frac{\psi_{\lambda} \cdot \lambda}{\lambda} d\lambda$. It is readily shown (Dobr, 1987) that $\| S_t \|^2$ converges to a finite limit in view of (A.4).

Therefore, for given $\epsilon > 0$ and $\Delta > 0$, there exists $t_1(\epsilon, \Delta)$ such that $\forall t \geq t_1$,

$$\| S_t \|^2 \leq \frac{4T}{2} \Delta^2 \epsilon$$

(A.13)

Combining (A.11), (A.12) and (A.13) yields

$$\| \tilde{\theta}_{t+T} - \tilde{\theta}_t \|^2 \leq \epsilon^2$$

(A.14)

This proves (iii). Q.E.D.