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Optimal Sensors Allocation Algorithm for Stochastic Distributed Systems : Numerical Performance

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Abstract

The minimum-variance state-estimator algorithm for a class of distributed parameter systems, with both process and measurement disturbances, is derived. A new algorithm is presented for the determination of the optimal sensors' locations that minimize the trace of the spatial integral of the state error covariance matrix. The algorithm can be applied to both the one-sensor allocation and the simultaneous multi-sensor allocation problems. An illustrative example is given to demonstrate the numerical performance of the algorithm.

Introduction

State estimation of distributed parameter systems has become of great importance in many engineering applications where it is required to estimate the state for the whole spatial domain using a finite number of transducers. The first contribution to the filtering problem of randomly disturbed distributed parameter systems was made by Balakrishnan and Lions [1] and Falb [2] . A rigorous derivation of the minimum-variance filter is presented by Kushner [3] .

Optimal selection of the transducers locations within the spatial domain has received only limited attention. The problem was first introduced by Goodson and Klein [4] ; they gave an observability definition based on restricted spatial domains and system normal modes. Yu and Seinfeld [5] developed an algorithm for determining a sub-optimal set of measurement locations for a class of linear distributed systems whose solutions can be expressed as eigenfunction expansions.

In the present work, an optimal sensors' allocation algorithm is presented. Given a space of admissible measurement locations, the spatial distributions of the measurement and process noises, and a fixed number of sensors, the minimum-variance state estimator is constructed with the sensors' positions as unknown parameters to be optimized. The optimal parameters are obtained via the minimization of a nonlinear functional of the error covariance matrix using a modified gradient algorithm. An important feature of the algorithm is that it can be applied to both the one-sensor allocation and the simultaneous multi-sensor allocation problems. The numerical performance of the algorithm has been studied as well as the influence of different disturbance processes. An example is given which illustrates the performance of the algorithm when applied to the optimal allocation of one and two sensors in a slab-type nuclear reactor.

Problem formulation

Consider the class of linear distributed parameter system whose dynamics is described by the following vector integral equation.

$$U(t,x) = \int_{\Omega} G(t,t',x,x')\, U(t',x')\, d\Omega' + \int_{\Omega}\int_{s=t'}^{t} G(t,s,x,x')\, d_s\beta(s,x')\, d\Omega', \quad t > t' \qquad (1)$$

where the space variable $x = \{x_1, \dots x_\nu\}$ ranges in an open set $\Omega \subset R^\nu$ with boundary $\partial\Omega$. $U(t,x)$ is the n-vector state; $\beta(t,x)$ is the n-vector Wiener process with zero-mean and incremental covariance,

$$E\{d_t\beta(t,x)\,(d_t\beta(t,x'))^T\} = C(t,x,x')\, dt \qquad (2)$$

$G(t,t',x,x')$, $t \geq t'$ is the system Green's function representing its response at (t,x) to an input source at (t', x').

The observations are taken at discrete time instants $t = k\tau$, $k = 0, 1, 2, \dots$ through a finite number of sensors located at N different points of the space,

$$Y_k(x_j) = M_k(x_j)\, U_k(x_j) + v_k(x_j)\ , \quad j=1,\dots,N \qquad (3)$$

where $U_k(x_j)$ is used to denote the state $U(k\tau, x_j)$, $Y_k(x_j)$ is the p-dimensional observation vector and $v_k(x_j)$ is a white Gaussian sequence with zero mean and covariance,

$$E\{\, v_k(x_i)\, v_l^T(x_j)\} = V_k(x_j)\, \delta_{i,j}\, \delta_{k,l} \qquad (4)$$

where $\delta_{i,j}$ is the Kronecher delta. Furthermore, it is assumed that both measurement noise and process noise are statistically independent.

The problem now can be formulated as follows :

1. <u>Filtering problem</u> : given the set of noisy measurements,

$$Y^k = \sigma\{Y_l(x_j)\ ,\ j=1,\dots,N\ ;\ l = 0,1,\dots,k\} \qquad (5)$$

it is required to find the minimum-variance estimate $\hat{U}_k(x)$ of the state $U_k(x)$ for all $x \in \Omega$, where the state-error covariance at time $k\tau$, will be denoted by,

$$P_k(x,x') = E\{\tilde{U}_k(x)\, \tilde{U}_k^T(x')\} \qquad (6)$$

and

$$\tilde{U}_k(x) = U_k(x) - \hat{U}_k(x) \qquad (7)$$

2. <u>Allocation problem</u> : for a given spatial distribution $P_{k-1}(x,x)$ of the state-error covariance at time $(k-1)\tau$, find the optimal positions $X_m(k) = \{x_j\ ,\ j=1,\dots,N\}\big|_{k\tau}$ of the sensor that minimizes the spatial integral of the error covariance at time $k\tau$,

$$\min_{X_m(k)} \int_{\Omega} P_k(x,x)\, d\Omega \qquad (8)$$

It must be noticed that the above formulation of two separate problems is valid, since the structure of the optimal filter (not the optimal gains) is independent of the positions. Therefore, the optimum filter gains will be calculated first as a function of the sensor positions; the resulting filter will then be optimized w.r.t. the sensor's positions.

Main results

In this section the final results will be stated without proof and their interpretation will be given. The reader who is interested in the mathematical derivations is referred to Aidarous et al [6] . A direct approach, in which all quantities are expanded in terms of a finite number of elements taken from a complete set of orthonormal basis, is used to obtain the minimum-variance state estimator. The estimator algorithm is given by [7] ,

$$\tilde{U}_k(x) = \tilde{U}_{k^-}(x) + K_k(x, X_m)\, \eta_k(X_m) \tag{9}$$

where the innovations $\eta_k(X_m)$ are given by,

$$\eta_k(X_m) = \begin{bmatrix} Y_k(x_1) - M_k(x_1)\, \hat{U}_{k^-}(x_1) \\ \vdots \\ Y_k(x_N) - M_k(x_N)\, \hat{U}_{k^-}(x_N) \end{bmatrix} \tag{10}$$

and the a priori state estimate $\hat{U}_{k^-}(x)$ is given by,

$$\hat{U}_{k^-}(x) = \int_{\Omega} G_{k-1}(x,x')\, \hat{U}_{k-1}(x')\, d\Omega' \tag{11}$$

where

$$G_{k-1}(x,x') \triangleq G((k-1)\tau, k\tau, x, x') \tag{12}$$

In the solution, the first r elements of a complete set of orthonormal basis $\{z_i(x),\ i=1,\dots,\infty\ \ x\in\Omega\}$ are used for the expansion of the state-error covariance matrix, the system Green's function and the process noise covariance matrix in the following form,

$$P_k(x,x') = Z^T(x)\, W_k\, Z(x') \tag{13}$$

$$G_k(x,x') = Z^T(x)\, A_k\, Z(x') \tag{14}$$

$$C_k(x,x') = Z^T(x)\, H_k\, Z(x') \tag{15}$$

where $Z(x)$ is nr x n matrix whose elements are the chosen orthonormal basis, and W, A, H are three nr x nr matrices of coefficients.

Define

$$E_k = A_k\left[W_k + \tau H_k\right] A_k^T \tag{16}$$

The filter optimal gain is obtained via the solution of the following matrix Riccati equation,

$$E_{k+1} = A_k E_k A_k^T - A_k E_k B_k^T(X_m)\left[B_k(X_m)\, E_k B_k^T(X_m) + Q_k(X_m)\right]^{-1} B_k(X_m)\, E_k A_k^T + \tau A_k H_k A_k^T \tag{17}$$

where,

$$Q_k(X_m) = \mathrm{diag}\left[V_k(x_1),\dots, V_k(x_N)\right] \tag{18}$$

$$B_k(X_m) = \begin{bmatrix} M_k(x_1) Z^T(x_1) \\ \vdots \\ M_k(x_N) Z^T(x_N) \end{bmatrix} \tag{19}$$

The examination of the Riccati equation (17) shows the dependence of its solution on the sensors' positions X_m, through the output matrix $M_k(x)$ and the measurement noise covariance $Q_k(X_m)$.

This dependence is used together with the orthonormality properties of the chosen basis to obtain the following nonlinear optimization problem,

$$\max_{X_m} \left\{ J(X_m) \mid E_{k-1} \right\} \tag{20}$$

where

$$J(X_m) = \mathrm{tr}\left[E_{k-1} B_k^T(X_m)\, \Phi_k^{-1}(X_m)\, B_k(X_m) E_{k-1}\right] \tag{21}$$

The solution of this optimization problem gives the optimum sensors' positions that minimize the cost given by (8) (cf. Aidarous et al [6]).

The maximization of the cost (21) is carried out using a modified gradient algorithm (M.G.A.), in which gains are increased according to changes of the cost and the gradient calculated at the previous iterations. In addition to this automatic gain adaptation, different stopping rules are associated with the algorithm to achieve optimal stopping criteria that depend on the geometry of the region of search. A complete discussion of those stopping rules is given in [6] .

The practical implementation of the above algorithm is demonstrated by the flow chart on Fig. 1

Numerical performance

The numerical performance of the proposed algorithm will be studied when applied to a reactor system. Consider a slab-type nuclear reactor whose dynamics is given by,

$$\frac{\partial u}{\partial t} = \alpha^2\, \frac{\partial^2 u}{\partial x^2} + c\, u + \xi$$

where u is the neutron flux distribution, $\alpha^2 = 1600$ is the diffusion coefficient and $c = 0.252$ is the neutron fission rate. The neutron flux must vanish at the extrapolated boundaries, i.e.,

$$u(t,o) = u(t,h) = 0$$

where the slab width $h = 250$ cm.

The process noise ξ is white Gaussian with zero mean and covariance

$$E\{\,\xi(t,x)\ \ \xi(t',x')\} = C(x)\ \delta_D(x-x')\ \delta_D(t-t')$$

Observations of the neutron flux are made at discrete time instants with a period $\tau = 0.1$ sec. using one or two transducers.

The measurement equation is given by,

$$y_k(x_i) = u_k(x_i) + v_k(x_i) \quad , \quad i=1 \text{ for one-sensor case.}$$
$$i=1,2 \text{ for two-sensor case.}$$

The measurement noise $v_k(x)$ is a zero mean white Gaussian sequence with covariance,

$$E\{\, v_k(x_i)\ v_l(x_j)\} = q(x_j)\ \delta_{k,l} \text{ if } |x_i - x_j| \le \delta$$
$$= 0 \text{ otherwise}$$

for some small δ . Here the whiteness-in-space assumption for the measurement noise is relaxed within a sphere of radius δ around the sensor position.

The following noise covariance profiles have been considered in testing the algorithm,

$$c(x) = a + b \sin 2\pi x$$
$$q(x) = d + f \sin 2\pi x$$

The geometry of the cost to be minimized is affected simultaneously by the system parameters, the process noise and the measurement noise covariance profiles. Typical cost profiles for different realizations of system and measurement noise processes are depicted in Fig. 2. This justifies the importance of incorporating the cost geometry (through automatically adapted gains and stopping rules) in the search algorithm. Absence of such automatic gain adaptation may cause slow convergence (if not oscillations) as shown in Fig. 3. On the contrary, when using our M.G.A., rapid convergence has been achieved without oscillations. Fig. 4 shows the convergence of the algorithm for the optimal location of one sensor; here there are two equivalent optimum positions that give the same minimum value to the cost.

In the case of multisensor allocation, the optimal solution obtained by the algorithm is achieved by simultaneously allocating all the sensors, while previously existing techniques [5] were seeking a suboptimal solution by using successive one-sensor allocations while keeping the other sensors fixed. An example where this suboptimal solution differs from our optimal algorithm is given in Fig. 5, where it is required to allocate two sensors in the spatial domain. The solid line represents the path followed using our optimal allocation algorithm, while the dashed line represents the suboptimal (sensor-by-sensor allocation) strategy.

Conclusions

A new algorithm is presented for the optimal simultaneous spatial allocation of sensors in linear distributed-parameter systems whith state and measurement disturbances. Using a direct approach, the problem has been reduced to the solution of recursive algebraic equations and the minimization of a non-linear function that is easily computable. The numerical performance of the algorithm has been studied as well as the effect of different disturbance processes. Rapid convergence and prevention of oscillations is achieved by using a modified gradient algorithm (M.G.A.) with automatic gain adaptations. An illustrative example is given to show the forementioned properties.

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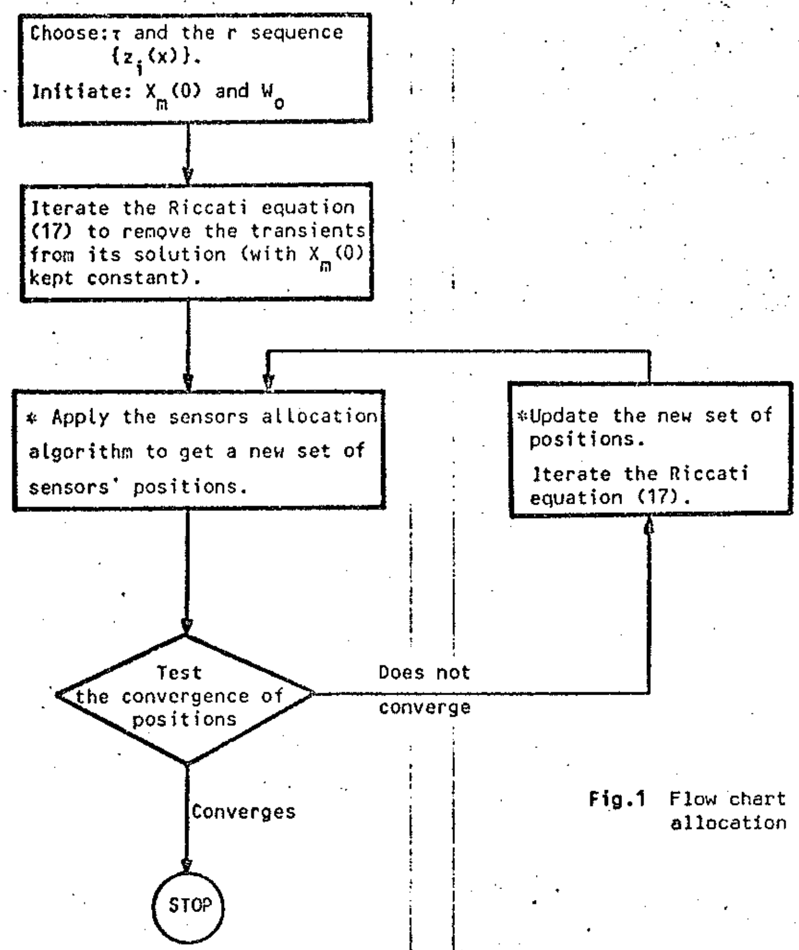

Fig.1 Flow chart of the sensors allocation algorithm.

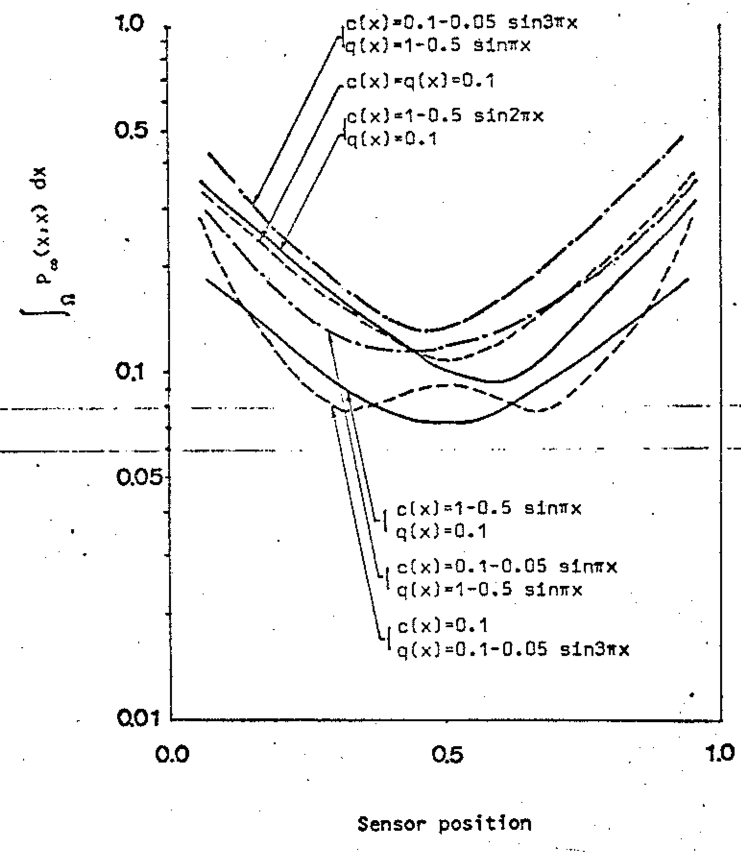

Fig.2 Noise processes effect on the cost.

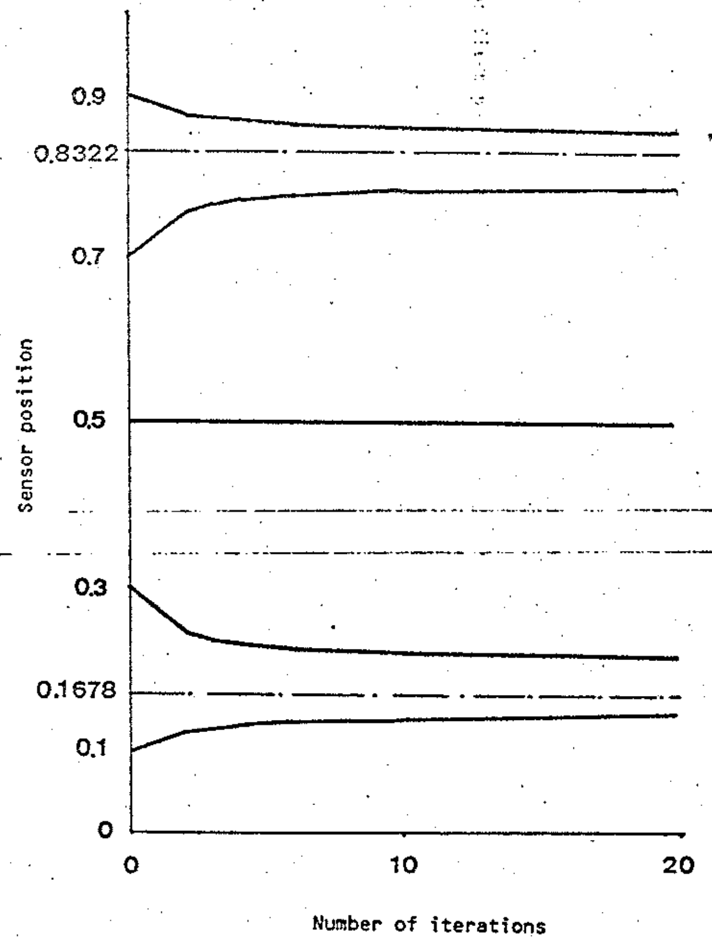

Fig.3 Performance when using a formel gradient algorithm.

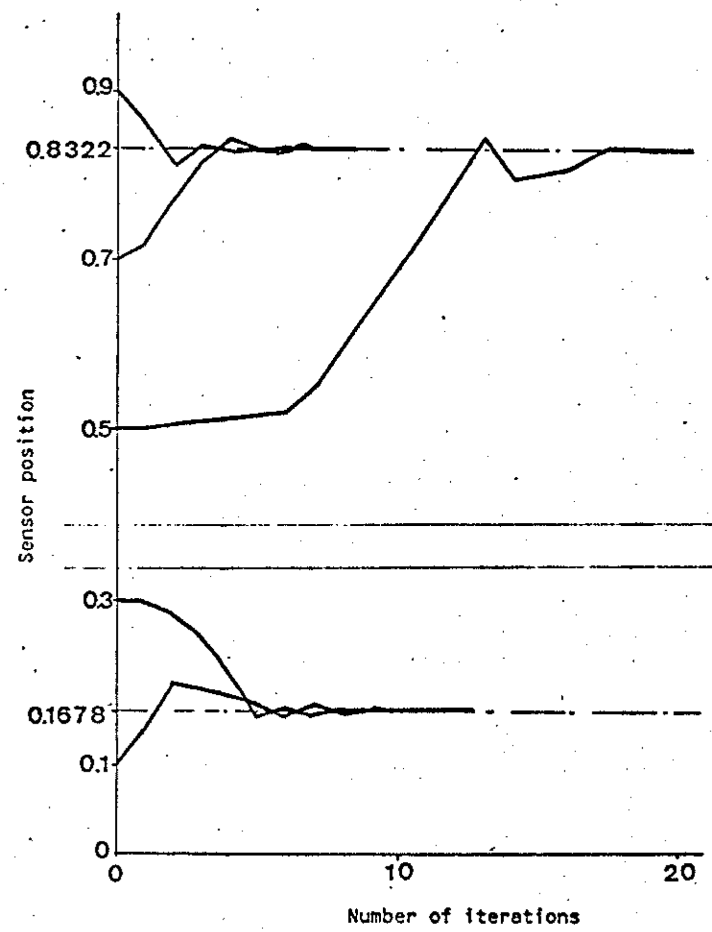

Fig.4 Convergence of the algorithm for one sensor allocation.

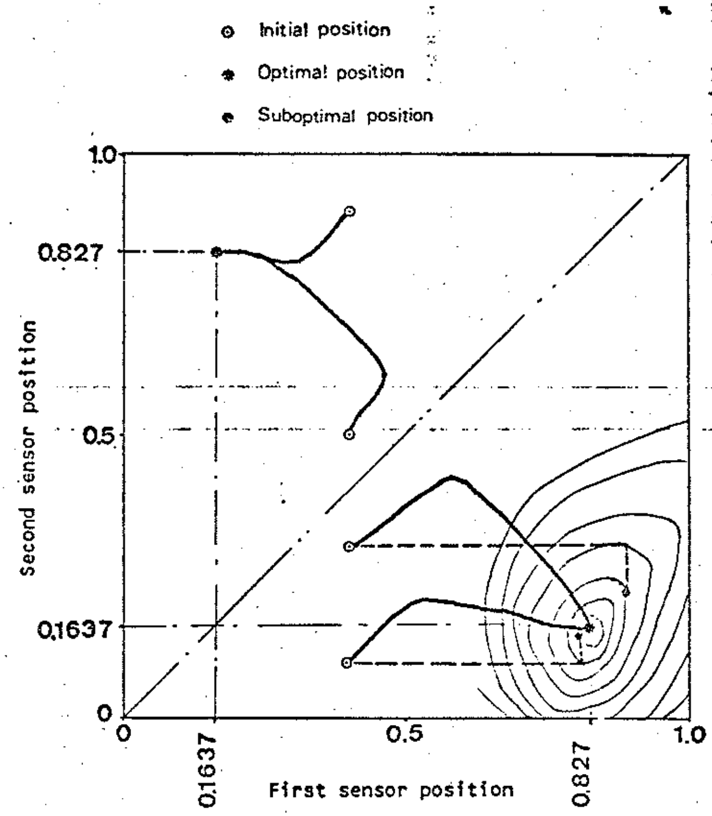

Fig.5 Convergence of the algorithm for two sensors allocation.