A PARAMETER ESTIMATION ALGORITHM FOR CONTINUOUS TIME STOCHASTIC ADAPTIVE CONTROL

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ABSTRACT

We present an adaptive parameter estimator for continuous time stochastic systems which results in globally stable adaptive control algorithms. Our analysis method is inspired by Goodwin and Mayne [1] in that properties of the parameter estimator are established that hold irrespective of the control law. Compared with comparable discrete-time schemes, the success of our algorithm hinges crucially on a modification of the parameter update law which ensures that the normalised regression vector remains uniformly bounded.

I. INTRODUCTION

A number of global stability results have been established for discrete stochastic adaptive control algorithms in the case of known model structure [2], [3]. However, a realistic continuous-time result has been elusive because of the difficulty of handling continuous-time white noise properly. Preliminary results have had to rely upon data dependent assumptions on the normalised regression vectors, that were unlikely to be true [4].

Here we present a globally stable continuous time stochastic adaptive control algorithm that does not require such assumptions. The main feature of our algorithm is a new normalisation technique which guarantees that the normalised regression vector $\mathbf{\Delta}_t$ is uniformly bounded. This allows us to establish properties of the parameter estimator which hold irrespective of the control law, in particular, the absence of finite escapes. The parameter estimator can be combined with a wide class of certainty equivalence control laws to establish almost sure global stability of the resultant algorithm.

Due to lack of space, we present the main ideas and results without proofs; the reader is referred to the full paper [5] for details and proofs.

II. THE MODEL

Consider the following single input single output state space innovations representation

$$
\begin{align*}
\dot{x}_t &= A x_t dt + B u_t dt + K \omega_t \\
\dot{v}_t &= C x_t dt + d \omega_t
\end{align*}
$$

(2.1)

where $\omega_t$ is a Wiener process with incremental covariance $\sigma^2 dt$. Notice that $v_t$ is the integral of the output. We denote by $\sigma_t$ the increasing $\sigma$-fields generated by $\omega_t$ and the unknown initial condition $x_0$.

We assume that $\omega_0$ bounded. Without loss of generality, we assume that the model (2.1) is in observer form:

$$
\begin{bmatrix}
a_{n-1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
a_0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
a_{n-1} \\
\vdots \\
a_0
\end{bmatrix}
\begin{bmatrix}
1 \\
\vdots \\
0
\end{bmatrix}
=
\begin{bmatrix}
b_{n-1} & \cdots & 0 \\
\vdots & \ddots & \vdots \\
b_0 & \cdots & 0
\end{bmatrix}
\begin{bmatrix}
b_{n-1} \\
\vdots \\
b_0
\end{bmatrix}
$$

(2.2)

The parameters $a_i$, $b_i$ and $k_i$ are assumed unknown but constant.

With proper care taken for the initial conditions, the model (2.1)-(2.2) can be rewritten in linear regression form as follows (see [5] for details):

$$
\mathbf{y}_t = \mathbf{t}_t^\top \mathbf{e} dt
$$

(2.3)

Here

$$
\mathbf{y}_t = \begin{bmatrix}
G \\
K \mathbf{G}
\end{bmatrix}
\mathbf{e}_t
$$

with $G = \begin{bmatrix}
0 \\
\vdots \\
0
\end{bmatrix}$

(2.4)

where $g_i \triangleq c_i - a_i$, $i = 0, \ldots, n-1$, and the $e_i$ are arbitrary subject to $E(\mathbf{e}_n) \triangleq \rho \mathbf{e}_{n-1} + \mathbf{e}_{n-2} + \cdots + \mathbf{e}_0$ having all its zeros in the open left half plane.

Using the operator notation $\rho \triangleq d \mathbf{G}/dt$, the regressor $\mathbf{r}_t$ is obtained as the solution of the following equations

$$
\mathbf{r}_t^\top = \frac{1}{E(\rho)} \left[ \rho^{-1} \mathbf{d} \mathbf{y}_t , \ldots , \mathbf{d} \mathbf{y}_t , \rho^{n-1} \mathbf{u} , \ldots , \mathbf{u} ; \rho^{n-1} \mathbf{d} \omega_t , \ldots , \mathbf{d} \omega_t \right]
$$

(2.5)

The equations (2.5) are, in fact, to be regarded as integral equations with appropriate choice of initial conditions: see [5].

The model (2.3)-(2.5) is not in a form suited for parameter estimation, because the last $n$ terms of $\mathbf{r}_t$ depend on the unmeasured noise $\omega_t$. Thus, assuming that $\mathbf{e}_t$ is an $\mathbf{r}_t$-measurable estimate of $\mathbf{e}_t$ we define the predicted output $\mathbf{d} \tilde{y}_t$ via a pseudo-regressor $\mathbf{v}_t$ as follows:

$$
\mathbf{d} \tilde{y}_t = \mathbf{v}_t^\top \hat{\mathbf{e}}_t dt = \left[ (\mathbf{v}_t^T), (\mathbf{v}_t^T)^2, \ldots , (\mathbf{v}_t^T)^n \right] \hat{\mathbf{e}}_t dt
$$

(2.6)

where $\mathbf{v}_t^T = \mathbf{v}_t^T$, $\mathbf{v}_t$ is the solution of

$$
\mathbf{v}_t^T = \frac{1}{E(\rho)} \left[ \rho^{-1} \mathbf{d} \mathbf{e}_t , \ldots , \mathbf{d} \mathbf{e}_t \right], \mathbf{v}_0 = 0
$$

(2.7)

with

$$
\mathbf{d} \tilde{e}_t \triangleq \mathbf{d} \tilde{y}_t - \mathbf{d} \mathbf{y}_t
$$

(2.8)

We now denote

$$
\mathbf{e}_t \triangleq \mathbf{v}_t^T \hat{\mathbf{e}}_t
$$

(2.9)

and notice that $\mathbf{e}_t$ is the "deterministic part" of the prediction error:

$$
\mathbf{d} \mathbf{e}_t \triangleq \mathbf{d} \omega_t = \mathbf{e}_t dt
$$

(2.10)

It can then be shown [5] that

$$
D(\rho) \mathbf{e}_t = E(\rho) [ - \mathbf{v}_t^T \hat{\mathbf{e}}_t]
$$

(2.11)
where $D(\rho) \triangleq \det (\sigma \Theta + K\Sigma)$ and $\eta_t = \dot{\theta}_t - \dot{\theta}_t - 0$. The scenario has now been set up to introduce a particular parameter estimator to construct $\hat{\theta}_t$.

### III. THE ESTIMATION ALGORITHM

We introduce the following estimation algorithm

$$\dot{\hat{\theta}}_t = \frac{\psi_t}{r_t} \left( d\hat{\theta}_t - \dot{\hat{\theta}}_t \right)$$

where

$$r_t = \sup \{ r_t \} \text{ s.t. } r_t \geq 0 \quad (3.1)$$

and $c_0$ is any positive constant. Note that $\dot{\theta}_t$ is obtained by an Ito integral which makes sense locally (and even globally as is established below) since by construction $\frac{\psi_t}{r_t}$ is $\mathcal{F}_t$-measurable and continuous. Using (3.1), (2.6), (2.8) and (2.10), we see that the parameter error, $\dot{\theta}_t$, is the solution of the following stochastic differential equation

$$d\hat{\theta}_t = \frac{\psi_t}{r_t} \eta_t dt + \frac{\psi_t}{r_t} d\xi_t$$

The analysis of our estimation algorithms now revolves around establishing properties of the solutions of (3.3). Recall that $\eta_t$ and $\frac{\psi_t}{r_t}$ are related by (2.11). The first result implies that the solutions $\dot{\theta}_t$ almost surely have no finite escape time.

**Lemma 3.1**

The solutions of (3.3) satisfy

$$E \left[ \sup_{0 \leq s \leq T} |\eta_s|^{2p} \right] < K + 3E (|\eta_0|^{2p})$$

for any $p > 0$, where $K$ depends only on $p$, $T$, $c_0$, and the bound on the initial condition $|\eta_0|$. We now present the main properties of our parameter estimator.

**Theorem 3.1**

Consider the model (2.1) and the estimator (3.1)-(3.2) with $\psi_t$ defined by (2.5)-(2.7). Assume that the zeros of $\tilde{D}(\rho)$ and $E(\rho)$ are strictly in the left half plane, and that $\tilde{D}$ is input strictly passive.

Then:

1. $\limsup_{t \to \infty} \eta_t = 0$ a.s.
2. $\limsup_{t \to \infty} \int_0^t \frac{\eta_s^2}{r_s} ds < \infty$ a.s.
3. For all finite $T$,

$$\limsup_{t \to \infty} \sup_{0 \leq \tau \leq T} \eta_t = 0$$

A key feature of our estimator is our choice of $r_t$, which differs from the choice usually made in the discrete time case. Our choice guarantees that $\frac{\psi_t}{r_t} \leq 1 \forall t$, and this is crucial in establishing (3.6). In [4] $r_t$ is defined as $\int_0^t \frac{\psi_t}{r_t}^2 ds$ and, as a consequence, all the convergence proofs require the assumption that $\frac{\psi_t}{r_t}$ is almost surely bounded. This is rather unrealistic given that $\psi_t$ contains signals driven by a Wiener process.

Given an adaptive control law that guarantees that the signals in the closed loop satisfy a certain growth condition, a further boundedness property can be established.

**Lemma 3.2**

Consider the model (2.1) or (2.3), and assume that the parameter estimator is such that (3.6) holds, and that the controller is such that the following growth condition holds

$$\frac{r_t}{r_{t/2}} \leq C + \frac{K}{t} \int_0^t \psi_s^2 ds$$

where $C$ and $K$ are finite positive constants. Then

1. $\limsup_{t \to \infty} \frac{r_t}{r_{t/2}} \leq K < \infty$ a.s.
2. $\lim_{t \to \infty} \frac{1}{t} \int_0^t \psi_s^2 ds = 0$ a.s.

Recall that $\eta_t$ can be thought of as the "deterministic part" of the prediction error. Comparing with the continuous-time deterministic algorithms of [1], we have not proved the uniform boundedness of $\frac{\psi_t}{r_t}$ or of $0$ (if that quantity existed). However, the parameter estimator properties that have just been described are all that is needed to establish global boundedness of a class of indirect certainty equivalence adaptive control algorithms, such as pole-placement algorithms. We refer the reader to [5] for a presentation and proof of these results.

### CONCLUSIONS

An adaptive parameter estimator for continuous time stochastic systems has been described. A key feature of this estimator is the normalization which ensures that no finite escapees occur. We also establish properties of the estimator which hold for very general classes of input signals. These properties can be used to establish global stability of continuous time stochastic adaptive control algorithms.

### References


