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INTERPLAY BETWEEN CONTROL LAW SELECTION AND CLOSED LOOP ADAPTATION

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Abstract. Adaptive control methods combine parametrized controller computation with a closed loop system parameter identifier. This control law design is typically carried out as a parametric version of a standard nonadaptive design, which possesses desirable stability, performance and robustness properties. Similarly, the identifier usually is identical to that used for open loop parameter estimation. Our subject here is to investigate the interconnection between the selection of the control law and the consequent achieved closed loop input signal spectrum presented to the plant. According to the recent theory of Ljung, this signal spectrum plus the measurement noise process spectrum dictate the frequency weighting implicit in the identified parametric plant model. In turn, this identified plant model affects the controlled plant's stability through the robustness of the control law at this nominal value. We shall present results demonstrating the potential coincidence of the control law's robustness perturbations to the plant and the implied closed loop identification frequency weighting.

Keywords. Adaptive control; Closed-loop identification; Robustness.

1. Introduction

A standard class of indirect adaptive control systems is comprised of the interconnection of a linear control law design and a recursive parameter estimator operating jointly in closed loop. The control law relies upon the identified parameters and the identification experiment is performed with input signals determined by the controller. This coupling of linear control laws and linear system identification is rooted in linear systems ideology even though the effected controller is manifestly nonlinear in the large [1], [2]. For slow adaptation and local behaviour, however, it may be the case that the apparent nonlinearity is ameliorated and a linearized behaviour pertains – certainly this is the thinking behind such adaptive designs. The question then arises as to the local performance and stability of this linearized system and, further, of the nonlinear connection between the parameter identifier and the controller. These are the issues which we address here.

The study of the ability of a fixed linear control law to preserve closed loop stability and performance in the face of inexact plant knowledge is the province of *robustness.* Because of the inherent inaccuracy of the identified plant model admitted in an adaptive context, robustness of the underlying linear control law is a critical ingredient of any adaptive controller. That is, should the adaptation be stayed at a nominal parameter value, the resulting linear controller should be capable of stabilizing a neighbourhood of plants about this nominal value. Of central importance in the consideration of linear system robustness is to define the appropriate topology within which this neighbourhood is described and hence to delineate the classes of allowable perturbations to the plant for which the controller can still provide robust stability. In this fashion the controller robustness properties can be tied in to the demands placed upon the identifier, which determines the accuracy of the nominal plant model.

By appealing to the methods of Lehtomaki *et al.* [3] we establish conditions required of the control law in order that the closed loop be robust to multiplicative plant perturbations. Inverting this, we show that for a fixed control law for a nominal plant, closed loop stability can be guaranteed provided the relative frequency response error between the actual plant and its identified nominal model is kept suitably small.

Parallel to the above connection between the controller robustness and the modelling requirements is the property that linear system identification involves a model fitting which reflects a weighting determined by the plant input signal features and the measurement noise properties, see e.g. Ljung [4]. That is, the plant input signal in closed loop is affected by the control law which, in turn, affects the fitting of the identified model. Thus the selection of the control law design exerts an influence upon the identification stage just as the identifier determines both the controller nominal parameter setting and affects its ability to satisfy a robustness criterion.

The thesis explored in this paper is that this interplay jointly between the identification rule and the control law selection is at the centre of adaptive control local robustness and, further, that there exist certain choices where this interplay is mutually supportive inasmuch as the controller engenders an input to the plant that leads to the fitting of a model which meets the robustness requirements for the control law.

The structure of the paper is as follows. In Section 2 we consider the linear robustness methods of Lehtomaki *et al.*, a version of gain margin arguments, which help us to specify a mixed modelling/control law criterion for stability with modelling error. Section 3 is devoted to the development of closed loop identification formulation, based on the work of Ljung, which indicates the connection between the fitting of a model with Least Squares in closed loop and the applied linear feedback law. In Section 4 this is then extended to treat the case of nonlinear adaptive adjustment of the feedback law. Section 5 concludes and summarises the nature of the interplay. An example of adaptive LQG/LTR control is used to support the major thrusts.

2. Linear System Robustness

We suppose that our linear plant system is described by

$$y_t = P(z)u_t + v_t \tag{1}$$

where y_t, u_t and v_t are the plant output, input and measurement noise signals, respectively, and $P(z)$ is the *true* or *actual* linear plant transfer function. Based on input-output measurements, however, we presume that we have an identified plant model or *nominal* plant transfer function $\hat{P}(z)$. For the moment we treat the case where $\hat{P}(z)$ is fixed and consider characterizations of circumstances under which stabilizing feedback controllers designed for $\hat{P}(z)$ maintain stability also for the actual plant $P(z)$. (For ease of development we consider only single input single output plants here. The results extend directly but tediously to MIMO systems. Further, we assume no unstable pole-zero cancellations in the nominal and actual plants.)

Suppose that the linear controller is described by

$$u_t = -C(z)y_t + w_t \tag{2}$$

where w_t is an external reference signal and $C(z)$ is the controller transfer function. Note that in an adaptive context such a controller would be designed based upon the nominal plant $\hat{P}(z)$ and applied to the actual plant $P(z)$. Hence we denote the cascaded plant/controller pairs as,

$$G(z) = P(z)C(z) \tag{3}$$
$$\hat{G}(z) = \hat{P}(z)C(z). \tag{4}$$

Further, we write the multiplicative error between the actual and nominal plants as $L(z)$, i.e. we have,

$$P(z) = L(z)\hat{P}(z) \tag{5}$$

or,

$$G(z) = L(z)\hat{G}(z). \tag{6}$$

Thus $\hat{G}(z)[1+\hat{G}(z)]^{-1}$ is the designed closed loop and $G(z)[1+G(z)]^{-1}$ is the achieved closed loop. The robust stability question is: under what conditions does stability of the designed closed loop imply stability of the achieved closed loop.

We have the following discrete-time counterpart to the results of Lehtomaki *et al.* [3].

Theorem 1 *The achieved closed loop system, $G(z)[1+G(z)]^{-1}$, will be asymptotically stable if,*

- *$P(z)$ and $\hat{P}(z)$ have the same number of unstable poles,*
- *$P(z)$ and $\hat{P}(z)$ have the same unit circle poles,*
- *the designed closed loop $\hat{G}(z)[1+\hat{G}(z)]^{-1}$ is asymptotically stable,*
- *at each z on the unit circle,*

$$|L^{-1}(z) - 1| < \min\left[|1+\hat{G}(z)|, 1\right]. \tag{7}$$

The import of this theorem is that it delivers explicit conditions upon the model and upon the controller in order to achieve robust closed loop stability. Specifically, $\hat{G}(z)$ is directly determined by the controller design and $L(z)$ is related to the plant modelling.

Robustness can be enhanced by choosing the controller $C(z)$ so that the designed return difference, $1+\hat{G}(z) = 1+\hat{P}(z)C(z)$, is well bounded away from zero for z on the unit circle. This is nothing else than a gain margin condition on the controlled plant since it states that the Nyquist diagram of the cascaded plant/controller should remain well clear of the minus one point. The left side of (7) is interpretable as a bound the relative modelling error because, from (5),

$$L^{-1}(z) - 1 = \left[\hat{P}(z) - P(z)\right]P^{-1}(z), \tag{8}$$

or, returning to (7)

$$\left|\left[\hat{P}(z) - P(z)\right]P^{-1}(z)\right| < \min\left(|1+\hat{P}(z)C(z)|, 1\right) \tag{9}$$

Thus we can see that robustness enhancement of a specific controller design can be achieved by jointly maintaining the gain margin large and/or keeping the relative modelling error small. Reinterpretting, provision of good gain margin provides robustness to multplicative (or relative) modelling error.

We may now state a set of prioritised criteria for controller design in an adaptive control scheme.

- Stabilize the nominal closed loop system.
- Achieve adequate closed loop performance for the nominal plant with respect to tracking of references and disturbance rejection.
- Maximize robustness so that stability and performance are preserved in the face of modelling errors.

In operational terms these requirements are reflected in conditions on the chosen control law schema for the adaptive controller. The final point above indicates that, after satisfaction of the first two conditions, a control law which endeavours to keep the return difference frequency response large is prefered.

3. Linear Closed Loop Identification

We develop here a closed-loop, fixed controller version of the frequency domain modelling ideas of Ljung [4], where Least Squares Prediction Error modelling ideas are reassessed via Parseval's theorem. This linear development follows closely that of Gunnarsson [5] and subsequent to this presentation we move on to consider the extension to adaptation and the nonlinear situation where the controller parameters are adjusted according to current parameter estimates. By *closed loop identification* here we mean that, while the input and output signals of the actual plant are measureable, the input is not freely assigned from outside but, rather, consists of external reference signals and feedback of the output measurements. In developing this theory we shall presume that all externally prescribed signals, i.e. references and noises, are *quasistationary* so that notions of averages and spectra are appropriate. We shall also be concerned with off-line and on-line (i.e. recursive) adaptation.

Associated with our actual plant description (1)

$$y_t = P(z)u_t + v_t$$

we take a class of linear time-invariant models parametrized by the vector θ,

$$y_t = \hat{P}(z,\theta)u_t + \hat{H}(z,\theta)q_t \qquad (10)$$

where $\hat{P}$, $\hat{H}$ are time-invariant, finite-dimensional linear systems for fixed θ values and q_t is a zero mean white process. The parameter vector θ may also include initial condition data appropriate for unstable systems. Directly from such models we may define associated parametrized one-step-ahead predictors for y_t via

$$\hat{y}_{t|t-1}(\theta) = \hat{H}^{-1}(z,\theta)\hat{P}(z,\theta)u_t + [1 - \hat{H}^{-1}(z,\theta)]y_t. \qquad (11)$$

Using standard techniques and parametrizations these predictions may also be written in pseudoregression form,

$$\hat{y}_{t|t-1} = \phi_t^T(\theta)\theta \qquad (12)$$

where $\{\phi_t(\theta)\}$ is the sequence of pseudoregressors. From the predictions one defines a one-step-ahead prediction error

$$\begin{aligned} \epsilon_t(\theta) &= y_t - \hat{y}_{t|t-1}(\theta) & (13)\\ &= \hat{H}(z,\theta)\left[[P(z) - \hat{P}(z)]u_t + v_t\right] & (14) \end{aligned}$$

and a filtered prediction error

$$\epsilon_t^f(\theta,\eta) = D(z,\eta)\epsilon_t(\theta) \qquad (15)$$

where $D(z,\eta)$ is a finite-dimensional, time-invariant, linear filter parametrized by the vector η.

On the basis of these prediction errors one may propose a model fitting criterion for the plant, such as the Least Squares criterion,

$$V_N(\theta,\eta) = \frac{1}{N}\sum_{t=1}^{N}[\epsilon_t^f(\theta,\eta)]^2 \qquad (16)$$

and define a best parameter estimate at time N,

$$\hat{\theta}_N = \arg\min_{\theta\in\Theta_D} V_N(\theta,\eta) \qquad (17)$$

where Θ_D is a domain of feasible θ values. Assuming closed loop stability, letting $N \to \infty$ and applying Parseval's theorem to (17) we may write the minimizing θ value as

$$\begin{aligned} \theta^* &= \arg\min_{\theta\in\Theta_D}\int_{-\pi}^{\pi}\left[|P(e^{j\omega}) - \hat{P}(e^{j\omega},\theta)|^2\Phi_u(\omega) + \Phi_v(\omega)\right] \\ &\quad\times\frac{|D(e^{j\omega},\eta)|^2}{|H(e^{j\omega},\theta)|^2}\,d\omega. & (18) \end{aligned}$$

Here Φ_u and Φ_v are the power spectral densities of the plant input and noise processes respectively.

In this formulation one sees directly the frequency weighting effects of the plant input spectrum, the noise spectrum, the prediction error filters D and the noise model H. For controlled systems we see that Φ_u will itself be a θ-dependent quantity, increasing the complexity of (18). Before attempting to deal with this case, however, we consider firstly the effect of a fixed linear control law

$$u_t = C_1(z,\rho)r_t - C_2(z,\rho)y_t \qquad (19)$$

with r_t an external reference signal and ρ a vector parametrizing the controller. In terms only of external signals, this control law is expressible as

$$u_t = W_1(z,\rho)r_t + W_2(z,\rho)v_t \qquad (20)$$

with $W_1 = C_1(1+C_2P)^{-1}$ and $W_2 = -C_2(1+C_2P)^{-1}$. The minimization (18) is altered to

$$\begin{aligned} \theta^* &= \arg\min_{\theta\in\Theta_D}\int_{-\pi}^{\pi}[|\Delta P(e^{j\omega},\theta)|^2|W_1(e^{j\omega},\rho)|^2\Phi_r(\omega) + & (21)\\ &\quad(|\Delta P(e^{j\omega},\theta)|^2|W_2(e^{j\omega},\rho)|^2 + 1)\Phi_v(\omega)]\frac{|D(e^{j\omega},\eta)|^2}{|H(e^{j\omega},\theta)|^2}\,d\omega \end{aligned}$$

where we write $P(z) - \hat{P}(z,\theta) = \Delta P(z,\theta)$. This is the closed frequency response formula of Gunnarsson [5], which may be abbreviated as

$$\theta^* = \theta^*(\rho,\eta). \qquad (22)$$

As stated above the explicit criterion minimization is an off-line procedure. That is, the selection of $\hat{\theta}_N$ is performed as a global minimization of (17) using the entire set of data $\{y_t, u_t; t \in 1,\ldots,N\}$ as a block. Using the pseudoregression form (12) we may also construct recursive (or *on-line*) estimators which update $\hat{\theta}_{N-1}$ to yield $\hat{\theta}_N$ given y_N and u_N, based upon gradient or Newton-Raphson methods. Ljung [6] and others have considered general circumstances for the convergence of on-line estimates to their off-line counterparts in deterministic and stochastic environments. We shall make contact with some of these ideas in the next section.

The formal solution (22) indicates that in linear closed loop identification the control parameters, ρ, and the prediction error filtering parameters, η, both play an important rôle in determining the model fit. If the controller in an adaptive context were designed according to some fixed schema such as pole-positioning or LQG then ρ would represent, for example, the specification of the LQG or the tracking objective. Returning to the robust stability objective (9) we can divine the complexity of the controller selection issue in adaptive control because $C(z)$ influences both sides of the expression (9). We have also made explicit the adaptive control design parameters available to the designer to attempt to achieve robust stabilityt of the linear system. In the next section we shall consier how the theory needs to be amended to remain applicable to (nonlinear) adaptive control.

4. Adaptive Closed Loop Identification

4.1 Existence of Stationary Points

The rewriting of the minimization (21) as (22) portrays the implicit involvement of the control objective and the error filtering in the determination of the off-line best θ value for the prediction error criterion (17) with fixed linear systems. In indirect adaptive control one maintains the prediction error identifier for the plant parameter but includes the simultaneous controller design and error computation based on this θ estimate. That is,

$$\rho_t = \rho(\hat{\theta}_t) \qquad \eta_t = \eta(\hat{\theta}_t).$$

In searching for potential convergence points of indirect adaptive control it is necessary to contemplate the solution of

$$\theta^* = \arg\min_{\theta\in\Theta_D} V\left(\theta, \rho(\theta), \eta(\theta)\right). \qquad (23)$$

This is a highly nonlinear problem and it is not apparent that it possesses a unique (or indeed any) solution, nor that such a solution θ^* yields a stabilizing closed loop when coupled with the control law $\rho(\theta^*)$. It is our aim to prescribe conditions under which stabilizing solutions will exist and be appropriate limit points for on-line adaptive solutions. Our methods here will be generalizations of those of Riedle and Kokotovic [7], [8] and [9] to the case of indirect adaptive control.

We begin by defining an *achieved control cost*, $J(\theta)$, associated with a θ value as follows. Compute the control cost (typically an LQ or tracking objective) associated with the closed loop operating with the fixed linear controller parametrized by $\rho(\theta)$. This cost is determined with the *actual* plant in the loop but with controller based on the model

$\hat{P}(z,\theta)$, $\hat{H}(z,\theta)$. In terms of control performance we may then define a best parameter value

$$\theta^{**} = \arg\min_{\theta\in\Theta} J(\theta). \tag{24}$$

In any reasonable problem $\rho(\theta^{**})$ must define a stabilizing controller since performance $J(\theta)$ is measured on the actual plant.

In the case of exact modelling, where the actual plant is included as part of the model set, and subject to persistently exciting signals in the closed loop (which we assume), the minimization of $J(\theta)$ and the minimization of $V(\theta)$ will yield the same model, i.e. $\theta^* = \theta^{**}$. With plant/model set mismatch, however, this need not be the case. One would hope though that θ^*, the stationary point of the adaptation, should be close to θ^{**} and further that $\rho(\theta^*)$ should therefore be close to $\rho(\theta^{**})$ and hence stabilizing.

Since the existence of θ^{**} satisfying (24) is not at issue and we wish to investigate the possibility of a θ^* lying close to it we write the true system (1) as

$$y_t = \phi_t^T\theta^{**} + \zeta_t + v_t \tag{25}$$

where ϕ_t is a regressor of past inputs and outputs, v_t is as in (1) and ζ_t denotes the unmodelled dynamics. With this description of the plant relative to control performance we are very much in a similar framework to Riedle and Kokotovic [8] who treat direct adaptive control methods.

For the plant operating under adaptive control the signals ϕ_t and ζ_t in (25) will depend upon the θ estimate. We can define *tuned* signals $\phi_t(\theta^{**})$ and $\zeta_t(\theta^{**})$ whose properties will determine the local behaviour of the adaptive system. Denote

$$\begin{aligned} m &= E[|\phi_t(\theta^{**})|] & (26)\\ \lambda &= \lambda_{min} E\left[\phi_t(\theta^{**})\phi_t^T(\theta^{**})\right], & (27)\end{aligned}$$

and introduce the following assumptions on the plant.

Assumption 1 *Let $B_r(\theta^{**}) = \{\theta : |\theta - \theta^{**}| \le r\}$ be a closed hypersphere centered on θ^{**} and with radius r such that*

1. *for all $\theta \in B_r(\theta^{**})$ the closed loop system is stable,*
2. *there exist positive constants α, β, δ, k such that for all $\theta \in B_r(\theta^{**})$*

$$E|\zeta_t(\theta^{**})| \le \alpha(E|\phi_t(\theta^{**})| + k), \qquad E|\frac{\partial\phi_t(\theta)}{\partial\theta}| < \beta,$$

$$E|\frac{\partial\zeta_t(\theta)}{\partial\theta}| < \delta \tag{28}$$

with α, β, and δ small enough that

$$r^2\lambda > 2r[\beta m r^2 + \delta m r + \beta\delta r^2 + \alpha r(m+k)(m+\beta r+\delta)] \tag{29}$$

Assumption 1 has the following implications

- Note that r refers to the radius of the ball in parameter space, λ to the signal excitation at the tuned value θ^{**}, m to the tuned regressor magnitude, α and k to the level of unmodelling in the plant description (25), β and δ to the smoothness properties of the situation;
- the first part assumes that around the "best" system θ^{**} there is a neighbourhood of stabilizing models. This is a very reasonable assumption if the closed loop regulator is computed using a robust design methodology;
- the second part is a constraint on keeping the unmodelled dynamics small enough, as well as an assumption of smoothness of both the regressor and the unmodelled dynamics of the closed loop system with respect to the model parameter θ; we notice that satisfaction of the constraint (29) also hinges on the amount of persistence of excitation of the regressor vector $\phi_t(\theta^{**})$ through the parameter λ;
- the two parts of the assumption really determine the existence of B_r and then define the radius of the hypersphere through the combined constraints of closed loop stability, small unmodelled dynamics and smoothness.

We now make the connection between (29) and the existence of a potential limit point in $B_r(\theta^{**})$. The proof has been omitted.

Lemma 1 *Let θ^{**} be defined as in (24) and let $B_r(\theta^{**})$ satisfy Assumption 1. Then there exists a θ^*, an interior point of $B_r(\theta^{**})$, defined as follows*

$$\theta^* = \arg\min_{\theta\in B_r(\theta^{**})} V(\theta, \rho(\theta), \eta(\theta)). \tag{30}$$

The point of Lemma 1 is to show that, under conditions of smoothness and limited unmodelled dynamics, there exists a closed hypersphere $B_r(\theta^{**})$ surrounding θ^{**}, the interior of which contains a stabilizing model θ^* that can be obtained as the solution of an off-line prediction error identification problem, with the search domain suitably restricted to that hypersphere. The model θ^{**} can then be seen as a mechanism for suitably defining the model θ^* as the solution of an off-line identification problem. Assumption 1 restricts the amount of allowable unmodelled dynamics. This causes the minimum of the prediction error criterion and the optimal control performance to be related. We should note that assumptions like (1) are very standard in the literature on robust indirect adaptive control: see e.g. [10].

4.2 Slow Adaptation and Convergence

Having defined θ^*, our analysis will proceed by demonstrating that, with a suitably restricted search domain and a sufficiently small adaptation gain, the solution of our recursive prediction error algorithm, used in an adaptive closed loop, will converge to a neighbourhood of stabilizing models around θ^*. Under suitable conditions similar to those of Assumption 1, the solution of the parameter-update equation can be shown by integral manifold arguments to converge, from a suitable region of initial conditions, to a limiting solution, which itself is close to θ_0. The application of integral manifold theory to the analysis of adaptive control systems is due to Riedle and Kokotovic [7], [8] and we shall refer to their work for proofs and details.

The full set of dynamical equations of the adaptive closed loop system can be written in compact form as follows:

$$\begin{aligned} \Xi_{t+1} &= A(\theta_t)\Xi_t + B(\theta_t)n_t, \qquad \Xi \in R^s, & (31)\\ \theta_{t+1} &= \theta_t + \gamma f_t(\theta_t, \Xi_t), \qquad \theta \in R^d, & (32)\end{aligned}$$

where Ξ includes the states of the plant model, of the regulator and of the filters, n_t denotes a vector made up of all the external signals (i.e. reference signals and noises), while the parameter update equation (32) is just another expression for the recursive least squares equation with a constant gain γ. Examples of adaptive control algorithms rewritten in this global form can be found in [8], [9] and elsewhere.

With θ^* defined as in the previous section, we now make the following assumption.

Assumption 2 *1. There exists a compact set Θ containing θ^* and constants $\lambda \in (0,1)$ and $K_1 \geq 1$ such that $\forall\theta \in \Theta$ and $\forall t \geq 0$*

$$|A(\theta)^t| \leq K_1\lambda^t \tag{33}$$

2. There exist constants c, c_1 and c_2 such that the frozen parameter response

$$\nu_t(\theta) = \sum_{j=0}^{\infty} A^j(\theta)B(\theta)n_{t-j-1} \tag{34}$$

and its sensitivity $\frac{\partial\nu_t}{\partial\theta}(\theta)$ satisfy

$$\begin{array}{c} |\nu_t(\theta)| \leq c, \quad |\frac{\partial\nu_t}{\partial\theta}(\theta)| \leq c_1 \\ |\frac{\partial\nu_t}{\partial\theta}(\theta) - \frac{\partial\nu_t}{\partial\theta}(\theta^*)| \leq c_2|\theta - \theta^*| \end{array} \tag{35}$$

for all $t \in Z$ and all $\theta, \theta^* \in \Theta$.

3. The function $f_t(\theta,\Xi)$ in (32) is bounded, Lipschitz in θ and Ξ uniformly with respect to t, $\theta \in \Theta$ and Ξ in compact sets.

The integral manifold theory of Riedle and Kokotovic is based on a time-scale separation between the dynamical equations for Ξ and for θ. The idea is that, if the gain γ is small enough, the solution of the parameter update law can be approximated by the solution of an "averaged" equation,

$$\bar{\theta}_{t+1} = \bar{\theta}_t + \gamma\bar{f}(\bar{\theta}_t), \tag{36}$$

where $\bar{f}$ is obtained by averaging f over t with $\bar{\theta}$ fixed. Asymptotically stable solutions of (36) can in turn be related to solutions of the ODE equation,

$$\frac{d\hat{\theta}}{dt} = \bar{f}(\hat{\theta}). \tag{37}$$

We then have the following important stability result, which we paraphrase in words, leaving out the precise values of the bounds: see [7], [8] for details and exact values of the bounds.

Theorem 2 *Suppose that θ^* is a local minimum,*

$$\theta^* = \arg \min_{\theta \in B_K(\theta^*)} V(\theta, \eta(\theta), \rho(\theta)). \tag{38}$$

of $V(\theta)$ with $B_K(\theta^*) \triangleq \left\{\theta \in R^d : |\theta - \theta^*| \leq K\right\}$ such that the filtered regression vector ϕ_t^f is persistently exciting $\forall\theta \in B(K,\theta)$. If $V(\theta^*)$ is small enough, then

1. the ODE (37) has an asymptotically stable equilibrium point θ^0 such that

$$|\theta^0 - \theta^*| \leq bV(\theta^*) \text{for some finite } b; \tag{39}$$

2. given $\chi > 0$, there exists a sufficiently small $\gamma^*(\chi)$ such that, for $\gamma \in (0,\gamma^*)$ the equation (32) possesses a bounded uniformly asymptotically stable solution $\tilde{\theta}_t(\gamma)$ which is close θ^0,

$$\lim_{\gamma\to 0} |\tilde{\theta}_t(\gamma) - \theta^0| = 0; \tag{40}$$

3. every solution $\theta_t(\gamma)$ of (32) with $\theta_0(\gamma) \in B_{K-\chi}(\theta^*)$, satisfies, for $\gamma \in (0,\gamma^*)$,

$$\theta_t(\gamma) \in B_K(\theta^*), \qquad \lim_{t\to\infty} |\theta_t(\gamma) - \tilde{\theta}_t(\gamma)| = 0. \tag{41}$$

The constant b in the first part of the theorem is proportional to the maximum value, over all the models in the hypersphere $B_K(\theta^*)$, of the average value of the Euclidean norm of the regressors, and is inversely proportional to the average amount of excitation in the filtered regressors $\psi_t^f(\theta)$.

The main conclusion to be drawn from this result is that the solution of the recursive parameter adaptation algorithm, implemented in an adaptive loop, will converge close to the solution θ^* of the off-line problem provided the following conditions hold.

1. The plant-model mismatch must be small enough; this is embodied in the condition that $V(\theta^*)$ must be small enough.
2. The filtered regressors must be persistently exciting.
3. The initial condition of the parameter update algorithm must be sufficiently close to the "optimal" value θ^*.
4. The models must be sufficiently smooth functions of θ around θ^*
5. The gain of the parameter update algorithm must be sufficently small.

Further, since under similar assumuptions θ^* is close to the performance criterion minimizing θ^{**}, we will have θ converging to the neighbourhood of θ^{**}.

5. The Interplay

Our presentation so far has focussed on two main themes; conditions for the robustness of linear control laws in terms of model accuracy, and conditions for local convergence of identification methods operating in adaptive closed loop. Assessing these issues at this point we see directly that (9) implicates model fit in the robust stability criterion and that Assumption 2 necessitates the choice of a low sensitivity control law in ensuring the viability of teh indirect parameter adaptation.

The natural question to pose at this juncture is whether these requirements for good adaptive control are potentially simultaneously satisfiable, or whether they represent a mutually confounding specification of desires. The answer is, of course, that the concurrent accommodation of both conditions is possible. Indeed, the thesis of deliberate selection of an insensitive control law or *detuning* in adaptive control is very familiar. Here, however, we shall indicate by an example that it is possible to propose a control law whose specific robustness requirements dovetail with its effect upon the closed loop identified model. This example is explored very much more fully in [11].

Operating under the conditions that the reference signal, r_t, dominates the measurement noise, v_t, we may approximate the closed loop frequency domain identification criterion (21) by

$$V(\theta) \;=\; \int_{-\pi}^{\pi} |\Delta P(e^{j\omega},\theta)|^2 |W_1(e^{j\omega},\rho)|^2 \Phi_r(\omega) \tag{42}$$

$$\times \frac{|D(e^{j\omega},\eta)|^2}{|H(e^{j\omega},\theta)|^2}\,d\omega.$$

Returning to the robustness criterion (9)

$$\frac{|\Delta P(e^{j\omega},\theta)|^2}{|P(e^{j\omega})|^2} < \min\left(|1+\hat{P}(e^{j\omega},\theta)C_2(e^{j\omega},\rho)|^2, 1\right), \quad (43)$$

we see that one desirable objective of the control design might be to attempt to achieve

$$|W_1|^2\Phi_r\frac{|D|^2}{|H|^2}|P|^2|1+\hat{P}C_2|^2 \approx 1. \quad (44)$$

In this fashion the (L^2) identification objective (42) and the (L^∞) robustness criterion (44) are made as compatible as is possible. The central difficulty in achieving this, of course, is that *a priori* $P(z)$ is not known nor indeed need a good approximation for $\hat{P}(z,\theta)$ be known. However, the selection of;

- white reference model during adaptation produces $\Phi_r = 1$,
- an almost minimum variance tracking control causes
  $$|W_1(e^{j\omega})| \approx |\hat{P}(e^{j\omega},\theta)|.$$
  This control law is singular LQG and, if implemented via Kalman Filter design based on $\hat{H}(z,\theta)$ followed by singular LQ control design, yields a controller of Loop Transfer Recovery type which, for a fixed α, keeps $|1+\hat{P}(e^{j\omega},\theta)C_2(e^{j\omega},\rho)| > \alpha$.
- $D(z,\eta) = H(z,\theta)$ or, if a useful *a priori* estimate of the final return difference is available, $D = H|1+\hat{P}C_2|^{-1}$.

The point made in [11] is that this prescription of an adaptive controller as the interconnection of such a control law and a least squares recursive parameter estimator is very close to Generalized Predictive Control of Clarke, Mohtadi and Tuffs [12]. This latter adaptive control law and its relatives have been the focus of many successful practical applications. The work of this paper helps to make better connection between the specification of these practical laws and more standard linear controller design problems.

We conclude our treatment of this problem by reiterating the salient points.

- In adaptive control there is a natural interplay between the parameter estimator local convergence properties and the robustness of the control law schema;
  - the identifier requires local smoothness of the controller behaviour.
  - the controller requires the robustness to cope with parameter variations due to the identifier.
- The selection of the control law impinges upon the robust stability conditions in two ways; it determines the magnitude of the return difference, $1+P(z)C(z)$, frequency response, through the effect on the closed loop input spectrum it affects the frequency weighting of the model fit to the plant.
- Subject to local smoothness properties of the closed loop behaviour and to the absolute model fitting capability, the on-line parameter identifier in indirect adaptive control can converge to a neighbourhood of the best value measured according to off-line nonadaptive control. These results hinge critically upon the persistence of excitation of the signals in the closed loop which, in turn, derives from the richness of the reference signal.
- An example of adaptive LQG control incorporating Least Squares parameter estimation, singular optimal control, Kalman Filtering and error filtering which demonstrated the feature of the robustness criterion and the implicit model fitting objective being coincident. That this controller is strongly related to emergent practically successful adaptive controllers but is more strongly theoretically supported, suggests to us that these interplays between control law selection and closed loop identification embody the connection between explanations of classes of applied adaptive control algorithms and generation of new procedures based on more classical control theories.

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