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Stabilizable by a stable and by an inverse stable but not by a stable and inverse stable

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Abstract

Our main result disproves conjectures on simultaneous stabilizability conditions by showing that, unlike the case of two plants, the existence of a simultaneous stabilizing controller for more than two plants is not guaranteed by the existence of a controller such that the closed loops have no *real* unstable poles. We include an example of a plant that is stabilizable by a stable controller and by a controller whose inverse is stable but not by a controller that is stable and has a stable inverse.

1 Introduction

This paper provides contributions to the *simultaneous stabilization problem*, which can be briefly stated as follows: under what conditions on k real rational transfer functions $p_i(s), i = 1, ..., k$, is it possible to find a single real rational controller $c(s)$ that simultaneously stabilizes these k plants. That is, we require that, with the chosen controller $c(s)$, the transfer functions $p_i c(1+p_i c)^{-1}, p_i(1+p_i c)^{-1}, c(1+p_i c)^{-1}$ and $(1+p_i c)^{-1}$ for $i = 1, ..., k$ have no poles in the extended right half plane. This problem has been formulated for some years now (see e.g. [5]) but has remained unsolved for $k \geq 3$.

In [5] the simultaneous stabilization of two plants was shown to be equivalent to the stabilization of a related single plant by a stable controller. The stabilizability of a single plant by a stable controller, known as the *strong stabilization problem*, was fully solved by Youla et al. [8]: a plant is stabilizable by a stable controller if and only if it has an even number of real unstable poles between each pair of real unstable zeros. Such plants are said to have the *parity interlacing property*. A most remarkable feature of this condition is that it involves only the *real* unstable poles and zeros of the plant.

The link between simultaneous stabilization of two plants and strong stabilization can be extended to the 3-plant case as follows. Modulo an avoidance condition that we shall specify later, the simultaneous stabilization of three plants is equivalent to the stabilization of a single plant by a stable controller whose inverse is also stable. Such a controller is called a *bistable* controller. The problem of finding a condition under which a rational plant p can be stabilized by a bistable controller can thus be seen as an intermediate step towards the solution of the simultaneous stabilization of three plants. It is easy to see that a *necessary condition* is that both p and p^{-1} must have the parity interlacing property. Such plants are said to have the *even interlacing property*. This even interlacing property ensures that there exists a bistable controller such that the closed loop transfer function has no *real* unstable poles and is therefore a *necessary condition* for stabilization of p by a bistable controller. In the same vein, [6], [7] and [1] give a condition on three plants p_1,p_2 and p_3 under which it is possible to find a single controller such that none of the closed loop transfer functions have *real* unstable poles.

In the two plant case, two plants p_1 and p_2 are simultaneously stabilizable if and only if there exists a single controller such that the closed loop transfer functions associated to p_1 and to p_2 have no *real* unstable poles. By analogy it was hoped that the simultaneous stabilization problem for three or more plants would also depend on the existence of a controller such that the closed loop transfer functions associated with these plants have no *real* unstable poles. Our main contribution is to disprove this with a counterexample in the case of three plants.

Modulo an avoidance condition, the simultaneous stabilization problem of three plants is equivalent to the stabilization of a single plant, $\bar{p}$, by a bistable controller. A necessary condition for this is that $\bar{p}$ has the even interlacing property and, if so, there exists a bistable controller such that the three feedback loops have no real unstable poles. Again, our contribution will be to show that this does not imply that there exists a bistable stabilizing controller.

Thus, our key contribution is a negative result showing that the simultaneous stabilizability question of more than two plants cannot be answered by just checking whether a controller exists such that the closed loop transfer functions have no *real* unstable poles.

This paper is a condensed version of [2] submitted to SIAM Journal of Control and Optimization. All proofs can be found in [2].

2 Definitions and classical results

First, we introduce some notations. We denote by $\mathbb{R}(s)$ the set of real rational functions. $\mathbb{C}_\infty$ is the extended complex plane, $\mathbb{C} \cup \{\infty\}$, and $\mathbb{R}_\infty$ is the extended real line, $\mathbb{R} \cup \{\infty\}$. $\mathbb{C}_{+\infty}$ is the positive real part of $\mathbb{C}_\infty$, while $\mathbb{R}_{+\infty}$ is the positive real part of $\mathbb{R}_\infty$. A real rational function $f(s) \in \mathbb{R}(s)$ is *stable* if it has no poles in $\mathbb{C}_{+\infty}$. S is the set of all stable rational functions.

S is a commutative ring. The invertible elements (or units) of S are the stable real rational functions whose inverse are stable, that is the real rational functions with no poles nor zeros in $\mathbb{C}_{+\infty}$; they are called *bistable rational functions.* Two elements of S are called *coprime* if they have no common zeros in $\mathbb{C}_{+\infty}$. Finally, the field of fractions of S is $\mathbb{R}(s)$. If $p(s) \in \mathbb{R}(s)$ then there exists $n_p(s), d_p(s) \in S$ and $x(s), y(s) \in S$ such that $p(s) = \frac{n_p(s)}{d_p(s)}$ and $n_p(s)x(s) + d_p(s)y(s) = 1$ (such a fractional factorization of $p(s)$ will be called a *coprime fractional factorization*).

We shall throughout this paper consider a controller to be within a unity feedback loop with the plant. We then adopt the following definitions.

- A controller $c(s) \in \mathbb{R}(s)$ *stabilizes* a plant $p(s) \in \mathbb{R}(s)$ if the four transfer functions $p(s)c(s)(1+p(s)c(s))^{-1}$, $c(s)(1+p(s)c(s))^{-1}$, $p(s)(1+p(s)c(s))^{-1}$ and $(1+p(s)c(s))^{-1}$ belong to S (i.e. they have no poles with nonnegative real part).

- A controller $c(s) \in \mathbb{R}(s)$ $R_{+\infty}$-*stabilizes* a plant if the same four transfer functions have no poles on the positive real axis.

- A plant $p(s) \in \mathbb{R}(s)$ is stabilizable if there exists a stabilizing controller. It is *strongly stabilizable* if it is stabilizable by a stable controller. It is *bistably stabilizable* if it is stabilizable by a bistable controller.

- A plant $p(s) \in \mathbb{R}(s)$ is $R_{+\infty}$-stabilizable if there exists a controller that $R_{+\infty}$-stabilizes the plant.

3 Stabilization of two plants and strong stabilization

We now recall a few results about the simultaneous stabilization of two plants and its equivalence with the stabilization of a single plant by a stable controller. We first need the following definitions.

Definition 1. *A rational function $p(s) \in \mathbb{R}(s)$ has the parity interlacing property if $p(s)$ has an even number of poles on $\mathbb{R}_{+\infty}$ between each pair of zeros on $\mathbb{R}_{+\infty}$. $p(s)$ has the even interlacing property if both $p(s)$ and $p^{-1}(s)$ have the parity interlacing property.*

We now state a collection of results on simultaneous stabilization of two plants and strong stabilization. Some of these results are well known, others are not.

Theorem 1.

1. The following three statements about a plant $\bar{p} \in \mathbb{R}(s)$ are equivalent:
   - $\bar{p}(s)$ is strongly stabilizable;
   - $\bar{p}(s)$ has the parity interlacing property;
   - $\bar{p}(s)$ is $\mathbb{R}_{+\infty}$-stabilizable by a stable controller.
2. Assume that $p_1(s), p_2(s) \in \mathbb{R}(s)$, let $\frac{n_1(s)}{d_1(s)}$ be any fractional factorization of $p_1(s)$ in S and let $x(s), y(s) \in S$ be solutions of the Bezout identity $n_1(s)x(s) + d_1(s)y(s) = 1$. The following three statements are equivalent:
   - $p_1(s)$ and $p_2(s)$ are simultaneously stabilizable;
   - $p_1(s)$ and $p_2(s)$ are simultaneously $\mathbb{R}_{+\infty}$-stabilizable;
   - $\bar{p}(s) \triangleq \frac{n_1(s)d_2(s)-n_2(s)d_1(s)}{n_2(s)x(s)+d_2(s)y(s)}$ is strongly stabilizable.
3. Suppose that $p_1(s)$ and $p_2(s) \in \mathbb{R}(s)$ have no common poles on $\mathbb{R}_{+\infty}$. Then they are simultaneously stabilizable if and only if $\bar{p}(s) \triangleq p_1(s) - p_2(s)$ is strongly stabilizable.

Proof: The proofs can be found in [8], [4] and [2]. ■

4 Stabilization of three plants and bistable stabilization

We now investigate the case of three plants. We first establish the connection between simultaneous stabilization of three plants and stabilization of one related plant by a bistable controller.

Theorem 2. *Let $p_i(s) \in \mathbb{R}(s)$, $i = 1,2,3$ and let $p_i(s) = \frac{n_i(s)}{d_i(s)}$, $i = 1,2,3$ be arbitrary coprime factorizations in S. Suppose that $p_1(s)$ avoids $p_2(s)$ in $\mathbb{C}_{+\infty}$, that is there is no $s_0 \in \mathbb{C}_{+\infty}$ for which $p_1(s_0) = p_2(s_0)$. Then $p_i(s)$, $i = 1,2,3$ are simultaneously stabilizable if and only if $\frac{n_3(s)d_1(s)-n_1(s)d_3(s)}{n_2(s)d_3(s)-d_2(s)n_3(s)}$ is stabilizable by a bistable controller.*

Proof. See [2].

It follows that, modulo an avoidance condition, the three plant problem can be reduced to one of finding a single bistable controller which stabilizes a single plant. The condition under which a single plant is $\mathbb{R}_{+\infty}$-stabilizable by a bistable controller is rather simple: it is the even interlacing condition.

Theorem 3. *Let $p(s) \in \mathbb{R}(s)$. There exists a bistable controller that $\mathbb{R}_{+\infty}$-stabilizes $p(s)$ if and only if $p(s)$ has the even interlacing property.*

5 Stabilization in the complex plane.

We have shown that the simultaneous stabilization of three plants, two of which do not intersect in $\mathbb{C}_{+\infty}$, is equivalent to the stabilization of a single related plant by a bistable controller, and we have found necessary and sufficient conditions for a single plant to be $\mathbb{R}_{+\infty}$-stabilizable by a bistable controller (the even interlacing property). Since two plants are simultaneously stabilizable if and only if they are simultaneously $\mathbb{R}_{+\infty}$-stabilizable, it was hoped that this property would flow on to the case $k \geq 3$. In this section we give counterexamples showing that $\mathbb{R}_{+\infty}$-stabilizability does not, in general, imply $\mathbb{C}_{+\infty}$-stabilizability.

Our first example covers the simultanous stabilization of three plants while the second deals with bistable stabilization. They both need advanced tools in analytic function theory for their proof: see [2].

Theorem 4. *The plants* $p_1(s) = 0, p_2(s) = \frac{s-1}{s+1}$ *and* $p_{3,\epsilon}(s) = \frac{(s-1)^2}{(1-\epsilon)s^2-2\epsilon s-(1+\epsilon)}$ *are simultaneously* $\mathbb{R}_{+\infty}$*-stabilizable for any* $\epsilon > 0$ *but are not simultaneously stabilizable for* $\epsilon < \frac{1}{16}$

.

We end this paper by providing an example of a plant which has the even interlacing property but which is not stabilizable by a bistable controller. For those familiar with the simultaneous stabilization problem, this is probably the most surprising contribution of our work.

Theorem 5. *There exists a positive integer* ϵ *such that the plant* $p_\epsilon(s) = \frac{s^2-1}{s^2-2s(\frac{1-\epsilon}{1+\epsilon})+1}$ *is not bistably stabilizable although it satisfies the even interlacing condition (i.e. it is stabilizable by a stable controller and by an inverse stable controller but not by a bistable controller).*

6 Conclusion

We have shown that, unlike the case of two plants, the existence of a simultaneous stabilizing controller for more than two plants cannot be guaranteed by the existence of a controller such that the closed loop transfer functions have no real unstable poles.

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