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COMPENSATOR REALIZATIONS THAT MINIMIZE THE CLOSED LOOP POLE SENSITIVITY

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Abstract. We investigate optimal realizations of systems and compensators in terms of minimizing a weighted pole sensitivity measure. Our main new result is to derive expressions for the pole sensitivity functions of a closed-loop system with respect to the parameters of the compensator realization and to give the necessary and sufficient condition that must be satisfied by all compensator realizations that minimize a weighted pole sensitivity measure of the closed-loop system. An algorithm is given to solve this optimization problem. Our weighted pole sensitivity minimization scheme is a contribution to the *Stability Robustness* theory: the optimal realizations have a maximal pole location robustness with respect to numerical errors.

Keywords: stability robustness, pole sensitivity, optimal realizations

1. Introduction

The *Stability Robustness* of a control system has been a focal point of attention in the last ten years. The most common point of view is to consider some measure of the 'distance' between the ideal (or true) system and some nominal system (or model). A designed closed loop system is said to be robustly stable if, by stabilizing the nominal model, it also stabilizes the true system, and indeed all systems that are contained within some specified set around the nominal system. In the case of state variable descriptions, the stability robustness is measured by the maximal perturbation distance defined with some matrix norm. The system becomes unstable when the perturbation is larger than this distance. Unfortunately, this maximum distance is hard to compute and only its upper bound is studied (see, for example, [KYF89], [KF90] and [LS88]).

In traditional robust stability analysis, the error between the ideal system and the nominal model is most often thought of as due to plant or plant/controller uncertainty. Here, we consider a special case where the plant is perfectly known, and where this error is due to the Finite Word Length (FWL) implementation of the controller. Thus, the uncertainty is entirely located in the controller. This can cause the closed-loop system to become unstable.

The stability of a system is determined by its pole pattern. When the system parameters are perturbed (due to FWL errors, say) the location of each individual pole is moved. The deviation of a pole due to the perturbation is approximately proportional to the sensitivity of this pole with respect to the errors in the system parameters. Therefore, the sensitivity of the poles that are close to the instability boundary (the unit circle for discrete time systems) is a relevant measure of the stability robustness. This is our main motivation for introducing a weighted pole sensitivity measure, and for studying its dependence vis-à-vis the realization of the compensator. But we shall do more: we shall show that one can actually compute compensator realizations that minimize a weighted sensitivity measure of the closed loop poles with respect to the parameters of the controller realization, thereby producing 'optimal compensator realizations'.

The outline of the paper is as follows. We first present some results in the *eigenvalue sensitivity* problem in Section 2. We then investigate the optimal compensator design problem in terms of minimizing the closed-loop pole sensitivity in Section 3. Section 4 is devoted to deriving the pole sensitivity functions of the closed loop system with respect to the parameters of a compensator realization. In Section 5, a weighted pole sensitivity measure is defined and an algorithm is given for computing the optimal compensator realizations that minimize this measure. Finally, some concluding remarks are given in Section 6.

2. Eigenvalue Sensitivity Problem

The computation of *eigenvalue sensitivity functions* is well known and can be found in e.g. [SS90], [SW84] and [GL93]. The following theorem presents the basic results.

Theorem 1 : *Let $M \in \Re^{m\times m}$ have only simple eigenvalues $\{\lambda_k\} = \lambda(M)$, x_k be a right eigenvalue of M with x_k corresponding to λ_k. Denote $X \triangleq (x_1 \ x_2 \ \dots \ x_m)$ and $Y = (y_1 \ y_2 \ \dots \ y_m) \triangleq X^{-H}$. Then*

$$(\frac{\partial \lambda_k}{\partial M})^T = x_k y_k^H, \quad \forall k \tag{1}$$

where y_k is called the reciprocal left eigenvector corresponding to x_k and 'H' denotes the transpose and conjugate operation.

Proof: The result can be proved in several ways. A self-contained derivation can be found in [GL93]. ■

Comment:

1. In the above theorem, it is assumed that M has only simple eigenvalues. If λ_k is a repeated eigenvalue of M, there exists at least an element of M, say $M(i,j)$, such that $\frac{\partial \lambda_k}{\partial M(i,j)}$ is *infinite* [GL93]. In the sequel, it is assumed that the system has only simple poles.

2. Let $m = 2n$, let x_k and y_k be partitioned as $x_k = (x_k^T(1) \ \ x_k^T(2))^T$ and $y_k = (y_k^T(1) \ \ y_k^T(2))^T$ with $x_k(i), y_k(i) \in \mathcal{C}^n, i = 1, 2, \forall k$, and let M be partitioned accordingly:
$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}, \tag{2}$$
with $M_{ij} \in \Re^{n\times n}$. It then follows from (1) that
$$(\frac{\partial \lambda_k}{\partial M_{ji}})^T = x_k(i) y_k^H(j). \tag{3}$$
This extension will be shown useful in the pole sensitivity minimization problem for a closed loop system later.

Consider a matrix M^0 that has a complete set of independent eigenvectors and any nonsingular matrix T, and denote $M = T^{-1}M^0T$. Clearly, $\lambda(M) = \lambda(M^0)$. Let x_k^0 be a right eigenvector of M^0 corresponding to the eigenvalue λ_k and let y_k^0 be its reciprocal left eigenvector, i.e. $y_k^{0H} x_k^0 = 1$. The corresponding eigenvectors of M for the same eigenvalue λ_k are:

$$x_k = T^{-1}x_k^0, \quad y_k = T^T y_k^0. \tag{4}$$

It then follows from (1) that the eigenvalue sensitivity in the new coordinate system is given by

$$(\frac{\partial \lambda_k}{\partial M})^T = T^{-1} x_k^0 y_k^{0H} T, \quad \forall k \tag{5}$$

which shows that similar matrices typically have different eigenvalue sensitivities. In [LG93], a series of results has been given for finding optimal transformations T to minimize some pole sensitivity measures. In the remainder of this paper, we extend those 'open loop' results to the optimal compensator realization problem, where the objective is to find a realization of a prescribed compensator such that the closed loop system has a minimal pole sensitivity with respect to perturbations in the parameters of the compensator realization.

3. The Ideal Pole Assignment Compensator

Consider that the plant to be controlled is described by a linear time-invariant proper discrete time state-space model:

$$\begin{aligned} x(t+1) &= A_0 x(t) + B_0 u(t) \\ y(t) &= C_0 x(t) \end{aligned} \tag{6}$$

with $A_0 \in \Re^{n\times n}, B_0 \in \Re^n$ and $C_0^T \in \Re^n$. We assume that this system is minimal, that is the pair $[A_0, B_0]$ is completely reachable and the pair $[C_0, A_0]$ is completely observable. The transfer function can be expressed in terms of the state space matrices as

$$H_0(z) = C_0(zI - A_0)^{-1}B_0. \tag{7}$$

When the state variables $x(t)$ are measurable, the poles can be assigned at any desired set of locations by a linear state feedback control law:

$$u(t) = -K_0 x(t) + r(t). \tag{8}$$

Here $r(t)$ is an external reference signal, and $K_0 \in \Re^n$ is the vector of feedback gains determined such that $\{\lambda_i(A_0 - B_0K_0)\}$ gives the set of desired closed loop poles (see e.g. [Kai80]). Since usually only the output process $y(t)$ is available, the state has to be estimated by a state observer. This is a dynamical system that generates $\hat{x}(t)$, an estimate of $x(t)$. The control law is then obtained by the combination of an observer and a state-estimate feedback controller:

$$\begin{aligned} \hat{x}(t+1) &= A_0\hat{x}(t) + B_0u(t) + L_0[y(t) - \hat{y}(t)] \\ \hat{y}(t) &= C_0\hat{x}(t) \\ u(t) &= -K_0\hat{x}(t) + r(t). \end{aligned} \tag{9}$$

The observer-controller (9) can be rewritten as a *two-input one-output* dynamical system:

$$\begin{aligned} \hat{x}(t+1) &= \Phi_0\hat{x}(t) + B_0r(t) + L_0y(t) \\ u(t) &= -K_0\hat{x}(t) + r(t) \end{aligned} \tag{10}$$

where $\Phi_0 \triangleq A_0 - B_0K_0 - L_0C_0$. The corresponding input/output relation is

$$\begin{aligned} u(t) &= [1 - K_0(zI - \Phi_0)^{-1}B_0]r(t) \\ &\quad -K_0(zI - \Phi_0)^{-1}L_0y(t) \\ &\triangleq C_1(z)r(t) - C_2(z)y(t) \\ &\triangleq C_0(z)(r(t) \ \ y(t))^T. \end{aligned} \tag{11}$$

In the sequel, we study the compensator with the parametrization (A_0, B_0, C_0, K_0, L_0) in (9). The input-output relation (11) admits an infinite number of realizations which, in infinite precision, are all equivalent. They are all related by similarity transformations; these transform (A_0,B_0,C_0,K_0,L_0) into a set $\{(T^{-1}A_0T, T^{-1}B_0, C_0T, K_0T, T^{-1}L_0)\}$ of similar compensator realizations. For any one of these realizations, the ideal (i.e. infinite precision) compensator $C_0(z)$ and hence the closed-loop transfer function $H_c^0(z)$ from $r(t)$ to $y(t)$ are the same:

$$\begin{aligned} H_c^0(z) &= \frac{C_1(z)H_0(z)}{1+C_2(z)H_0(z)} \\ &= C_0[zI-(A_0-B_0K_0)]^{-1}B_0. \quad (12) \end{aligned}$$

All these similar compensator realizations form what we call the compensator realization set S_c.

In practice, the compensator is implemented with Finite Word Length, that is a perturbed version, $(A,B,C,K,L) \neq (A_0,B_0,C_0,K_0,L_0)$, is actually implemented. As a consequence of this precision limit in the actual implementation of the compensator, we have the following undesired results:

1. the poles of the actual closed-loop transfer function $H_c(z)$, obtained by replacing the ideal compensator (A_0,B_0,C_0,K_0, L_0) by its perturbed version (A,B,C,K,L), differ from the desired prespecified poles.

2. the actual observer dynamics cannot be cancelled and the actual closed loop system is therefore of order $2n$ instead of n.

These undesired results can lead to a serious performance degradation of the closed-loop system, which could even become unstable. We will show that the sensitivity of the poles of the closed loop system with respect to parameter errors in the compensator realization depends on the particular realization. Our intuition tells us that *a compensator realization that makes the closed loop system have a low pole sensitivity will better resist perturbations in this realization and should thus yield a better stability robustness performance for the closed- loop system.* This is indeed the case.

4. Pole Sensitivity Analysis for the Closed Loop System

We call (A_0, B_0, C_0) the exact (infinite precision) implementation of $H_0(z)$ in that coordinate space (see (6)), (A_0, B_0, C_0, K_0, L_0) the infinite precision desired compensator (9) in that same coordinate system and (A,B,C,K,L) the corresponding actual implemented realization of this compensator. The state equations of the closed loop system are then

$$\begin{aligned} \begin{pmatrix} x(t+1) \\ \hat{x}(t+1) \end{pmatrix} &= \begin{pmatrix} A_0 & -B_0K \\ LC_0 & A-BK-LC \end{pmatrix} \times \\ & \quad \begin{pmatrix} x(t) \\ \hat{x}(t) \end{pmatrix} + \begin{pmatrix} B_0 \\ B \end{pmatrix} r(t) \\ y(t) &= \begin{pmatrix} C_0 & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ \hat{x}(t) \end{pmatrix}. \quad (13) \end{aligned}$$

We denote:

$$\begin{aligned} \bar{A} &= \begin{pmatrix} A_0 & -B_0K \\ LC_0 & A-BK-LC \end{pmatrix}, \; \bar{B} = \begin{pmatrix} B_0 \\ B \end{pmatrix} \\ \bar{C} &= \begin{pmatrix} C_0 & 0 \end{pmatrix}. \quad (14) \end{aligned}$$

The closed loop transfer function with FWL compensator coefficients is $H_c(z) = \bar{C}(zI-\bar{A})^{-1}\bar{B}$. Since $(A,B,C,K,L) \neq (A_0,B_0,C_0,K_0,L_0)$, $H_c(z) \neq H_c^0(z)$. In particular, the observer dynamics do not cancel in the closed loop transfer function.

Of Theorem 1 with $M = \bar{A}$, one can derive the sensitivity functions of the poles of the closed loop system w.r.t. the compensator realization matrices:

$$\begin{aligned} \frac{\partial\lambda_k}{\partial A} &= \frac{\partial\lambda_k}{\partial M_{22}}, \; (\frac{\partial\lambda_k}{\partial B})^T = -K(\frac{\partial\lambda_k}{\partial M_{22}})^T \\ \frac{\partial\lambda_k}{\partial C^T} &= -(\frac{\partial\lambda_k}{\partial M_{22}})^T L \\ \frac{\partial\lambda_k}{\partial K^T} &= -(\frac{\partial\lambda_k}{\partial M_{12}})^T B_0 - (\frac{\partial\lambda_k}{\partial M_{22}})^T B \\ (\frac{\partial\lambda_k}{\partial L})^T &= C_0(\frac{\partial\lambda_k}{\partial M_{21}})^T - C(\frac{\partial\lambda_k}{\partial M_{22}})^T \quad (15) \end{aligned}$$

Now, let λ_k be a pole of the closed loop system, x_{pk} its right eigenvector and y_{pk} the corresponding left reciprocal eigenvector. It then follows from (3) that all pole sensitivity functions with respect to (A,B,C,L,K) can be expressed in terms of these eigenvectors as follows:

$$\begin{aligned} (\frac{\partial\lambda_k}{\partial A})^T &= x_{pk}(2)y_{pk}^H(2), \; (\frac{\partial\lambda_k}{\partial B})^T = \alpha_p(k)y_{pk}^H(2) \\ \frac{\partial\lambda_k}{\partial C^T} &= \beta_p(k)x_{pk}(2), \quad \frac{\partial\lambda_k}{\partial K^T} = \gamma_p(k)x_{pk}(2) \\ (\frac{\partial\lambda_k}{\partial L})^T &= \eta_p(k)y_{pk}^H(2) \quad (16) \end{aligned}$$

where the four scalar sets $\{\alpha_p(k)\}$, $\{\beta_p(k)\}$, $\{\gamma_p(k)\}$ and $\{\eta_p(k)\}$ are given by

$$\begin{aligned} \gamma_p(k) &= y_{pk}^H \begin{pmatrix} -B_0 \\ -B \end{pmatrix}, \eta_p(k) = x_{pk}^T \begin{pmatrix} C_0^T \\ -C^T \end{pmatrix} \\ \alpha_p(k) &= (0 \; -K)x_{pk}, \beta_p(k) = y_{pk}^H \begin{pmatrix} 0 \\ -L \end{pmatrix} (17) \end{aligned}$$

With $(A,B,C,K,L) = (A^0,B^0,C^0,K^0,L^0)$ one can get the pole sensitivity functions for the ideal compensator realization.

5. Optimal Compensator Structures

Let $(A^0, B^0, C^0, K^0, L^0)$ be some initial compensator realization in S_c and let $(A,\ B,\ C,\ K,\ L) \in S_c$ be obtained from $(A^0, B^0,\ C^0,\ K^0,\ L^0)$ through a similarity transformation T. Let $\{x_p^0(k)\}$ and $\{y_p^0(k)\}$ be the right and left eigenvectors corresponding to a pole λ_k of the closed loop system for this initial realization. Note that applying a similarity transformation T to the initial realization means to perform a similarity transformation $T_c = diag(I_n, T)$ to the closed loop realization, which yields the corresponding eigenvectors $x_{pk} = T_c^{-1}x_{pk}^0$ and $y_{pk} = T_c^T y_{pk}^0$, respectively, (see (??)). This means that $x_{pk}(1) = x_{pk}^0(1)$, $x_{pk}(2) = T^{-1}x_{pk}^0(2)$ and $y_{pk}(1) = y_{pk}^0(1)$, $y_{pk}(2) = T^T y_{pk}^0(2)$. The pole sensitivity functions in the new realization can be obtained directly from (??). Now, it is clear that *different compensator realizations yield different pole sensitivities for the closed loop system, and are therefore no longer equivalent in the finite precision case.*

We define the partial pole sensitivity measure as follows:

$$\Psi_p(k) = \sum_E ||\frac{\partial \lambda_k}{\partial E}||_F^2 \qquad (18)$$

where $||.||_F$ denotes the *Frobenius* norm and $E = \{A_0,\ B_0,\ C_0,\ K_0,\ L_0\}$. This is a sensitivity measure of an individual pole of the closed loop system. We define the overall pole sensitivity measure as the sum of each partial sensitivity measure $\Psi_p(k)$ weighted with some positive weight $w_k > 0$:

$$\Psi_p = \sum_{k=1}^{2n} w_k \Psi_p(k). \qquad (19)$$

Such measure was initially proposed and minimized for open loop systems in [GL93]. Since the stability margin of a discrete time control system depends on the pole that is nearest to the unit circle, a good FWL compensator design will make the sensitivity of that particular pole (or set of poles) as small as possible. This can be achieved by applying a large weighting factor to its sensitivity measure.

Denote

$$\begin{aligned} H_{xy}(k) &= y_{pk}^0(2)x_{pk}^{0H}(2), H_x(k) = x_{pk}^0(2)x_{pk}^{0H}(2) \\ H_y(k) &= y_{pk}^0(2)y_{pk}^{0H}(2) \qquad (20) \end{aligned}$$

and $P = TT^T$. With some manipulations it can be shown that

$$\begin{aligned} \Psi_p(k) &= tr\{PH_{xy}(k)P^{-1}H_{xy}^H(k)\} + tr\{[|\alpha_p(k)|^2 \\ &+ |\eta_p(k)|^2]PH_y(k)\} + tr\{[|\beta_p(k)|^2 \\ &+ |\gamma_p(k)|^2 P^{-1}H_x(k)\}. \qquad (21) \end{aligned}$$

Therefore, once an initial compensator realization is chosen the overall pole sensitivity measure is a function of $P = TT^T$. The optimal realization design in terms of pole sensitivity is equivalently to find those P and hence T that minimize the overall pole sensitivity measure $\Psi_p(P)$:

$$\min_{(A,B,C,K,L)\in S_C} \Psi_p \Longleftrightarrow \min_{P=TT^T>0} \Psi_p(P) \qquad (22)$$

where the solution T is the transformation from the arbitrary initial realization $(A^0,\ B^0,\ C^0,\ K^0,\ L^0) \in S_c$. The solution P and hence T of the optimization problem (??) can be solved and due to the space limit the details are not given. We refer to the full version of this paper for a detailed discussion.

6. Conclusions

In this paper, we have investigated the pole sensitivity problem for closed loop systems. We have argued that pole sensitivity is a relevant measure for stability robustness. Our main contribution has been to derive the sensitivity expressions for the poles of a closed loop system and to compute controller realizations that optimize a weighted sum of these closed loop pole sensitivities.

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