~~Submitted to~~ MTNS 93.

C 71

The simultaneous stabilizability question of three linear systems is rationally undecidable

V. Blondel (+) and M. Gevers (++)

October 28, 1992

Abstract

We show that the simultaneous stabilizability of three linear systems, that is the question of knowing whether three linear systems are simultaneously stabilizable, is rationally undecidable. By this we mean that it is not possible to find necessary and sufficient conditions for simultaneous stabilization of the three systems that involve only a combination of arithmetical operations (additions, substractions, multiplications and divisions), logical operations ('and' and 'or') and sign test operations (equal to, greater than, greater than or equal to,...) on the coefficients of the three systems.

Key words: simultaneous stabilization, decidability, decidable question.

(+) University of Oxford, Mathematical Institute, 24-29 St Giles', OX1 3LB Oxford, UK. Tel: +44-865-270509. Fax: +44-865-270515.
E-mail: blondel@maths.oxford.ac.uk
(++) Université Catholique de Louvain, Cesame, Place du Levant 2, B-1348 Louvain-La-Neuve, Belgium. Tel: +32-10-472590. Fax: +32-10-478667.
E-mail: gevers@auto.ucl.ac.be

Please address all correspondence to M. Gevers.

1 Introduction

When is it possible to find a single rational controller that simultaneously stabilizes three, or more, linear systems? At present nobody is capable of giving a comprehensive answer to this question and this paper is devoted to it [1].

We restrict our attention to single-input single-output linear, time invariant systems that are rational but not necessarily causal ($p_i(s) \in \mathbb{R}(s), i = 1, ..., k$) and we allow ourselves the use of a linear, time invariant rational controller. Our goal is to achieve closed loop internal stability with the controller. That is, we require that the four closed loop transfer functions $p_i(s)c(s)(1+p_i(s)c(s))^{-1}, p_i(s)(1+p_i(s)c(s))^{-1}, c(s)(1+p_i(s)c(s))^{-1}$ and $(1 + p_i(s)c(s))^{-1}$ associated to the k systems have no poles in the extended right half plane. A controller that satisfies that condition is said to be a *simultaneous stabilizing controller* of $p_i(s)$. The search for conditions on the $p_i(s)$ for the existence of a stabilizing controller was first adressed in [15] and [18].

The simultaneous stabilization question of two systems can be reformulated into one of strong stabilization –stabilization with a stable controller– of a single system [17]. The strong stabilization question was solved in 1974 by D. Youla *et al.* [21] and has an elegant solution: a system is stabilizable by a stable controller if and only if it has an even number of real unstable zeros between each pair of real unstable poles. B.D.O. Anderson [1] proved that this condition on real poles and zeros can be checked by performing only elementary arithmetic operations (additions, substractions, multiplications and divisions) on the coefficients of the system: the real poles and zeros do not have to be computed explicitly.

The picture is different for three systems. The simultaneous stabilization

[1]Consider for instance the three linear systems $p_1(s) = 0, p_2(s) = \frac{2}{17}\frac{s-1}{s+1}$ and $p_3(s) = \frac{(s-1)^2}{(9s-8)(s+1)}$. It is at this stage not known whether these three systems are simultaneously stabilizable or not. A bottle of good French champaign is offered by the authors to the first person who either gives the expression of a stabilizing controller or proves that no such controller exists.

question for three (or more) systems is recognized as one of the hard open problems in linear system theory and has attracted much attention this last decade. The presently available results are in the form of necessary conditions [8], [19], sufficient conditions [20], [5], [12] or untractable necessary and sufficient conditions [8], [7]. Despite all these efforts, *there exist at present no tractable necessary and sufficient conditions for testing the simultaneous stabilizability of three (or more) systems.*

In the central theorem of the paper – Theorem 5 – we show that contrary to the case of two systems the simultaneous stabilizability question for three linear systems is rationally undecidable. It is not possible to find a general criterion that involves only the coefficients of the three linear systems, arithmetical operations (additions, substractions, multiplications and divisions), logical operations ('and' and 'or') and sign test operations (equal to, greater than, greater than or equal to,...) and that is necessary and sufficient for simultaneous stabilizability of the three systems.

Section 2 sets out the notations. Our notion of a rationally decidable question is presented in Section 3. Section 4 introduces a result on the range of analytic functions that is used in our central Section 5. In that final section, we show in Theorem 5 that the simultaneous stabilization question of three systems is not rationally decidable.

This paper contains some of the material from the first author's PhD thesis: see [6].

2 Notations

$\mathbb{R}$ is the set of real numbers and $\mathbb{Q}$ is the set of rational numbers. $\mathbb{R}[s]$ is the set of real polynomials in the variable s. $\mathbb{R}(s)$ is the set of real rational functions. $\mathbb{Q}(\beta)$ is the set of rational functions in the variable β and with coefficients in $\mathbb{Q}$. $\mathbb{C}_\infty$ is the extended complex plane $\mathbb{C} \cup \{\infty\}$ topologized with the Riemann sphere topology and $\mathbb{R}_\infty$ is the extended real line, $\mathbb{R} \cup \{\infty\}$. D is the open unit disc $\{s \in \mathbb{C} : |s| < 1\}$, $\overline{D}$ is the closed unit disc $\{s \in \mathbb{C} : |s| \leq 1\}$ and $\mathbb{C}_{+\infty} = \{s \in \mathbb{C} : \Re(s) \geq 0\} \cup \{\infty\}$ is the extended closed right half plane.

Assume that Ω is a subset of $\mathbb{C}_\infty$. A real rational function $f(s) \in \mathbb{R}(s)$ is Ω-*stable* if it has no poles in Ω [2]. $S(\Omega)$ is the set of all Ω-stable rational functions. We use $U(\Omega)$ to denote the set of functions in $S(\Omega)$ whose inverse are in $S(\Omega)$ and we call such rational functions Ω-*bistable* rational functions. Finally, to shorten the notations, we denote $U = U(\mathbb{C}_{+\infty})$ and $S = S(\mathbb{C}_{+\infty})$.

3 Rational decidability and algebraic numbers

This section is in three parts. We first give our definition of rational decidability, then that of algebraic and transcendental numbers and finally we prove a result that links rational decidable questions and algebraic numbers.

3.1 Rational decidability

The test for polynomial stability is a good example of what we mean by a rationally decidable question. A polynomial is called stable iff all its zeros have negative real part. For fourth order polynomials the Routh Hurwitz test is:

the polynomial

$$p(s) = a_1 + a_2 s + a_3 s^2 + a_4 s^3 (a_4 \neq 0)$$

is stable if and only if the logical sentence

$$(((a_1 > 0) \wedge (a_2 > 0) \wedge (a_3 > 0) \wedge (a_4 > 0) \wedge (a_2 a_3 - a_1 a_4 > 0))$$

$$\vee((a_1 < 0) \wedge (a_2 < 0) \wedge (a_3 < 0) \wedge (a_4 < 0) \wedge (a_2 a_3 - a_1 a_4 < 0)))$$

is true. Here the symbols $\wedge$ and $\vee$ stand for the logical operations 'and' and 'or', respectively .

[2]We draw the reader's attention to the fact that this is pure convention. Other authors define Ω-stability in exactly the opposite way.

The binary question of deciding whether a third order polynomial is stable by using only its four coefficients is a typical example of what we mean by a *rationally decidable* question: it can be answered by using a finite number of elementary operations where elementary operations are defined as follows.

Definition 1 *An elementary operation is any one of*

1. *the four arithmetic operations: addition, substraction, multiplication and division. These are commonly referred to as rational operations,*
2. *the two logical operations: 'and' and 'or',*
3. *the five test operations:* $=, >, <, \geq$ *or* $\leq$.

We say that the polynomial stability question is rationally decidable because it is rationally decidable for each fixed polynomial degree n.

The abstract notion of rational decidability is a formalization of this idea.

Definition 2 *A binary question Q associated to an n-uple $(a_1, ..., a_n) \in \mathbb{R}^n$ is rationally decidable if and only if there exists a logical sentence L of finite length that involves only elementary operations on the entries a_i of the n-uple and such that L is true if and only if Q is.*

With this definition the following questions are rationally decidable: the stability of a polynomial, the positive definiteness of a matrix, the coprimeness of two polynomials or the simultaneous stabilizability of two linear systems. On the other hand, we show in Section 5 that the question: 'when are three systems simultaneously stabilizable?' is *not* rationally decidable.

3.2 Algebraic numbers

Algebraic numbers are numbers that are roots of polynomials whose coefficients are integers (see for example [4]).

Definition 3 *A real number is algebraic if and only it is the root of a polynomial that has integer (or rational) coefficients. A real number that is not algebraic is transcendental.*

For example, $-1, \sqrt{2}, i=\sqrt{-1}$ and $\frac{\sqrt{\sqrt{7}+31}}{\sqrt{13}-5}$ are algebraic numbers whereas π, e and $\Gamma(\frac{1}{4})$ are transcendental. It is in general not true that the ratio of two transcendental numbers is a transcendental number. For our simultaneous stabilization purposes we need the next non-trivial result. The proof of this theorem is independent of the rest of the paper.

Theorem 1 *The real number* $\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}$ *is transcendental.*

Proof
Our proof is based on a result contained in the third section of the last chapter of 'Transcendental number theory' (A. Baker, p. 158, [4]). This result states: "The transcendence degree of the field L generated by $\omega_1 = \frac{\Gamma^2(\frac{1}{4})}{\sqrt{8\pi}}$, $\omega_2 = i\omega_1$, $\eta_1 = \frac{\pi}{\omega_1}$, and $\eta_2 = -i\eta_1$ over the rationals $\mathbb{Q}$ is at least 2." Since $\omega_1^2 + \omega_2^2 = 0$ and $\eta_1^2+\eta_2^2 = 0$ this means that ω_1 and η_1 are transcendental and algebraically independent. But then $\frac{\eta_1}{2\omega_1^3} = \frac{4\pi^2}{\Gamma^4(\frac{1}{4})}$ is transcendental and so the theorem is proved. ■

3.3 Rational decidability and algebraic numbers

In this section we establish a link between rational decidability and algebraic numbers. We first illustrate our point with the example of polynomial stability and then generalize the concept in an abstract setting.

Assume that $\beta \in \mathbb{R}$. By the previous section we know that the polynomial

$$p(s) = 1 + \beta s + \beta s^2 + 2s^3$$

is stable if and only if the logical sentence

$$(((1>0) \wedge (\beta>0) \wedge (\beta>0) \wedge (2>0) \wedge (\beta^2-2>0))$$
$$\vee((1<0) \wedge (\beta<0) \wedge (\beta<0) \wedge (2<0) \wedge (\beta^2-2<0)))$$

is true. Some trivial simplifications show that this logical sentence is true if and only if

$$\beta \in (\sqrt{2}, \infty).$$

In this formulation the stability condition is expressed by means of an open interval $(\sqrt{2},\infty)$ whose endpoints are the point at infinity and the algebraic number $\sqrt{2}$. A similar feature remains true in the abstract general case. Recall that $\mathbb{Q}(\beta)$ denotes the set of rational functions of β with coefficients in $\mathbb{Q}$.

Theorem 2 *If $Q(a_1,...,a_n)$ is a rationally decidable binary question associated to an n-uple $(a_1,...,a_n)$ and if all the entries a_i of the n-uple are in $\mathbb{Q}(\beta)$ $(a_i(\beta)\in\mathbb{Q}(\beta))$, then there exist values $\overline{\sigma}_{k,j}$ and $\underline{\sigma}_{k,j}$ ($k=1,2$ and $j=1,...,m_k$) that are either equal to $\pm\infty$ or to algebraic numbers, such that*

$$\begin{aligned}&Q(a_1(\beta),...,a_n(\beta)) \textit{ is true}\\ \Leftrightarrow\quad &\beta\in\left(\cup_{j=1}^{m_1}(\underline{\sigma}_{1,j},\overline{\sigma}_{1,j}]\right)\ \cup\ \left(\cup_{j=1}^{m_2}[\underline{\sigma}_{2,j},\overline{\sigma}_{2,j})\right).\end{aligned}$$

Proof
Since the question $Q(a_1,...,a_n)$ is rationally decidable, there exists a logical sentence $L(a_1,...,a_n)$ of finite length that involves only elementary operations on the entries a_i of the n-uple and such that $L(a_1,...,a_n)$ is true if and only if $Q(a_1,...,a_n)$ is. Thus

$$\forall\beta\in\mathbb{R}\ :\ (Q(a_1(\beta),...,a_n(\beta)) \text{ is true } \Leftrightarrow L(a_1(\beta),...,a_n(\beta)) \text{ is true}).$$

It remains to show that there exist values $\overline{\sigma}_{k,j}$ and $\underline{\sigma}_{k,j}$ ($k=1,2$ and $j=1,...,m_k$) that are either equal to $\pm\infty$ or to algebraic numbers, such that

$$\forall\beta\in\mathbb{R}\ : L(a_1(\beta),...,a_n(\beta)) \text{ is true} \Leftrightarrow$$

$$\beta\in\left(\cup_{j=1}^{m_1}(\underline{\sigma}_{1,j},\overline{\sigma}_{1,j}]\right)\ \cup\ \left(\cup_{j=1}^{m_2}[\underline{\sigma}_{2,j},\overline{\sigma}_{2,j})\right).$$

To prove this we proceed by induction on the size of the logical sentence $L(a_1,...,a_n)$.

The logical sentence $L(a_1,...,a_n)$ is either made up of two smaller logical sentences $L_1(a_1,...,a_n)$ and $L_2(a_1,...,a_n)$ linked by an 'and' or an 'or' logical operation ($L(a_1,...,a_n)=L_1(a_1,...,a_n)\wedge L_2(a_1,...,a_n)$ or $L(a_1,...,a_n)=L_1(a_1,...,a_n)\vee L_2(a_1,...,a_n)$) or is a nucleus expression of the form $L(a_1,...,a_n)=R_1(a_1,...,a_n)\boxdot R_2(a_1,...,a_n)$ where $R_1(a_1,...,a_n)$ and $R_2(a_1,...,a_n)$ are rational expressions of the coefficients $a_1,...,a_n$ $(R_i(a_1,...,a_n)\in\mathbb{Q}(a_1,...,a_n)$

for $i = 1, 2$) and $\boxdot$ is any one of the five sign test operations $<, \leq, >, \geq, =$.

We analyse these two cases successively.

First, if $L(a_1, ..., a_n)$ is a nucleus expression, then $L(a_1(\beta), ..., a_n(\beta))$ is true if and only if

$$R_1(a_1(\beta), ..., a_n(\beta)) \boxdot R_2(a_1(\beta), ..., a_n(\beta))$$

for some $\boxdot \in \{<, \leq, >, \geq, =\}$. By hypothesis $a_i(\beta)$ are rational expressions of β ($a_i(\beta) \in \mathbb{Q}(\beta)$ for $i = 1, 2, ..., n$) and $R_j(a_1, ..., a_n)$ are rational expressions of $a_1, ..., a_n$ ($R_j(a_1, ..., a_n) \in \mathbb{Q}(a_1, a_2, ..., a_n)$ for $j = 1, 2$). Hence, $R'_j(\beta) \triangleq R_j(a_1(\beta), ..., a_n(\beta))$ are also rational expressions of β. The condition $R_1(a_1(\beta), ..., a_n(\beta)) \boxdot R_2(a_1(\beta), ..., a_n(\beta))$ is satisfied if and only if $R'_1(\beta) \boxdot R'_2(\beta)$ is, and this last condition is equivalent to

$$\beta \in \left(\cup_{j=1}^{m_1} (\underline{\sigma}_{1,j}, \overline{\sigma}_{1,j}]\right) \cup \left(\cup_{j=1}^{m_2} [\underline{\sigma}_{2,j}, \overline{\sigma}_{2,j})\right)$$

for some $\overline{\sigma}_{k,j}$ and $\underline{\sigma}_{k,j}$ ($k = 1, 2$ and $j = 1, ..., m_k$) that are equal to $\pm\infty$ or to algebraic numbers. Thus the theorem is proved in the case of a nucleus expression.

Secondly, suppose that $L(a_1, ..., a_n)$ is made up of two logical sentences $L_1(a_1, ..., a_n)$ and $L_2(a_1, ..., a_n)$ linked by an 'and' or an 'or' logical operation. By induction hypothesis assume that the values $\overline{\sigma}^1_{k,j}$ and $\underline{\sigma}^1_{k,j}$ ($k = 1, 2$ and $j = 1, ..., m^1_k$) and $\overline{\sigma}^2_{k,j}$ and $\underline{\sigma}^2_{k,j}$ ($k = 1, 2$ and $j = 1, ..., m^2_k$) are equal to $\pm\infty$ or to algebraic numbers and are such that

$$L_1(a_1(\beta), ..., a_n(\beta)) \text{ is true} \Leftrightarrow \beta \in \left(\cup_{j=1}^{m_1} (\underline{\sigma}^1_{1,j}, \overline{\sigma}^1_{1,j}]\right) \cup \left(\cup_{j=1}^{m_1} [\underline{\sigma}^1_{2,j}, \overline{\sigma}^1_{2,j})\right)$$

and

$$L_2(a_1(\beta), ..., a_n(\beta)) \text{ is true} \Leftrightarrow \beta \in \left(\cup_{j=1}^{m_1} (\underline{\sigma}^2_{1,j}, \overline{\sigma}^2_{1,j}]\right) \cup \left(\cup_{j=1}^{m_2} [\underline{\sigma}^2_{2,j}, \overline{\sigma}^2_{2,j})\right).$$

Then, if $L(a_1, ..., a_n) = L_1(a_1, ..., a_n) \wedge L_2(a_1, ..., a_n)$ we have

$$L(a_1(\beta), ..., a_n(\beta)) \text{ is true} \Leftrightarrow \beta \in \left(\left(\cup_{j=1}^{m_1} (\underline{\sigma}^1_{1,j}, \overline{\sigma}^1_{1,j}]\right) \cup \left(\cup_{j=1}^{m_2} [\underline{\sigma}^1_{2,j}, \overline{\sigma}^1_{2,j})\right)\right)$$

$$\cap \left(\left(\cup_{j=1}^{m_1} (\underline{\sigma}_{1,j}^2, \overline{\sigma}_{1,j}^2] \right) \cup \left(\cup_{j=1}^{m_2} [\underline{\sigma}_{2,j}^2, \overline{\sigma}_{2,j}^2) \right) \right)$$

whereas, if $L(a_1, ..., a_n) = L_1(a_1, ..., a_n) \vee L_2(a_1, ..., a_n)$ we have

$$L(a_1(\beta), ..., a_n(\beta)) \text{ is true} \Leftrightarrow \beta \in \left(\left(\cup_{j=1}^{m_1} (\underline{\sigma}_{1,j}^1, \overline{\sigma}_{1,j}^1] \right) \cup \left(\cup_{j=1}^{m_2} [\underline{\sigma}_{2,j}^1, \overline{\sigma}_{2,j}^1) \right) \right)$$

$$\cup \left(\left(\cup_{j=1}^{m_1} (\underline{\sigma}_{1,j}^2, \overline{\sigma}_{1,j}^2] \right) \cup \left(\cup_{j=1}^{m_2} [\underline{\sigma}_{2,j}^2, \overline{\sigma}_{2,j}^2) \right) \right).$$

It is trivial to see that, in both cases we can rewrite the unions and intersections involved under the form

$$\left(\cup_{j=1}^{m_1} (\underline{\sigma}_{1,j}, \overline{\sigma}_{1,j}] \right) \cup \left(\cup_{j=1}^{m_2} [\underline{\sigma}_{2,j}, \overline{\sigma}_{2,j}) \right)$$

for some $\overline{\sigma}_{k,j}$ and $\underline{\sigma}_{k,j}$ ($k = 1, 2$ and $j = 1, ..., m_k$) equal to $\pm\infty$ or to algebraic numbers. Thus, by induction on the size of L, the theorem is proved. ■

4 Analytic functions

The results that we need are contained in two books on analytic functions (see Nehari [13] and Goluzin [10]). We pick out a result from each of these sources and then merge them into a single formulation that is more suitable for our subsequent treatment. In all what follows we define $A \triangleq \frac{4\pi^2}{\Gamma^4(\frac{1}{4})} = 0.228....$

Theorem 3 (Goluzin, [10], p.89) *Suppose that the function $F(z) = z^q + a_{q+1}z^{q+1} + a_{q+2}z^{q+2} + ...$, for $q \geq 1$, is regular (= analytic) in the disk $|z| < 1$. Then the image of that disk under the mapping $\xi = F(z)$ completely covers some segment of arbitrary predetermined slope that contains the point $\xi = 0$ and is of length no less than $2A$. The number A cannot be increased without additional restrictions on $F(z)$.* ■

The proof of this theorem is not contained in the book itself but in a Russian journal [3] referenced in [10].

That the bound A is the best achievable can be seen from a result contained in Nehari [13]. The function $f_e(z)$ – denoted by $f(z)$ and introduced at the bottom of page 330 in [13] – is connected to the so-called elliptic modular function and is defined by the converging infinite product

$$f_e(z) \triangleq \frac{4\pi^2}{\Gamma^4(\frac{1}{4})}\left(32e^{-\pi\frac{1+z}{1-z}}\prod_{n=1}^{\infty}\left(\frac{1+e^{-2n\pi\frac{1+z}{1-z}}}{1-e^{-(2n-1)\pi\frac{1+z}{1-z}}}\right)-1\right).$$

It is shown in [13] that $f_e(z)$ is an analytic function on $D = \{z \in \mathbb{C} : |z| < 1\}$ such that $f_e(\overline{z}) = \overline{f_e(z)}$, $f_e(0) = 0$, $f_e'(0) = 1$ and that $f_e(z)$ does not take the values $\pm A$ on D.
For further purposes we transform the formulation of Theorem 3 slightly by making use of the properties of the function $f_e(z)$.

Theorem 4 *Assume that $\beta \in \mathbb{R}$. There exists an analytic function on D such that $f(\overline{z}) = \overline{f(z)}$, $f(0) = 0$, $f'(0) = 1$ and that leaves out the values $\pm\beta$ if and only if $|\beta| \geq A$.*

Proof
We first prove sufficiency. Let $f_e(z)$ be the function defined above, assume that $\beta \geq A$ and define

$$f(z) \triangleq \frac{\beta}{A} f_e(\frac{A}{\beta}z).$$

Due to the properties of $f_e(z)$ it is easy to check that $f(z)$ satisfies the conditions of the theorem. For necessity, assume by contradiction that $f(z)$ satifies the conditions of the theorem and that $0 < \beta < A$. By assumption, the image of the disc D under the mapping $\xi = f(z)$ contains neither the value β nor the value $-\beta$. Thus, the image does not cover any segment of the real line that contains the origin and is of length $2A$. This contradicts Theorem 3, hence the result. ∎

This theorem is the crucial result that is needed for proving Theorem 5 below.

5 Simultaneous stabilization of three systems: a question that is rationally undecidable

From our definition of stabilization it is easy to see that a controller stabilizes the null plant if and only if it is stable. Therefore the three rational systems 0,

p_1 and p_2 are simultaneously stabilizable if and only if the two systems p_1 and p_2 are simultaneously stabilizable by a stable controller. In the next theorem we prove that the simultaneous stabilizability question of two systems by a stable controller is rationally undecidable. By the short discussion above this proves the simultaneous stabilizability of three systems to be a rationally undecidable question.

Theorem 5 *The simultaneous stabilizability of two systems by a stable controller is rationally undecidable.*

Proof
Discrete and continuous time stability regions are mapped into one another by the usual bilinear transformation. We present our proof in a discrete time set up, in which a plant is called stable if it has no poles in the closed unit disc $\bar{D}$.
Assume that $\beta \in \mathbb{R}$ and consider the two systems $p_{1,\beta}(z) = \frac{z^2}{z^2-\beta}$ and $p_{2,\beta}(z) = \frac{z^2}{z^2+\beta}$. We proceed in two steps.

First, we show that when $\beta = 0$ or $|\beta| > A$ the two systems are simultaneously stabilizable by a stable controller, whereas when $0 < |\beta| < A$ they are *not* simultaneously stabilizable by a stable controller. Note that we leave out the analysis of the case $\beta = A$.

Second, we show that the first step contradicts the fact that the simultaneous stabilizability question of two systems by a stable controller is a rationally decidable question.

Step 1.
If $\beta = 0$ then $p_{1,\beta}(z) = p_{2,\beta}(z) = z$ and a stable stabilizing controller is given, for example, by $c(z) = 2$.
If $\beta \neq 0$ then the two systems are simultaneously stabilizable by a stable controller if and only if there exists a rational function $c(z)$ that has no poles in $\overline{D}$ and such that

$$z^2 c(z) + z - \beta$$

and

$$z^2c(z) + z + \beta$$

have no zeros in $\overline{D}$.

It remains to show that, when $\beta < A$ such a function $c(z)$ does not exist whereas it does exist when $\beta > A$. We prove these two points in parts (a) and (b), respectively.

(a) Assume, by contradiction, that $\beta < A$, that $c(z)$ has no poles in $\overline{D}$ and that

$$z^2c(z) + z - \beta$$

and

$$z^2c(z) + z + \beta$$

have no zeros in $\overline{D}$. Then, the function defined by

$$f(z) \triangleq z^2c(z) + z$$

satisfies all the conditions of Theorem 4 and leaves out the values $\pm\beta$ with $\beta < A$. A contradiction is achieved and this part is proved.

(b) Assume that $\beta > A$. We construct a rational function $c(z)$ that satisfies all the requested conditions.
By Theorem 4, there exists an analytic function $f(z)$ on D such that $f(0) = 0$, $f'(0) = 1$ and that does not take the values $\pm A$ on D. We define the function $g(z)$ by

$$g(z) \triangleq \frac{\beta}{A} f\left(\frac{A}{\beta} z\right).$$

Due to the properties of $f(z)$, the function $g(z)$ is such that

1. $g(\overline{z}) = \overline{g(z)}$,
2. $g(z)$ is analytic on $|z| < \frac{\beta}{A}$ (and $1 < \frac{\beta}{A}$),
3. $g(0) = 0$ and $g'(0) = 1$,
4. $g(z)$ leaves out the values $\pm A$ on $|z| < \frac{\beta}{A}$.

With the help of this function $g(z)$, we construct a real *polynomial* $p(z) \in \mathbb{R}[z]$ such that $p(0) = 0$, $p'(0) = 1$ and $p(z) \neq \pm A$ on D.

Because of the points 2 and 4, the real number μ defined by

$$\mu \triangleq \min\{\inf_{z \in \overline{D}} |g(z) - A|, \inf_{z \in \overline{D}} |g(z) + A|\}$$

is strictly positive. Because of the first three points, the function $h(z)$ defined by

$$h(z) \triangleq \frac{g(z) - z}{z^2}$$

is real and analytic in $\{z : |z| < \frac{\beta}{A}\}$. By Runge's theorem (see Rudin [14]), there exists a real polynomial $q(z)$ such that

$$|h(z) - q(z)| < \mu \left(\frac{A}{\beta}\right)^2, \; z \in \overline{D}.$$

This polynomial is then also such that

$$|g(z) - z - z^2 q(z)| < \mu, \; z \in \overline{D}.$$

Defining the polynomial $p(z) \triangleq z + z^2 q(z) \in \mathbb{R}[z]$, we have $p(0) = 0$ and $p'(0) = 1$. But also, because

$$|g(z) - p(z)| < \mu, \; z \in \overline{D}$$

and

$$\mu \leq \min\{\inf_{z \in \overline{D}} |g(z) - A|, \inf_{z \in \overline{D}} |g(z) + A|\},$$

it follows that

$$|g(z) - p(z)| < |g(z) \pm A|, \; z \in \overline{D}.$$

Hence,

$$p(z) \neq \pm A, \; z \in \overline{D},$$

as requested.

A polynomial is a rational function with no poles of module less than or equal to one and, thus, point (b) is proved.

Step 2.
Assume, by contradiction, that the simultaneous stabilizability of two systems by a stable controller is a rationally decidable question. Then, so is the simultaneous stabilizability of the two systems $p_{1,\beta}(z) = \frac{z^2}{z^2-\beta}$ and $p_{2,\beta}(z) = \frac{z^2}{z^2+\beta}$ by a stable controller.
But then, using Theorem 2, there exist values $\overline{\sigma}_{k,j}$ and $\underline{\sigma}_{k,j}$ ($k = 1, 2$ and $j = 1, ..., m_k$) that are either equal to $\pm\infty$ or that are algebraic numbers, such that our three systems are simultaneously stabilizable if and only if

$$\beta \in \left(\cup_{j=1}^{m_1}(\underline{\sigma}_{1,j}, \overline{\sigma}_{1,j}]\right) \cup \left(\cup_{j=1}^{m_2}[\underline{\sigma}_{2,j}, \overline{\sigma}_{2,j})\right).$$

This contradicts our first step since we know from there that the two systems are simultaneously stabilizable by a stable controller if and only if

$$\beta \in (-\infty, -\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}) \cup [0,0] \cup (\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}, +\infty)$$

or if and only if

$$\beta \in (-\infty, -\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}] \cup [0,0] \cup [\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}, +\infty).$$

By Theorem 1, $\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}$ is a transcendental number, a contradiction is achieved and the theorem is proved. ∎

6 Conclusion

We believe that our result closes much of the research on the simultaneous stabilization problem. Indeed, we have shown that there exists no criterion for simultaneous stabilizability that involves only elementary operations on the coefficients. In particular, it is not possible to find a criterion that involves only, say, solving systems of linear equations, solving a Nevanlinna type interpolation problem or evaluating a Cauchy index, because all these operations are conducted by performing elementary operations only.

Acknowledgments

We wish to thank B.D.O. Anderson for his creative comments on the original ideas developed here. The central reference used in Theorem 1 for proving the transcendence of $\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}$ was given to us by J. Meinguet, whereas the proof itself is due to P. Delsarte.

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