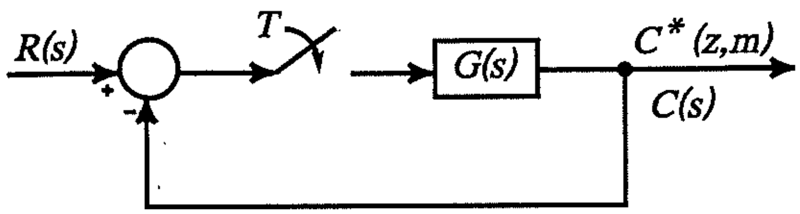
FUNDAMENTALS OF DISCRETE-TIME SYSTEMS

A Tribute to Professor Eliahu I. Jury



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pp. 88–101. NewYork: Va
958.
data control systems wit
l. 27, pp. 705–710, 1991.
constraints for systems d
963.
l-data control systems,"
, 1991.
ftfahrzeugen," *Automob*
st Γ-stability analysis in
5, 1991.
a family of plant models
, pp. 240–244, 1982.
omatic steering," in *Pro*
00, 1990.
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Optimal FWL Design of a Digital Controller*

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ABSTRACT

The optimal Finite Word Length (FWL) state-space digital compensator design problem is investigated. Instead of the usual sensitivity measure, it is argued that it may be desirable to minimize a frequency weighted sensitivity measure over all similarity transformations. The set of optimal realizations minimizing this weighted sensitivity is completely characterized, and an algorithm is proposed to find the optimal solution set.

*Dedicated to Professor Eli Jury, on the occasion of his 70th birthday, and with deepest appreciation for his technical leadership and personal encouragement to younger workers, especially the first author, over decades of outstanding professional activity

1. INTRODUCTION

It has long been recognized that in realizing digital transfer functions, finite word length (FWL) effects can manifest themselves. It is well known that any linear system can be represented by state-space equations and that this state-space model is not unique. In the infinite precision case, all these realizations are equivalent since they yield one and the same transfer function. But different realizations have different numerical properties such as sensitivity and error propagation. This means that they are no longer equivalent in the finite precision case. The optimal FWL state-space design task is to identify those realizations which minimize the degradation of the system performance due to the FWL effects, [1-5].

In [3] a global sensitivity measure of a transfer function w.r.t. the parameters of the state space model was proposed by Tavsanoglu and Thiele, and a reasonable and easily computable upper bound for this measure was studied. It was shown in [5] that the realizations that minimize the upper bound also minimize the sensitivity measure itself and that, under a dynamic range constraint, this sensitivity measure and the roundoff noise gain are simultaneously optimized. The set of optimizing structures was characterized in [1]-[3] and [5].

In controller design, it may be that large errors in the controller realization at one frequency have little effect on the closed-loop gain, while small errors at other frequencies may have a large effect. In quantitative terms, let $R(z), P(z)$ and $T(z) = PR(1+PR)^{-1}$ denote the controller, plant and closed-loop transfer functions (the scalar case only will be considered). If α is a controller parameter, then

$$\frac{\partial T}{\partial \alpha} = (1+PR)^{-2} P \frac{\partial R}{\partial \alpha} \tag{1}$$

Evidently, the magnitude of $(1+PR)^{-2}P$ has the effect of frequency-weighting errors in the controller transfer function.

Evidently, a procedure is needed which can select an optimum FWL realization, having regard to the introduction of frequency weighting. This is the subject of the paper.

A frequency weighted measure has already been introduced by Thiele [5], but with a specific relationship between the weightings of the various terms of the measure. This relationship can virtually never be satisfied when the weightings are derived as in (1).

2. WEIGHTED SENSITIVITY MEASURE OF A REALIZATION

In this paper we consider the implementation of a discrete linear time-invariant single input, single output compensator having the following transfer function :

$$R(z) = \frac{\sum_{i=0}^{n} b_i z^{-i}}{1 + \sum_{i=1}^{n} a_i z^{-i}}. \tag{2}$$

This system can be implemented by a minimal state-space realization:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + du(t) \end{aligned} \tag{3}$$

with $A \in \gtrless^{n\times n}$, $B \in \gtrless^{n}$, $C^T \in \gtrless^{n}$ and $d \in \gtrless$. The transfer function can be expressed in terms of the state matrices as

$$H(z) = C(zI - A)^{-1}B + d. \tag{4}$$

We now define a realization set S_H of this system as follows:

$$S_H = \{(A, B, C, d) : (A, B, C, d) \text{ satisfies } (4)\}.$$

Clearly, if (A, B, C, d) belongs to S_H, so does $(T^{-1}AT, T^{-1}B, CT, d)$ for any similarity transformation T. This means that S_H is an infinite set.

There are of course several ways of defining an overall sensitivity measure. Here we present a measure proposed by Tavsanoglu and Thiele [3]. It is an absolute rather than a relative sensitivity measure and is therefore based on a fixed-point arithmetical implementation; alternative floating-point implementations are discussed in [6, 7].

Definition 1. : Let $M \in \gtrless^{n\times m}$ be a matrix and let $f(M) \in \triangleleft$ be a scalar complex function of M, differentiable w.r.t. all the elements of M. We then define the sensitivity function of f w.r.t. M as

$$S_M \triangleq \frac{\partial f}{\partial M} \text{ with } (S_M)_{ij} \triangleq \frac{\partial f}{\partial m_{ij}} \tag{5}$$

where m_{ij} denotes the $(i,j)^{th}$ element of the matrix M. ∎

With these notations it is easy to show [3] that

$$\begin{aligned} S_A(z) &\triangleq \frac{\partial R(z)}{\partial A} = G(z)F^T(z) \\ S_B(z) &\triangleq \frac{\partial R(z)}{\partial B} = G(z) \\ S_C(z) &\triangleq \frac{\partial R(z)}{\partial C^T} = F(z) \end{aligned} \tag{6}$$

where

$$F(z) \triangleq (zI - A)^{-1}B = [f_1(z) \dots f_n(z)]^T$$
$$G^T(z) \triangleq C(zI - A)^{-1} = [g_1(z) \dots g_n(z)] \tag{7}$$

Note that the direct term d and the sensitivity function w.r.t. it are coordinate independent, so they have nothing to do with the optimal realization problem.

Definition 2. : Let $f(z) \in \lhd^{n \times m}$ be any complex matrix valued function of the complex variable z. We then define the L_p-norm of $f(z)$ as

$$\| f \|_p \triangleq \left(\frac{1}{2\pi} \int_0^{2\pi} \| f(e^{j\omega}) \|_F^p \, d\omega \right)^{1/p} \tag{8}$$

where $\| f(e^{j\omega}) \|_F$ is the Frobenius norm of the matrix $f(e^{j\omega})$:

$$\| f(e^{j\omega}) \|_F \triangleq \left(\sum_{i=1}^n \sum_{k=1}^m |f_{ik}(e^{j\omega})|^2 \right)^{1/2} = \{tr[f^T(e^{-j\omega})f(e^{j\omega})]\}^{1/2}. \tag{9}$$

■

Tavsanoglu and Thiele [3] have proposed the following overall sensitivity measure of the transfer function $R(z)$ w.r.t. the parameters in the realization A, B, C:

$$M_a \triangleq \| \frac{\partial R}{\partial A} \|_1^2 + \| \frac{\partial R}{\partial B} \|_2^2 + \| \frac{\partial R}{\partial C^T} \|_2^2 . \tag{10}$$

The mixing of different measures (L_1 and L_2) in the overall sensitivity measure above is motivated by the analytic properties of the first term on the right of (10), which allow one to derive an analytic minimization procedure for M_a: see [3] and [5]. The optimization of a more logical L_2 measure is much harder and has only recently been solved in [8] and, independently, by Helmke and Moore [9].

Note that the measure in Definition 2 is in fact a frequency independent mean value of a matrix function in the whole frequency range. Therefore, the sensitivity measure M_a defined in (10) considers the sensitivity behavior of the transfer function at one frequency point to be as important as at another frequency point. However, we want to introudce frequency weighting. This is done via the definition of a weighted sensitivity and hence a weighted sensitivity measure. We shall provide a procedure applying for arbitrary weights.

Let $W_A(z)$, $W_B(z)$ and $W_C(z)$ be three integrable scalar functions of the complex variable z. Then the weighted sensitivity functions corresponding to those given in

(6) are defined as

$$\begin{aligned}\frac{\delta R(z)}{\delta A} &\triangleq W_A(z)\frac{\partial R(z)}{\partial A}\\ \frac{\delta R(z)}{\delta B} &\triangleq W_B(z)\frac{\partial R(z)}{\partial B}\\ \frac{\delta R(z)}{\delta C^T} &\triangleq W_C(z)\frac{\partial R(z)}{\partial C^T}.\end{aligned} \tag{11}$$

Note that the notation is not meant to suggest that δ is a derivative operator.

$$W_A(z) = W_1(z)W_2(z) \tag{12}$$

be any factorization of $W_A(z)$. With Definition 2, the over-all weighted L_1/L_2 sensitivity measure is defined as

$$M_a^* \triangleq \| \frac{\delta R(z)}{\delta A} \|_1^2 + \| \frac{\delta R(z)}{\delta B} \|_2^2 + \| \frac{\delta R(z)}{\delta C^T} \|_2^2 . \tag{13}$$

Now using (6), (11) and (12), M_a^* can be rewritten as

$$M_a^* = \| W_1(z)G(z)(W_2(z)F(z))^T \|_1^2 + \| W_B(z)G(z) \|_2^2 + \| W_C(z)F(z) \|_2^2 . \tag{14}$$

A similarity transformation $x = Tz$ transforms $(A, B, C, F(z), G(z))$ into $(T^{-1}AT$,-$T^{-1}B, CT, T^{-1}F(z), T^TG(z))$. The optimal FWL state-space design can then be formulated as follows:

$$\min_{(A,B,C)\in S_H} M_a^* \tag{15}$$

3. OPTIMAL FWL REALIZATIONS

The difficulty in solving (15) is due to the fact that the first term on the right of (13) is a complicated function of the realization (A, B, C). To overcome this, note that by the Cauchy-Schwartz inequality

$$\| \frac{\delta H(z)}{\delta A} \|_1^2 = \| W_1(z)G(z)(W_2(z)F(z))^T \|_1^2 \leq \| W_1(z)G(z) \|_2^2 \| W_2(z)F(z) \|_2^2 \tag{16}$$

where equality holds if and only if

$$\rho^2 G^H(z)G(z)|W_1(z)|^2 = F^H(z)F(z)|W_2(z)|^2 \; \forall z \in \{|z| = 1\}, \tag{17}$$

for some $\rho \neq 0 \in \gtreqless$. We will study the following upper bound of M_a^* :

$$M_a^* \leq \bar{M}_a^* \triangleq \| W_1(z)G(z) \|_2^2 \| W_2(z)F(z) \|_2^2 + \| \frac{\delta R(z)}{\delta B} \|_2^2 + \| \frac{\delta R(z)}{\delta C^T} \|_2^2 . \tag{18}$$

We consider methods for minimizing $\bar{M}_a^*$. It is easy to show with (8)-(9) that

$$\bar{M}_a^* = tr(K_{o1})tr(K_{c2}) + tr(K_{oB}) + tr(K_{cC}), \tag{19}$$

where K_{o1}, K_{c2}, K_{oB} and K_{cC} can be obtained by the following general expression:

$$K = \frac{1}{2\pi j}\oint_{|z|=1} X(z)X^H(z)z^{-1}dz \tag{20}$$

with $X(z) = G(z)W_1(z), F(z)W_2(z), G(z)W_B(z)$ and $F(z)W_C(z)$, respectively. We call these four matrices $K_{o1}, K_{c2}, K_{oB}, K_{cC}$ weighted Gramians. Several algorithms for computing a weighted Gramian are available in [5] and [10]. A similarity transformation $x = Tz$ transforms $(A, B, C, K_{cC}, K_{c2}, K_{oB}, K_{o1})$ into $(T^{-1}AT, T^{-1}B, CT, T^{-1}K_{cC}T^{-T}, T^{-1}K_{c2}T^{-T}, T^TK_{oB}T, T^TK_{o1}T)$. So, the optimal FWL design problem of (15) is replaced by the following upper bound minimization:

$$\min_{T:\det T\neq 0}\{\tilde{M}_a^* = tr(T^TK_{o1}T)tr(T^{-1}K_{c2}T^{-T}) + tr(T^TK_{oB}T) + tr(T^{-1}K_{cC}T^{-T})\}. \tag{21}$$

Now, with $P = TT^T$, it is easy to see that

$$\tilde{M}_a^* = tr(K_{o1}P)tr(K_{c2}P^{-1}) + tr(K_{oB}P) + tr(K_{cC}P^{-1}) \triangleq \mathcal{M}(P) \tag{22}$$

$$\min_{T:\det T\neq 0}\tilde{M}_a^* \iff \min_{P:P=TT^T,\det P\neq 0}\mathcal{M}(P). \tag{23}$$

We can now state (without proof here) the following result

Main Result: Suppose that K_{oB} and K_{cC} are nonsingular.[1] Then the minimum of $\mathcal{M}(P)$ defined in (23) exists and is achieved by a nonsingular P. Further, the optimum P can be obtained through a gradient algorithm iteration

$$P(k+1) = P(k) - \mu\frac{\partial\mathcal{M}(P)}{\partial P}\bigg|_{P=P(k)} \tag{24}$$

$$\frac{\partial\mathcal{M}(P)}{\partial P} = -tr(K_{o1}P)P^{-1}K_{c2}P^{-1} - P^{-1}K_{cC}P^{-1} + tr(K_{c2}P^{-1})K_{o1} + K_{oB} \tag{25}$$

Remarks

1. Thiele considered the case $W_1(z) = W_B(z)$ and $W_2(z) = W_0(z)$. In this case, the minimization is analytically achievable. If further, $W_1(z) = W_2(z)$, then $M_a^* = \tilde{M}_a^*$. In compensator implementation, we have $W_A = W_B = W_C = (1+PR)^{-2}P$.

2. Knowledge of the optimum P does not define an optimum coordinate basis change T through $P = TT'$; rather T is only unique up to right multiplication by an orthogonal matrix. The additional freedom in T can be used to ensure the advantageous incorporation in the matrices A, b, c of $\frac{1}{2}n(n-1)$ zero elements.

[1] This condition is generally satisfied. Space limitations prevent fuller analysis here

3. There are appropriate, easily computable, initial conditions for (24)

4. THE NEXT STEP - SAMPLED DATA CONTROL

A discrete time controller $C(z)$ will normally be used with a continuous-time plant $P(s)$, there being also present in the loop an antialiasing filter $F(s)$, sampler S and hold element H. The analysis of the loop using a discrete-time transfer function approach alone suffers well known disadvantages, particularly the neglect of inter-sample ripple. A scheme is needed for selection of an optimal FWL realization of the controller $R(z)$ which allows consideration of intersample behaviour. We outline very briefly how this can be done. *Formally,* we have a closed-loop operator $T = PHCSF(1+PHCSF)^{-1}$ and

$$\begin{aligned}\frac{\partial T}{\partial \alpha} &= (1+PHCSF)^{-1}PH\ \frac{\partial C}{\partial \alpha}SF(1+PHCSF)^{-1} \\ &= W_1\,\frac{\partial C}{\partial \alpha}\,W_2\end{aligned}$$

The operators W_1, W_2 do not have transfer function representations, and this is a difficulty. The difficulty can be tackled by a two-step procedure: very fast sampling [at a multiple of the sampling frequency of $C(z)$] of the continuous-time parts of W_1 and W_2, followed by "blocking" or "lifting" to turn the resulting multirate discrete-time system into a single rate system. The use of very fast sampling followed by blocking has been used also for the problem of passing from a continuous-time to a discrete time controller, and for solving the sampled-data H_∞ problem, [11-12].

The details of these ideas will be presented in a forthcoming publication, together with examples, [13]

5. CONCLUSIONS

The main ideas of the paper are these: Optimum FWL realization of a digital controller can be viewed as a frequency weighted sensitivity minimization problem; this problem has a unique solution, obtainable via a gradient algorithm iteration; a more sophisticated problem statement, reflecting inter-sample performance, is possible. A fuller version of many of the ideas of this paper is available as [14].

REFERENCES

C.T. Mullis and R.A. Roberts (1976), "Filter structures which minimize roundoff noise in fixed-point digital filters", Proc. ICASSP, pp. 505-508.