Proceedings of the
American Control Conference
Seattle, Washington • June 1995

C80



# OPTIMALITY AND SUB-OPTIMALITY OF ITERATIVE IDENTIFICATION AND CONTROL DESIGN SCHEMES [1]

Håkan Hjalmarsson#, Svante Gunnarsson# and Michel Gevers$
#Dept. of Electrical Engineering, Linköping University, S-581 83 Linköping, Sweden
$CESAME, Louvain University, B-1348 Louvain-la-Neuve, Belgium

### Abstract

We demonstrate that some recently proposed iterative identification and control design schemes do not necessarily converge to a local minimum of the design objective in the case of a restricted complexity model. There is, however, a link between these approaches and a recently proposed iterative optimization based control design procedure based on experimental data. We show that if the achieved and the desired output responses are perfectly matched, the schemes are (essentially) equivalent under noise free conditions.

## 1. Introduction

Recently, so called iterative identification and control design schemes have received considerable attention. See e.g. [1], [2] and [3]. These schemes iteratively perform plant model identification and model-based controller update in the closed loop. The work in [4] and [5] is a continuation of these ideas, and there it is shown that for certain control criteria, *e.g.* LQG, it is possible to carry out the optimization using measurements from the plant collected during (essentially) normal operating conditions. No models of the plant and the disturbance are required. In this contribution we show that there is a close relation between the optimization based tuning algorithm in [4] and the indirect schemes proposed for example in [6] and [1]. When Gauss-Newton steps are used, the optimization procedure can be approximately expressed as an identification criterion. These two identification criteria become identical only if the achieved and designed closed loops perfectly match each other, i.e. only if the true system has been perfectly identified. We show that when the model is too simple, the indirect schemes fail to get close to the minimum of the control criterion.

[1] This paper presents research results of the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with its authors.

## 2. Criterion minimization

Let the true system be given by

$$y(t) = G_0(q)u(t) + v(t) \tag{1}$$

where $\{v(t)\}$ is a (process) disturbance. The output, $\{y(t)\}$, from the true system will be called the achieved response. We will use the following two degrees of freedom controller:

$$u(t) = C_r(q,\rho)r(t) - C_y(q,\rho)y(t) \tag{2}$$

where $\{r(t)\}$ is an external reference signal and $\rho$ represents the controller parameters. To ease the notation somewhat we will from now on omit the time argument of the signals. In addition, whenever signals are obtained from the closed loop system with the controller $\{C_r(\rho), C_y(\rho)\}$ operating, we will indicate this by using the $\rho$-argument; thus, $y(\rho)$ will denote the output of the system (1) in feedback with the controller (2). Let $T_d$ be a desired stable closed loop response from reference signal to output signal

$$y_d = T_d r. \tag{3}$$

The error between the achieved and desired response $\tilde{y}(\rho) = y(\rho) - y_d$ is given by

$$\tilde{y}(\rho) = \frac{C_r(\rho)G_0}{1+C_y(\rho)G_0}r - T_d r + \frac{1}{1+C_y(\rho)G_0}v. \tag{4}$$

It is natural to formulate the design objective as a minimization of some norm of $\tilde{y}(\rho)$, i.e.

$$\rho^* = \arg\min_\rho J(\rho) = \arg\min_\rho \frac{1}{2}\mathrm{E}\left[\tilde{y}^2(\rho)\right]. \tag{5}$$

Here E denotes expectation over $v$ and $r$ which we assume to be realizations of stationary stochastic processes. With $T_0(\rho)$ and $S_0(\rho)$ denoting the achieved closed loop response and the sensitivity function with the controller $\{C_r(\rho), C_y(\rho)\}$, and given the statistical independence of $r$ and $v$, $J(\rho)$ can be written as

$$J(\rho) = \frac{1}{2}\mathrm{E}\left[\{(T_d - T_0(\rho))r\}^2 + \{S_0(\rho)v\}^2\right] \quad (6)$$

The first term is the tracking error, the second term is the contribution due to the disturbance. It is evident from (4) that $J(\rho)$ depends in a fairly complicated way on $\rho$. Furthermore, the true system $G_0$ and the spectrum of $\{v\}$ are unknown. The problem we would like to solve is to find a solution for $\rho$ to the equation

$$0 = J'(\rho) = \mathrm{E}\left[\tilde{y}(\rho)\tilde{y}'(\rho)\right]. \quad (7)$$

This is done by taking repeated steps in a descent direction

$$\rho_{i+1} = \rho_i - \gamma_i R_i^{-1} J'(\rho_i). \quad (8)$$

Here $R_i$ is some appropriate positive definite matrix, typically an estimate of the Hessian of $J$, such as a Gauss-Newton approximation of this Hessian. As stated this problem is intractable since it involves taking expectation. It is, however, exactly a problem that can be attacked with stochastic approximation procedures, where one replaces $J'$ with an approximation based on the current samples. In order to do this, the signal $\tilde{y}(\rho_i)$ its gradient $\tilde{y}'(\rho_i)$ is required. If a model of the plant is available, then this model can be used to compute this quantities. However, in [4] and [5] it is shown that $\tilde{y}(\rho_i)$ can be computed exactly and that $\tilde{y}'(\rho_i)$ can be computed approximately using experimental data from (essentially) normal operating conditions only. No explicit model is needed.

## 3. Model based criterion minimization

Instead of going directly for the controller parameters $\rho$, one may want to estimate a model $G(\theta)$ of the open loop as an intermediate step. This is essentially only a question of reparametrization of the controller as a function of model parameters: $\rho = \rho(\theta)$. The idea of optimization remains the same: it is the control performance criterion $J(\rho(\theta))$ of (5) that is minimized and not an identification criterion. The gradients $C_r'(\rho(\theta))$ and $C_y'(\rho(\theta))$ are now with respect to the parameters $\theta$ of the model. To compute these can be quite complicated, *c.f.* pole placement and LQG designs which involve Diophantine and Riccati equations. In, for example the case of a pole placement design, the reference model $T_d$ is (possibly) also a function of $\theta$[1]:

$$T_d(\theta) = \frac{C_r(\theta)G(\theta)}{1 + C_y(\theta)G(\theta)}; \quad S_d(\theta) = \frac{1}{1 + C_y(\theta)G(\theta)}, \quad (9)$$

[1]This will be the case when some of the zeros of the system are preserved in the design.

so that $y_d = y_d(\theta)$. As pointed out in [6] it is possible to write

$$\tilde{y}(\theta) = y(\theta) - y_d(\theta) = S_d(\theta)(y(\theta) - G(\theta)u(\theta)). \quad (10)$$

Even after this rewriting, the minimization of $E[(\tilde{y}(\theta))^2]$ cannot be viewed as a frequency weighted identification problem because the frequency weighting $S_d(\theta)$ and the input and output signals all depend on the unknown parameter vector $\theta$. However, an iterative procedure, that can be interpreted as identification using closed loop signals followed by control design, is obtained by keeping $\theta$ fixed to $\theta_i$, say, in all places in this expression except for $G(\theta)$, *i.e.* at the $i$th identification iteration one could minimize a square norm of the following errors:

$$e(\theta, \theta_i) = S_d(\theta_i)\left(y(\theta_i) - G(\theta)u(\theta_i)\right). \quad (11)$$

Here $y(\theta_i)$ and $u(\theta_i)$ are the data obtained from the true system with the $i$th controller (computed from $G(\theta_i)$) operating on the system. The identification step now is

$$\theta_{i+1} = \arg\min_{\theta} F(\theta, \theta_i) \quad (12)$$

where $F$ is the quadratic function

$$F(x, z) = \frac{1}{2}\mathrm{E}\left[e^2(x, z)\right] \quad (13)$$

The new parameter value $\theta_{i+1}$ and the new model $G(\theta_{i+1})$ is used to update the controller, the new controller is applied to the true plant, new data are collected and the procedure is repeated. This procedure is suggested in [6]. The schemes found in [1] and [2] also employ the same iterative identification/control procedure. They use control design objectives that do not involve a prespecified reference model $T_d$. An $H_\infty$ criterion of the closed loop transfer functions is used in [2], while Zang *et.al.* [1] consider an LQG criterion.

## 4. Comparing the minima of the objective functions

We now examine the possibility that iterative identification and control schemes based on the above idea can converge to the minimum of $J(\theta)$. A convergence point $\theta^{\#}$ of the procedure satisfies

$$\theta^{\#} = \arg\min_{\theta} F(\theta, \theta^{\#}). \quad (14)$$

A necessary condition for this is

$$F_x(\theta^{\#}, \theta^{\#}) = 0. \quad (15)$$

This should be compared with the necessary condition for a minimizing point $\theta^*$ of $J(\theta) = F(\theta, \theta)$:

$$J'(\theta^*) = F_x(\theta^*, \theta^*) + F_z(\theta^*, \theta^*) = 0. \quad (16)$$

Thus, unless $F$ is chosen properly, $\theta^*$ does not have to satisfy (15) in order to be a minimizing point for $J(\theta)$. The following example gives some insight.

**Example:** Let

$$y(t) = G_0(q)u(t) + H_0(q)e(t),$$

where $\{e\}$ is a zero mean white noise sequence with unit variance and where

$$G_0(q) = \frac{bq^{-1}}{1 - aq^{-1}}; \quad H_0(q) = 1 + aq^{-1}; \quad |a| < 1.$$

For simplicity we shall assume that the noise model $H_0$ is known, *i.e.* $H = H_0$. Suppose now that the purpose is to construct a minimum variance controller

$$C_y(\theta) = \frac{H - 1}{G(\theta)} \quad (17)$$

When the true system is in the model set, *i.e.* there is a $\theta^*$ such that $G(\theta^*) = G_0$, it is easy to verify that $\theta^*$ satisfies (15). Thus it is possible for the iterative scheme to converge to the (non-restricted) optimal controller. However, the situation changes if we consider a restricted complexity model. Let the model structure now be

$$G(\theta) = \theta q^{-1}; \quad H(q) = H_0(q).$$

Then (17) gives $C_y = a/\theta$ and the output of the closed loop system then becomes

$$y(t) = \frac{1 - a^2 q^{-2}}{1 - \alpha q^{-1}} e(t)$$

where $\alpha = a(1 - b/\theta)$. This gives the output variance

$$J(\theta) = \mathrm{E}[y^2(t)] = \frac{1 + a^4 - 2a^2\alpha^2}{1 - \alpha^2}$$

Minimizing this expression gives $\theta^* = b$. Hence the optimal restricted complexity control law is

$$u^* = -\frac{a}{b} y(t)$$

which gives the optimal closed loop pole $\alpha^* = 0$. Turning to the iterative identification and control design schemes the expression (11), in this example, is

$$\begin{aligned} e(x, z) &= S_d(z)(y(z) - G(x)u(z)) \\ &= \frac{1 + \frac{ax}{z} q^{-1}}{1 + aq^{-1}} y(z) \end{aligned}$$

Hence, the partial derivative of (13) w.r.t. its first argument becomes

$$\begin{aligned} F_x(\theta, \theta) &= \mathrm{E}\left[e(\theta, \theta) \cdot e_x(\theta, \theta)\right] \quad (18) \\ &= \frac{a}{\theta(1 - \alpha^2)} \left(-a\alpha^2 + (1 - a^2)\alpha + a^3\right) \end{aligned}$$

A convergence point $\theta = \theta^{\#}$ must satisfy (15), and hence $\alpha$ has to satisfy

$$a\alpha^2 - (1 - a^2)\alpha - a^3 = 0 \quad (19)$$

It is easy to show that the stable solution ($|\alpha| < 1$) is

$$\alpha^{\#} = \frac{1 - a^2 - \sqrt{1 - 2a^2 + 5a^4}}{2a} \qquad a \neq 0$$

and $\alpha^{\#} = 0$ for $a = 0$. As a function of $a$, $\alpha^{\#}$ is continuous in the interval $-1 < a < 1$. Furthermore

$$\lim_{a \to -1_-} \alpha^{\#} = 1 \quad \text{and} \quad \lim_{a \to 1_+} \alpha^{\#} = -1.$$

Thus, the identification based iterative scheme for this restricted complexity model situation has a stationary point which corresponds to a closed loop pole which can be anywhere in the interval $(-1, 1)$, depending on the open loop pole $a$ whereas the optimal closed loop pole is at the origin regardless of $a$.

## 5. Numerical comparison

Let us now consider the problem of finding a controller for a third order system. In the indirect approach it will be identified using output error (OE) models of first order. The control design will be done using pole placement, and the design variable in the controller design will be the closed loop bandwidth $\omega_B$. The reference signal is white noise filtered through a low pass filter with bandwidth $\omega_B$. The iterative design schemes starts by identifying a model from open loop data. The model is then used in the design of a pole placement regulator which is used for collecting a new set of data from the system, now acting in closed loop. This procedure is repeated until convergence and the value of the criterion after the last iteration is used for comparison. In each identification step the designed sensitivity function from the previous iteration is used as prefilter. The results of the simulations can be found in Table 1 where the achieved cost $J$ of (5) are shown. It can be seen that for low designed bandwidths the iterative identification and control scheme performs very well. However, as the bandwidth is increased, the performance deteriorates dramatically. A reason is that the optimal (reduced complexity) control laws for the higher bandwidths correspond to unstable models, while OE models always are stable. Hence, it is in these cases impossible for these schemes to give the optimal controller. Unstable models can of course be obtained using the ARX model structure, but only minor improvements are obtained using this structure.

| $\omega_B$ | Iterative Optimization | Iter.Id./Control |
|---|---|---|
| 1 | $2.4 \cdot 10^{-5}$ | $2.4 \cdot 10^{-5}$ |
| 2 | $2.2 \cdot 10^{-4}$ | $2.6 \cdot 10^{-4}$ |
| 3 | $8.0 \cdot 10^{-4}$ | $9.8 \cdot 10^{-4}$ |
| 4 | $2.3 \cdot 10^{-3}$ | $2.6 \cdot 10^{-3}$ |
| 5 | $4.8 \cdot 10^{-3}$ | $8.8 \cdot 10^{-3}$ |
| 6 | $8.6 \cdot 10^{-3}$ | $7.7 \cdot 10^{-2}$ |
| 7 | $1.5 \cdot 10^{-2}$ | $1.0 \cdot 10^{-1}$ |
| 8 | $2.2 \cdot 10^{-2}$ | $1.2 \cdot 10^{-1}$ |
| 9 | $3.1 \cdot 10^{-2}$ | $1.5 \cdot 10^{-1}$ |
| 10 | $4.2 \cdot 10^{-2}$ | $1.7 \cdot 10^{-1}$ |

**Table 1:** Loss function using Gauss-Newton optimization and iterative identification and control

## 6. Approximating the Gauss Newton minimization step by an identification step

We will now show that each Gauss Newton iteration in the direct minimization procedure can be approximated by an identification step. For simplicity we consider only the disturbance rejection problem, *i.e.* $r \equiv 0$. The servo problem gives the same result but the technical details are a lot more involved. We use the following simple model reference scheme

$$C_y(\theta) = \frac{S_d^{-1} - 1}{G(\theta)}. \tag{20}$$

Here $S_d$ represents a desired sensitivity function and is therefore a fixed ($\theta$-independent) quantity. For design methods such as LQG, $S_d(\theta)$ is the result of a model based optimization procedure and the results below are only valid approximately. Recall that in the $H_2$ iterative identification and control procedure, the identification step minimizes (w.r.t. $\theta$) the quadratic norm of (11),

$$e(\theta, \theta_i) = S_d(y(\theta_i) - G(\theta)u(\theta_i)), \tag{21}$$

where $S_d$ is now independent of $\theta$, and remember that $\tilde{y}(\theta) = e(\theta, \theta)$. The key to link the minimization of the quadratic norm of this signal to the optimization procedure is the following technical result, which is proven in [5].

**Lemma:** Let $\tilde{z}_i(\theta)$ be a first order Taylor expansion of $\tilde{y}(\theta)$ around $\theta_i$,

$$\tilde{z}_i(\theta) = \tilde{y}(\theta_i) + \tilde{y}'(\theta_i)(\theta - \theta_i). \tag{22}$$

Then the Gauss-Newton update $\theta_{i+1}$, with $\gamma_i = 1$, is the solution of the minimization problem

$$\theta_{i+1} = \arg\min_{\theta} \frac{1}{N} \sum_{1}^{N} \left[\tilde{z}_i^2(t, \theta)\right]. \tag{23}$$

■

Let us now derive an expression for $\tilde{z}_i(\theta)$. For the present case of disturbance rejection (*i.e.* $r \equiv 0$), we have:

$$\tilde{y}(\theta) = S_d(y(\theta) - G(\theta)u(\theta)) = \frac{1}{1 + G_0 C_y(\theta)} v = y(\theta),$$

with $C_y$ as given in (20). Simple manipulations then show that

$$\tilde{y}'(\theta) = S_0(\theta)(1 - S_0(\theta)) \frac{1}{G(\theta)} G'(\theta) v \tag{24}$$

Next notice that

$$G'(\theta_i)(\theta - \theta_i) \approx G(\theta) - G(\theta_i) \tag{25}$$

for $\theta$ close to $\theta_i$. Thus,

$$\tilde{y}'(\theta_i)(\theta - \theta_i) \approx S_0(\theta_i)(1 - S_0(\theta_i)) \frac{1}{G(\theta_i)} (G(\theta) - G(\theta_i)) v \tag{26}$$

which, after some simplifications, gives

$$\tilde{z}_i(\theta) \approx S_0(\theta_i) \left[ S_0(\theta_i) + G(\theta)(1 - S_0(\theta_i)) \frac{1}{G(\theta_i)} \right] v. \tag{27}$$

We now compare this expression with that of $e(\theta, \theta_i)$ in (21), and we observe that, if the achieved and desired sensitivity functions coincide, i.e.

$$S_0(\theta_i) = \frac{1}{1 + C_y(\theta_i)G(\theta_i)} = \frac{1}{1 + C_y(\theta_i)G_0} = S_d,$$

then (27) gives

$$\tilde{z}_i(\theta) \approx S_d\left(y(\theta_i) - G(\theta)u(\theta_i)\right) = e(\theta, \theta_i). \tag{28}$$

In view of (23) and (28), it follows that identifying a model $G(\theta)$ by minimizing $\sum_1^N (e(\theta, \theta_i))^2$ with $e(\theta, \theta_i)$ as in (21) is approximately equivalent to taking a Gauss-Newton step in the minimization of $J(\theta)$ provided the true sensitivity function $S_0(\theta_i)$ coincides with the designed $S_d$. We conclude that the least squares identification step used in the iterative $H_2$ identification and control schemes approximates the Gauss Newton step in the direct minimization scheme only if the present closed loop model is very close to the true closed loop system. A question that now remains is whether the Gauss-Newton step can be expressed as an identification step also in the case when the model does not coincide with the true system. We will thus attempt to formulate the minimization of $\sum_1^N \tilde{z}_i^2(\theta)$ as an identification problem no matter what $G(\theta_i)$ is. Denote the right hand side of (27) by $w_i(\theta)$,

$$w_i(\theta) = S_0^2(\theta_i)v + G(\theta)S_0(\theta_i)(1 - S_0(\theta_i))\frac{1}{G(\theta_i)}v. \tag{29}$$

The signals $S_0^2(\theta_i)v$ and $S_0(\theta_i)(1 - S_0(\theta_i))\frac{1}{G(\theta_i)}v$ can be obtained from two closed loop experiments. In the first experiment, the reference signal is zero, since we are doing disturbance rejection. In the second experiment one should use the output signal from the first experiment as reference signal. Let as usual superscript $j$ denote the experiment number, then

$$\begin{aligned} \bar{y}_i &\stackrel{def}{=} y_i^1 - y_i^2 = S_0^2(\theta_i)v_i^1 - S_0(\theta_i)v_i^2 && (30)\\ \bar{u}_i &\stackrel{def}{=} -\frac{1}{G(\theta_i)}y_i^2 = -(1 - S_0(\theta_i))S_0(\theta_i)\frac{1}{G(\theta_i)}v_i^1 \\ &- \frac{1}{G(\theta_i)}S_0(\theta_i)v_i^2. && (31) \end{aligned}$$

Thus, neglecting the disturbance in the second experiment, we have

$$\begin{aligned} \bar{y}_i &- G(\theta)\bar{u}_i = S_0^2(\theta_i)v_i^1 \\ &+ G(\theta)S_0(\theta_i)(1 - S_0(\theta_i))\frac{1}{G(\theta_i)}v_i^1 \qquad (32)\\ &- S_0(\theta_i)(\frac{G(\theta)}{G(\theta_i)} - 1)v_i^2 \approx w_i(\theta), \end{aligned}$$

and therefore, $\tilde{z}_i(\theta) \approx \bar{y}_i - G(\theta)\bar{u}_i$. This shows that the minimization of (23) can be interpreted as an identification problem. We observe that two experiments with different reference signals are needed. Furthermore, we have neglected the influence of the disturbance in the second experiment. Taking this term into account we have

$$\bar{y}_i - G(\theta)\bar{u}_i = w_i(\theta) + \frac{G(\theta_i) - G(\theta)}{G(\theta_i)}S_0(\theta_i)v_i^2.$$

Thus, if $\theta_{i+1}$ is chosen as the minimum of the quadratic norm of $\bar{y}_i - G(\theta)\bar{u}_i$, this new parameter will be biased towards the previous one, $\theta_i$, since the term

$$\mathrm{E}\left[\left(\frac{G(\theta_i) - G(\theta)}{G(\theta_i)}S_0(\theta_i)v_i^2\right)^2\right]$$

is minimized by $\theta = \theta_i$. New parameter values will tend to stick to old ones and this can cause the procedure to converge, not to the desired local minimum of $J(\theta)$, but to some other point. Thus, we conclude that it is better to use an explicit Gauss-Newton step instead of the identification based procedure presented in this section. With a direct Gauss-Newton step one also avoids having to solve an identification step which itself requires an iterative minimization procedure. Before we close this section let us point out that for design methods other than (20) the derivations are only approximate. Several additional terms are then involved in the expression (24).

## 7. Conclusions

We have compared two approaches to iterative controller design. It has been shown that the schemes proposed in [6] and [1] do not necessarily converge to a local minimum of the design criterion if the modeling error is non-zero. With the iterative optimization approach in [4] and [5], convergence to a local minimum does indeed take place under the assumption of boundedness of the signals in the loop. When the method from [4] is used in an indirect (model based) scheme, this approach becomes an iterative model update and control design procedure. With a Gauss Newton parameter update and a model reference control design procedure, we have shown that the model update step can be approximated by a least squares identification step, but with a bias error due to a disturbance. This identification step differs from the identification steps in *e.g.* [6] and [1], in the way that the least-squares criterion contains an additional term which is obtained from a second experiment. This term is the explanation why the optimization based method does converge regardless of the model error. This term vanishes when the achieved and desired loops are identical. Thus, under this condition the Gauss-Newton identification step becomes (essentially) identical to the corresponding identification steps in the algorithms of [6] and [1].

**References**

[1] Z. Zang, R.R. Bitmead. and M. Gevers. $H_2$ iterative model refinement and control robustness enhancement. In *Proc. of the 30th Conference on Decision and Control*, pages 279–284, Brighton, 1991.

[2] R.J.P. Schrama. *Approximate Identification and Control Design.* PhD thesis, Delft University of Technology, 1992.

[3] W.S. Lee, B.D.O. Anderson, R.L. Kosut, and I.M.Y Mareels. A new approach to adaptive robust control. *Int. Journal of Adaptive Control and Signal Processing*, 7:183–211, 1993.

[4] H. Hjalmarsson, S. Gunnarsson, and M. Gevers. A convergent iterative restricted complexity control design scheme. In *Proc. 33nd CDC*, Orlando, Florida, 1994.

[5] H. Hjalmarsson, S. Gunnarsson, and M. Gevers. Model free data driven optimal tuning of controller parameters. Technical report, Report LiTH-ISY-R-1680, Department of Electrical Engineering, Linköping University, Sweden, 1994.

[6] K.J. Åström. Matching criteria for control and identification. In *Proc. ECC*, pages 248–251, Groningen, The Netherlands, 1993.