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LQG control in the Youla parametrization[1]

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Abstract: Using the Youla parametrization of all two-parameter compensators stabilizing a given scalar plant, we provide a computational procedure for computing an optimal infinite horizon Linear Quadratic Gaussian (LQG) two-degree of freedom controller from any stabilizing controller without having to solve any Riccati equation. Our procedure is extended to the solution of the frequency weighted LQG problem, and to the more general problem of LQG control with a prescribed domain of stability.

1 Introduction

Suppose a scalar plant P, described as a proper rational transfer function, is specified, and that it is desired to stabilize P using some feedback compensator. If u and y denote the plant input and output, respectively, v is a disturbance signal and r denotes the external input, then the most general linear time invariant feedback controller stabilizing the system

$$y = Pu + v \tag{1.1}$$

is given by

$$u = C_1 r - C_2 y. \tag{1.2}$$

The equations that describe the closed loop system are

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} \frac{C_1}{1+PC_2} & -\frac{C_2}{1+PC_2} \\ \frac{PC_1}{1+PC_2} & \frac{1}{1+PC_2} \end{bmatrix} \begin{bmatrix} r \\ v \end{bmatrix}, \tag{1.3}$$

provided of course $1 + PC_2 \not\equiv 0$. Let

$$H(P, C_1, C_2) = \begin{bmatrix} \frac{C_1}{1+PC_2} & -\frac{C_2}{1+PC_2} \\ \frac{PC_1}{1+PC_2} & \frac{1}{1+PC_2} \end{bmatrix} \tag{1.4}$$

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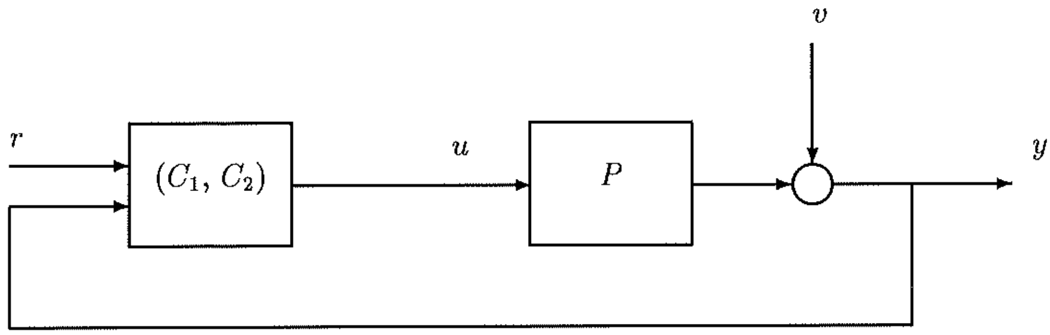

Figure 1.1: Two-degree of freedom control loop

denote the 2×2 transfer matrix from $[r\, v]^T$ to $[u\, y]^T$. Then we say that the triplet (P, C_1, C_2) is stable, and that the pair (C_1, C_2) stabilizes P, if and only if each of the four elements of $H(P, C_1, C_2)$ represents a stable system[2].

It is well known that if one stabilizing controller is available, then the set of all stabilizing controllers can be expressed as a function of the plant and of this initial controller using the so-called Youla parametrization: see e.g. [7]. This requires that the plant P and the initial controller $C_0 = [C_{10}\, C_{20}]$ both be factored as ratios of proper stable rational transfer functions.

In this paper, we shall exploit the Youla parametrization of all stabilizing two-degree of freedom compensators to provide a simple solution to a range of optimal Linear Quadratic Gaussian (LQG) control design problems, namely:

- the LQG tracking and/or disturbance rejection problem,
- the frequency weighted LQG problem,
- and the LQG problem with a prescribed degree of stability (closed loop poles must lie in some half plane $\{s : \mathrm{Re}\, s < -\alpha\}$ with α a positive real (in continuous time) or in a disk of radius r, say, with $0 < r < 1$ (in discrete time)), or with a prescribed domain of stability (closed loop poles must lie within a prescribed subdomain of the left half plane (in continuous time) or the unit disk (in discrete time)).

We first recall the basic results concerning Youla parametrizations for two-degree of freedom controllers [7]. All results apply for both the discrete and continuous time case and are expressed for scalar systems; the extension to the multivariable case is straightforward.
Let $P = N_P D_P^{-1}$ be a coprime factorization of P, where $N_P, D_P \in \mathbf{S}$, the ring of proper stable rational functions. In continuous time, a rational function belongs to $\mathbf{S}$ if and only if it has no poles in $\mathbf{C}_{+e}$, the extended right half plane; in discrete time if and only if it has no poles in $|z| \geq 1$. The pair N, D is said to be coprime in $\mathbf{S}$ if and only if either there exist X, $Y \in \mathbf{S}$ such that $XD + YN$ is a unit in $\mathbf{S}$ (i.e. its inverse belongs to $\mathbf{S}$) or, equivalently, if they have no common zeros in $\mathbf{C}_{+e}$ (in continuous time) or in $|z| \geq 1$ (in discrete time).

[2]It is clear that the present notion of stability requires all "internal" quantities to remain bounded if the input vector is bounded.

Let $C_0 = [C_{10}\, C_{20}] = [N_{C_1} N_{C_2}] D_C^{-1}$ be a coprime factorization of some two-degree of freedom controller C_0, where $N_{C_1}, N_{C_2}, D_C \in \mathbf{S}$. The coprimeness of D_C and $[N_{C_1} N_{C_2}]$ means that N_{C_1}, N_{C_2} and D_C have no common zeros in $\mathbf{C}_{+\mathbf{e}}$ (in continuous time) or in $|z| \geq 1$ (in discrete time). It does not imply that D_C and N_{C_1} are coprime; i.e. (D_C, N_{C_1}) need not be a coprime factorization of C_{10}. Similar remarks apply to the pair (D_C, N_{C_2}). It is now routine to verify that

$$H(P, C_{10}, C_{20}) = \frac{1}{N_P N_{C_2} + D_P D_C} \begin{bmatrix} D_P N_{C_1} & -D_P N_{C_2} \\ N_P N_{C_1} & D_P D_C \end{bmatrix}. \tag{1.5}$$

Theorem 1.1 [7] *The triplet (P, C_{10}, C_{20}) is stable if and only if $N_P N_{C_2} + D_P D_C$ is a unit of* $\mathbf{S}$ *(i.e. its inverse belongs to* $\mathbf{S}$*).* ■

Theorem 1.2 [7] *Let $P = N_P D_P^{-1}$ with $N_P, D_P \in \mathbf{S}$ and (N_P, D_P) coprime. Let (N_{C_2}, D_C) be any two elements of* $\mathbf{S}$ *such that the following Bezout equation holds*

$$N_P N_{C_2} + D_P D_C = 1. \tag{1.6}$$

Then the set (denoted $\mathcal{C}(R,S)$) of all two-parameter compensators that stabilize P is given by

$$\mathcal{C}(R,S) = \left\{ C_1 = \frac{R}{D_C + S N_P}, C_2 = \frac{N_{C_2} - S D_P}{D_C + S N_P} : R, S \in \mathbf{S},\ \textit{and}\ D_C + S N_P \not\equiv 0 \right\}. \tag{1.7}$$

This parametrization of all stabilizing two-degree of freedom controllers by two "Youla parameters" R and S is called the two-parameter Youla parametrization. ■

Comment: For $S = 0$ and $R = N_{C_1}$, we obtain $[C_{10}\, C_{20}] = [N_{C_1} N_{C_2}] D_C^{-1}$. It is clear that for $N_P N_{C_2} + D_P D_C$ to be a unit, (D_C, N_{C_2}) must be coprime. There is no constraint on N_{C_1} apart from being stable and proper, therefore it can be used as a Youla parameter. For all $C = [C_1\, C_2] \in \mathcal{C}(R,S)$,

$$N_P(N_{C_2} - S D_P) + D_P(D_C + S N_P) = N_P N_{C_2} + D_P D_C = 1.$$

It follows from Theorem 1.1 that C stabilizes P.

Remark: Condition (1.6) is a normalization assumption that can be relaxed by letting the second member of the equality be any unit transfer function. It is also possible to use a normalized coprime description of the plant (or the controller) by imposing an additional constraint of the type

$$|N_P|^2 + |D_P|^2 = 1 \quad \text{or} \quad |N_P|^2 + \lambda |D_P|^2 = 1 \ \text{ on the plant} \tag{1.8}$$

$$|N_{C_2}|^2 + |D_C|^2 = 1 \quad \text{or} \quad |N_{C_2}|^2 + \lambda |D_C|^2 = 1 \ \text{ on the controller} \tag{1.9}$$

However, it is only possible to use two of the three normalization assumptions (1.6), (1.8) and (1.9) at the same time: the Bezout assumption and a normalized coprime description of the plant or the controller, or, a normalized coprime description of both the plant and the controller.

The previous Theorem provides powerful tools. It says that, once we know one stabilizing controller for a plant, we can easily generate the family of all stabilizing two-degree of freedom controllers, by means of fractional representations. In this paper we use this parametrization to solve a control design problem where the design criterion is Linear Quadratic Gaussian (LQG). Our basic two-degree of freedom control loop is that of Figure 1.1, and our basic control design criterion is the following LQG index (expressed here in discrete time)

$$J_{LQG} = \lim_{N\to\infty} \frac{1}{N} E\left\{\sum_{t=1}^{N} \left\{[y_{t+d} - r_t]^2 + \lambda u_t^2\right\}\right\} \tag{1.10}$$

where d is the delay[3] in the plant, y_t is the plant output, u_t is the control signal designed to force the output signal y_t to track a given reference trajectory r_t as close as possible. We shall always assume $d \geq 1$. The signals r_t and v_t are, respectively, modelled as the output of a reference model and a noise model driven by independent white noise sequences.

Using the two-parameter Youla parametrization, we first solve the following problem: for a given plant, P, and the LQG criterion (1.10) we compute the optimal LQG controller as a function of an arbitrary stabilizing controller, $C_0 = [C_{10}\, C_{20}]$, and of the optimal Youla parameters, R_{opt} and S_{opt}, without having to solve a Riccati equation (Spectral factorization is however required). Our method for the computation of a two-degree of freedom controller is an extension of the one proposed in [4] when $C_1 = 1$: we show that the two-degree of freedom LQG control problem can be seen as the minimization of two separate costs, the disturbance rejection and the tracking contributions to the LQG cost. We then show that this computational procedure, which does not require the solution of a Riccati equation, can be readily applied to the solution of a frequency weighted optimal LQG problem.

Now, a major benefit of the Youla parametrization is that constraints on the closed loop poles can easily be imposed by replacing the usual stability domain over which the coprime factors of P and C_0 are expressed by more restricted domains. More precisely, if the coprime factors N_P, D_P of P, and the coprime factors $[N_{C_1} N_{C_2}]$ and D_C of C_0 have their poles in some restricted domain $\mathbf{D}$, and if, similarly, R and S have their poles in $\mathbf{D}$, then the set $\mathcal{C}(R,S)$ of (1.7) describes all two-degree of freedom controllers such that the closed loop systems have their poles in $\mathbf{D}$. We use this property to examine the possibility of solving the constrained LQG problem using the Youla parametrization. We show that an optimal solution does not exist in general, but that the infimum of the cost in the constrained optimization problem can be approximated as closely as desired using a sequence of controllers of increasing order that all achieve the constraint on the closed loop poles. A continuous time numerical example shows that a near optimal solution can often be obtained using very low order controllers. For the LQG design problem with a prescribed stability margin (i.e. all closed loop poles to the left of $-\alpha$), we show that our new computational procedure achieves a significantly lower cost than the "classical" approach of Anderson and Moore [2] that introduces an exponential weighting

[3]If a system is described by the following difference equation $A(q)y(k) = B(q)u(k)$ where q is the forward-shift operator, $A(z) = z^{n_a} + a_1 z^{n_a-1} + \cdots + a_{n_a}$ and $B(z) = z^{n_b} + b_1 z^{n_b-1} + \cdots + b_{n_b}$, then it can be rewritten in terms of the backward-shift operator as $y(k) + a_1 y(k-1) + \cdots + a_{n_a} y(k-n_a) = b_0 u(k-d) + \cdots + b_{n_b} u(k-d-n_b)$ where d, the delay in the system, is defined as $d = n_a - n_b$, the pole excess of the system, or relative degree of the system transfer function [3].

term into the original criterion.

The outline of our paper is as follows. In Section 2 we solve the LQG controller design problem in the Youla parametrization framework starting from the plant model P and any stabilizing controller C_0, using the set $\mathcal{C}(R,S)$ of all stabilizing two-degree of freedom controllers for P, i.e. we show how to compute $R_{\rm opt}$ and $S_{\rm opt}$. In Section 3, the problem of LQG control with a prescribed general domain of stability is tackled. We prove that the optimal cost achieved by the classical "unconstrained" controller can be approached as closely as desired by a sequence of high order controllers that put the closed loop poles in the domain of stability. In Section 4, we show with a numerical example that the classical method for the computation of an LQG controller with a prescribed stability margin [1]-[3] is far from optimal with respect to the original unmodified LQG cost and that by choosing an appropriate approximation of the Youla parameters, we can achieve a much lower LQG cost.

2 Optimal LQG control in the two-parameter Youla parametrization

Let $P = N_P D_P^{-1}$ and $C_{20} = N_{C_2} D_C^{-1}$ be coprime factorizations of the plant P and of the feedback part of an arbitrary stabilizing controller $C_0 = [C_{10}\, C_{20}]$, such that the Bezout equation (1.6) holds.
According to Theorem 1.2, the set of all two-degree of freedom controllers stabilizing P is given by

$$\mathcal{C}(R,S) = \left\{ C_1 = \frac{R}{D_C + SN_P}, C_2 = \frac{N_{C_2} - SD_P}{D_C + SN_P} : S, R \in \mathbf{S}, \text{ and } D_C + SN_P \not\equiv 0 \right\}. \quad (2.1)$$

Observe again that the stabilizing controller $C_0 = [C_{10}\, C_{20}]$ is obtained for the choice $R = N_{C_1}$, $S = 0$. Let $C = [C_1\, C_2]$ be any controller in the set $\mathcal{C}(R,S)$ defined above. The transfer functions corresponding to (1.3), with Bezout identity (1.6) holding, are now given by

$$\begin{aligned} u &= D_P R r - D_P(N_{C_2} - D_P S)\, v\,, \\ y &= N_P R r + D_P(D_C + N_P S)\, v. \end{aligned} \quad (2.2)$$

It is interesting to see that the Youla parameter R influences only the transfer functions from the reference signal to the output and the input signal, while the independent Youla parameter S influences only the transfer functions from the noise signal to the output and the input signal.

Using Parseval's theorem[4], we get an expression for the LQG index that is integral in R and S:

$$\begin{aligned} J_{LQG} &= \frac{1}{2\pi} \int \mathrm{d}\omega \Big\{ (|z^d N_P R - 1|^2 + \lambda |D_P R|^2)\phi_r \\ &\quad + (|D_C + N_P S|^2 + \lambda |N_{C_2} - D_P S|^2)|D_P|^2 \phi_v \Big\}\,. \end{aligned} \quad (2.3)$$

[4]The integration bounds have been omitted to stress the fact that the expressions are valid in both the continous ($\int_{-\infty}^{\infty}$) and discrete time case ($\int_{-\pi}^{\pi}$). The delay z^d in the expression of J_{LQG} and in all the corresponding expressions that will follow has to be discarded in the continuous time case.

If C_0 is the optimal LQG controller for P, then the choices $R = N_{C_1}$ and $S = 0$ must minimize J_{LQG} over all stable R and S.

We now consider that $C_0 = [C_{10}\, C_{20}]$ is an arbitrary stabilizing controller of P, and we compute the stable transfer functions R and S that minimize the previous LQG index.

The minimization of the LQG index with respect to all stable R and S is achieved by separately minimizing the first term with respect to all stable R and the second term with respect to all stable S. We therefore split the LQG index in two and consider each minimization separately.

$$\begin{aligned} J_{LQG} &= J_{tr} + J_{dr} \\ &= \frac{1}{2\pi}\int \mathrm{d}\omega \left\{ |z^d N_P R - 1|^2 + \lambda |D_P R|^2 \right\} \phi_r \\ &+ \frac{1}{2\pi}\int \mathrm{d}\omega \left\{ (|D_C + N_P S|^2 + \lambda |N_{C_2} - D_P S|^2)|D_P|^2 \right\} \phi_v \end{aligned}$$

J_{tr} is the tracking error contribution to the cost while J_{dr} is the disturbance rejection cost.

2.1 Minimization of J_{dr} with respect to all stable S

We first consider the minimization of J_{dr}[5].

$$\begin{aligned} \text{Integrand of } J_{dr} &= \Big\{ S^\star S[|N_P|^2 + \lambda |D_P|^2] + S[N_P D_C^\star - \lambda D_P N_{C_2}^\star] \\ &+ S^\star[N_P^\star D_C - \lambda D_P^\star N_{C_2}] + [|D_C|^2 + \lambda |N_{C_2}|^2] \Big\} |D_P|^2 \phi_v \end{aligned}$$

Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be defined as follows:

$$\mathcal{A}\mathcal{A}^\star = \left[|N_P|^2 + \lambda |D_P|^2 \right] |D_P|^2 \phi_v \tag{2.4}$$

$$\mathcal{B} = [N_P^\star D_C - \lambda D_P^\star N_{C_2}] |D_P|^2 \phi_v \tag{2.5}$$

$$\mathcal{C} = \left[|D_C|^2 + \lambda |N_{C_2}|^2 \right] |D_P|^2 \phi_v \tag{2.6}$$

where $\mathcal{A}$ is minimum phase, stable and of relative degree zero[6]. Then the integrand of J_{dr} is of the form

$$\begin{aligned} & S^\star S \mathcal{A}^\star \mathcal{A} + \mathcal{B}^\star S + S^\star \mathcal{B} + \mathcal{C} \\ = \; & [\mathcal{A}^\star S^\star + \mathcal{A}^{-1}\mathcal{B}^\star][\mathcal{A}S + \mathcal{A}^{-\star}\mathcal{B}] + \mathcal{C} - (\mathcal{A}^\star\mathcal{A})^{-1}\mathcal{B}^\star\mathcal{B}. \end{aligned}$$

Let $T \triangleq \mathcal{A}S$. Minimizing J_{dr} with respect to all stable S is equivalent to minimizing the following index with respect to all stable T:

$$\overline{J}_{dr} = \frac{1}{2\pi}\int \mathrm{d}\omega\, [T^\star + \mathcal{A}^{-1}\mathcal{B}^\star][T + \mathcal{A}^{-\star}\mathcal{B}] = \|T + \mathcal{A}^{-\star}\mathcal{B}\|_2^2 \tag{2.7}$$

[5]If $S = \frac{B}{A}$ of relative degree d, with A and B polynomials, then $S^\star$ is defined as $\frac{B(-s)}{A(-s)}$ in continuous time and as $\frac{z^d B^\star(z)}{A^\star(z)}$ in discrete time which ensures that $(S^\star)^\star = S$, and that $S^\star(e^{j\omega})$ is the complex conjugate of $S(e^{j\omega})$. $A^\star(z)$ is defined here as the reciprocal polynomial of $A(z)$, i.e. if $A(z) = z^{n_a} + a_1 z^{n_a - 1} + \cdots + a_{n_a}$, then $A^\star(z) = z^{n_a} A(z^{-1}) = 1 + a_1 z + \cdots + a_{n_a} z^{n_a}$.

[6]In the continuous time case, this is not always possible. This case is treated subsequently.

The minimizing T is clearly given by $-[\mathcal{A}^{-\star}\mathcal{B}]_{\text{st}}$ where $[\ ]_{\text{st}}$ denotes the stable part. Then

$$S_{\text{opt}} = -\mathcal{A}^{-1}[\mathcal{A}^{-\star}\mathcal{B}]_{\text{st}} \tag{2.8}$$

The optimal cost J_{dr}^{opt}, that corresponds to the optimal control cost in the disturbance rejection case, is

$$\begin{aligned} J_{dr}^{\text{opt}} &= \frac{1}{2\pi}\int \mathrm{d}\omega \left\{ |[\mathcal{A}^{-\star}\mathcal{B}]_{\text{unst}}|^2 + \mathcal{C} - (\mathcal{A}^\star\mathcal{A})^{-1}\mathcal{B}^\star\mathcal{B} \right\} \\ &= \frac{1}{2\pi}\int \mathrm{d}\omega \left\{ |[\mathcal{A}^{-\star}\mathcal{B}]_{\text{unst}}|^2 + \left[(|D_C|^2 + \lambda|N_{C_2}|^2) - \frac{|N_P^\star D_C - \lambda D_P^\star N_{C_2}|^2}{|N_P|^2 + \lambda|D_P|^2} \right] |D_P|^2\phi_v \right\} \\ &= \frac{1}{2\pi}\int \mathrm{d}\omega \left\{ |[\mathcal{A}^{-\star}\mathcal{B}]_{\text{unst}}|^2 + \frac{\lambda}{|N_P|^2 + \lambda|D_P|^2}|D_P|^2\phi_v \right\} \end{aligned} \tag{2.9}$$

Further simplifications occur in both the first and the second term of the integrand if the second normalization of (1.8) is applied. This is explored subsequently.

Remark: Every finite rational transfer functiom H can be decomposed into the sum of its stable and unstable part, $H = [H]_{\text{st}} + [H]_{\text{unst}}$, as follows. Expand H into partial fractions (unique decomposition) and a polynomial; then $[H]_{\text{st}}$ (respectively $[H]_{\text{unst}}$) is the sum of the terms corresponding to poles in the open left half plane (respectively in the closed right half plane) in continuous time and inside (respectively on or outside) the unit circle in discrete time. The improper[7] part of H is assigned to the unstable part. In the continous time decomposition of $\mathcal{A}^{-\star}\mathcal{B}$, it is necessary to take the unique solution with the constant part assigned to the stable part in order to make the cost (2.9) finite.

In discrete time, one can either assign the constant part to the stable or unstable part, or partly to the stable and the unstable part: all these solutions lead to a finite cost. Suppose we restrict attention to optimizing over all strictly proper controllers. We shall assume that the arbitrary stabilizing controller C_{20} is strictly proper, so that in the fractional representation of the controller $N_{C_2}D_C^{-1}$, N_{C_2} is strictly proper. Then the optimal controller, which has the structure $(N_{C_2} - D_P S)(D_C + N_P S)^{-1}$ for some stable S, must be strictly proper. This holds if and only if:

$$\begin{gathered} N_{C_2} - D_P S \text{ is strictly proper} \\ \Leftrightarrow \\ D_P S \text{ is strictly proper} \\ \Leftrightarrow \\ S \text{ is strictly proper} \end{gathered}$$

$S_{\text{opt}} = -\mathcal{A}^{-1}[\mathcal{A}^{-\star}\mathcal{B}]_{\text{st}}$ and the optimal controller will be strictly proper if and only if the constant term of $\mathcal{A}^{-\star}\mathcal{B}$ is assigned to $[\mathcal{A}^{-\star}\mathcal{B}]_{\text{unst}}$, i.e. $[\mathcal{A}^{-\star}\mathcal{B}]_{\text{st}}|_{z=\infty} = 0$ and there is a unique decomposition. If we optimize over all proper controllers, $[\mathcal{A}^{-\star}\mathcal{B}]_{\text{unst}}$ has to reflect just that part of the associated impulse $\{h_k\}$ corresponding to $k < 0$, so that $[\mathcal{A}^{-\star}\mathcal{B}]_{\text{unst}} = \sum_{k<0} h_k z^{-k}$. The constant term in the partial fraction expansion must be so partitioned between $[\mathcal{A}^{-\star}\mathcal{B}]_{\text{unst}}$ and $[\mathcal{A}^{-\star}\mathcal{B}]_{\text{st}}$ that $[\mathcal{A}^{-\star}\mathcal{B}]_{\text{unst}}$ has $z = 0$ as a zero, i.e. there is again a unique decomposition.

[7]The only case where $\mathcal{A}^{-\star}\mathcal{B}$ is improper occurs in the discrete time case for $\lambda = 0$.

2.1.1 Conditions for S_{opt} to be zero

We now consider the conditions on the coprime factors of P and C_{20} under which C_{20}, the feedback part of the controller, is optimal, i.e. the conditions under which $S_{\text{opt}} = 0$ is optimal. First, we note the following result.

Lemma 2.1 *Let X be minimum phase and stable. Then $[X^{-\star}Y]_{\text{st}} = 0$ if and only if $[Y]_{\text{st}} = 0$.*

Proof:

$$X^{-\star}Y = [X^{-\star}Y]_{\text{st}} + [X^{-\star}Y]_{\text{unst}}$$

Therefore,

$$Y = X^{\star}[X^{-\star}Y]_{\text{st}} + X^{\star}[X^{-\star}Y]_{\text{unst}}$$

If $[X^{-\star}Y]_{\text{st}} = 0$, then Y is the product of two unstable functions; hence, $[Y]_{\text{st}} = 0$.

On the other hand, $X^{-\star}Y = X^{-\star}[Y]_{\text{st}} + X^{-\star}[Y]_{\text{unst}}$. If $[Y]_{\text{st}} = 0$, then $X^{-\star}Y$ is the product of two unstable functions, and hence $[X^{-\star}Y]_{\text{st}} = 0$. ∎

As a consequence, $S_{\text{opt}} = 0$ if and only if

$$[\mathcal{B}]_{\text{st}} = [(N_P^{\star}D_C - \lambda D_P^{\star}N_{C_2})D_P^{\star}D_P M_{\phi_v}]_{\text{st}} = 0, \tag{2.10}$$

where M_{ϕ_v} is the minimum phase stable spectral factor of ϕ_v. The first equality results from (a second application of) Lemma 2.1.

If the plant P is **stable**, D_P is minimum phase and as a consequence of Lemma 2.1 this condition simplifies to:

$$[\mathcal{B}]_{\text{st}} = [(N_P^{\star}D_C - \lambda D_P^{\star}N_{C_2})D_P M_{\phi_v}]_{\text{st}} = 0 \tag{2.11}$$

Finally, we compute the optimal "disturbance rejection" LQG cost in the case where C_{20} is optimal. In such case, $S_{\text{opt}} = 0$, hence $[\mathcal{A}^{-\star}\mathcal{B}]_{\text{st}} = 0$, and therefore $[\mathcal{A}^{-\star}\mathcal{B}]_{\text{unst}} = \mathcal{A}^{-\star}\mathcal{B}$. It follows from (2.3) that

$$J_{dr}^{\text{opt}} = \frac{1}{2\pi}\int d\omega \left\{|D_C|^2 + \lambda|N_{C_2}|^2\right\}|D_P|^2\phi_v. \tag{2.12}$$

2.1.2 Normalization assumption

We now consider how the normalization assumptions can be used to simplify the expression of S_{opt}. If we assume that the Bezout identity (1.6) holds and that in addition

$$|N_P|^2 + \lambda|D_P|^2 = 1 \tag{2.13}$$

then, using $D_P D_C = 1 - N_P N_{C_2}$ and $-\lambda D_P^{\star}D_P = N_P^{\star}N_P - 1$, we obtain

$$D_P[N_P^{\star}D_C - \lambda D_P^{\star}N_{C_2}] = N_P^{\star} - N_P N_P^{\star}N_{C_2} - N_{C_2} + N_{C_2}N_P N_P^{\star} = N_P^{\star} - N_{C_2}.$$

This enables us to rewrite the previous equations:

$$\mathcal{A}\mathcal{A}^\star = |D_P|^2\phi_v = K^\star K \tag{2.14}$$

$$\mathcal{B} = (N_P^\star - N_{C_2})D_P^\star\phi_v = \frac{N_P^\star - N_{C_2}}{D_P}K^\star K \tag{2.15}$$

$$\mathcal{A}^{-\star}\mathcal{B} = (N_P^\star - N_{C_2})D_P^{-1}K \tag{2.16}$$

$$S_{\text{opt}} = -K^{-1}[(N_P^\star - N_{C_2})D_P^{-1}K]_{\text{st}} \tag{2.17}$$

where K is the minimum phase spectral factor of $|D_P|^2\phi_v$. If in addition P is stable, then $K = D_P M_{\phi_v}$ and

$$S_{\text{opt}} = -(D_P M_{\phi_v})^{-1}[(N_P^\star - N_{C_2})M_{\phi_v}]_{\text{st}} \tag{2.18}$$

The optimal control cost reduces to:

$$J_{dr}^{\text{opt}} = \frac{1}{2\pi}\int d\omega \left\{ \left|[(N_P^\star - N_{C_2})D_P^{-1}K]_{\text{unst}}\right|^2 + \lambda\,|D_P|^2\,\phi_v \right\} \tag{2.19}$$

Note: It is impossible here to do any normalization on the coprime factorization of the controller since we already assume that the Bezout equation (1.6) holds.

2.1.3 Relative degree zero constraint

The restriction on the relative degree of $\mathcal{A}$ being zero can always be met in the discrete time case since the spectral factor can always be multiplied by an arbitrary power of z. In the continuous time case, however, it is not always possible to find a spectral factor of relative degree zero. In particular, this is the case when the spectral quantity $[|N_P|^2 + \lambda|D_P|^2]\,|D_P|^2\phi_v$ is strictly proper. This depends on the relative degree of the noise spectrum and, for $\lambda = 0$, on the relative degree of the plant[8].

Consider the following approximation problem

$$\inf_{S\in\mathbf{S}} J(S) = \inf_{S\in\mathbf{S}} \|\mathcal{A}S - \mathcal{A}^{-\star}\mathcal{B}\|_2^2 \tag{2.20}$$

in the case where $\mathcal{A}$ is strictly proper, i.e. $\mathcal{A}^{-1} \notin \mathbf{S}$. It turns out that in such cases, the subspace $\{\mathcal{A}S : S \in \mathbf{S}\}$ is not closed in $\mathbf{S}$ for the $L_2(d\omega)$ norm [7]. As a result, there might not exist a closest point to $\mathcal{A}^{-\star}\mathcal{B}$ in this subspace. In such cases, the infimum of $J(.)$ is not attained by any $S \in \mathbf{S}$. It is nevertheless possible to compute the infimum of $J(.)$, and to construct a family $\{S_\epsilon \in \mathbf{S}\}$ such that $J(S_\epsilon)$ approaches the infimum as $\epsilon \longrightarrow 0^+$.

$$\min_S J(S) = \|[\mathcal{A}^{-\star}\mathcal{B}]_{\text{unst}}\|_2^2$$

and this minimum is achieved by $S_{\min} = -\mathcal{A}^{-1}[\mathcal{A}^{-\star}\mathcal{B}]_{\text{st}} \notin \mathbf{S}$.

The only possible $\mathbf{C}_{+e}$ zeros of spectral factor $\mathcal{A}$ are at infinity or on the $j\omega$-axis. Let

[8]Notice that in conventional LQG control [1], $\lambda = 0$ corresponds to a control weighting matrix that is singular while a strictly proper noise spectrum ϕ_v (no direct feedthrough term in the noise model) corresponds to a singular output noise covariance matrix in the state estimation problem.

$j\omega_1, \cdots, j\omega_l$ denote the $j\omega$-axis zeros of $\mathcal{A}$ with multiplicities $m_1, \cdots, m_l$, respectively, and suppose that $\mathcal{A}$ has relative degree d. Then

$$\mathcal{A}(s) = u(s) \prod_{i=1}^{l} \left(\frac{s - j\omega_i}{s+1}\right)^{m_i} \cdot \left(\frac{1}{s+1}\right)^d \tag{2.21}$$

for some unit $u(s) \in \mathbf{S}$. Construct a real rational family $b_\epsilon(s)$

$$b_\epsilon(s) = \prod_{i=1}^{l} \left(\frac{s - j\omega_i}{s + \epsilon - j\omega_i}\right)^{m_i} \cdot \left(\frac{1}{\epsilon s+1}\right)^d . \tag{2.22}$$

Note that $b_\epsilon(j\omega) \longrightarrow 1$ uniformly on all compact subsets of $(-\infty, \infty)$ that do not contain any of the ω_i when $\epsilon \longrightarrow 0^+$. The family

$$S_\epsilon = -\mathcal{A}^{-1} b_\epsilon [\mathcal{A}^{-\star}\mathcal{B}]_{\rm st} \in \mathbf{S}$$

approaches $S_{\rm min}$ as $\epsilon \longrightarrow 0^+$ and

$$J(S_\epsilon) = \|[\mathcal{A}^{-\star}\mathcal{B}]_{\rm unst}\|_2^2 + \|\mathcal{A}S_\epsilon - [\mathcal{A}^{-\star}\mathcal{B}]_{\rm st}\|_2^2 \longrightarrow \|[\mathcal{A}^{-\star}\mathcal{B}]_{\rm unst}\|_2^2 \quad \text{as} \quad \epsilon \longrightarrow 0^+.$$

The preceding procedure makes it also possible to cope with unit circle zeros of $\mathcal{A}$ in the discrete time case.

2.2 Minimization of J_{tr} with respect to all stable R

The minimization of J_{tr} with respect to all stable R is very similar to that of J_{dr} with respect to all stable S. The first part of the integrand of expression (2.3) can be rewritten as

$$\begin{aligned} &\left(R^\star R\{|N_P|^2 + \lambda |D_P|^2\} - z^d R N_P - z^{-d} R^\star N_P^\star + 1\right) \phi_r \\ =\ & [R^\star \mathcal{D}^\star - z^d \mathcal{D}^{-1} N_P\, \phi_r]\,[R\mathcal{D} - z^{-d}\mathcal{D}^{-\star} N_P^\star\, \phi_r] + \phi_r - (\mathcal{D}^\star\mathcal{D})^{-1} |N_P|^2 (\phi_r)^2 \end{aligned}$$

where $\mathcal{D}\mathcal{D}^\star = (|N_P|^2 + \lambda|D_P|^2)\phi_r$, $\mathcal{D}$ being minimum phase, stable and of relative degree zero[9]. Minimizing J_{tr} with respect to all stable R is equivalent to minimizing the following index:

$$\overline{J}_{tr} = \frac{1}{2\pi} \int d\omega\, [R^\star \mathcal{D}^\star - z^d \mathcal{D}^{-1} N_P\, \phi_r]\,[R\mathcal{D} - z^{-d}\mathcal{D}^{-\star} N_P^\star\, \phi_r] = \|R\mathcal{D} - z^{-d}\mathcal{D}^{-\star} N_P^\star\, \phi_r\|_2^2 \tag{2.23}$$

The minimizing R is now clearly given by

$$R_{\rm opt} = \mathcal{D}^{-1}[z^{-d}\mathcal{D}^{-\star} N_P^\star \phi_r]_{\rm st} \tag{2.24}$$

The optimal cost $J_{tr}^{\rm opt}$ is given by

$$\begin{aligned} J_{tr}^{\rm opt} &= \frac{1}{2\pi}\int d\omega \left\{ |[z^{-d}\mathcal{D}^{-\star} N_P^\star \phi_r]_{\rm unst}|^2 + \phi_r - (\mathcal{D}^\star\mathcal{D})^{-1}|N_P|^2(\phi_r)^2 \right\} \\ &= \frac{1}{2\pi}\int d\omega \left\{ |[z^{-d}\mathcal{D}^{-\star} N_P^\star \phi_r]_{\rm unst}|^2 + \frac{\lambda |D_P|^2}{|N_P|^2 + \lambda|D_P|^2}\phi_r \right\} \end{aligned} \tag{2.25}$$

[9]Problems can again occur in continuous time; they can be solved by the technique used in subsection 2.1.3.

3 LQG control with a prescribed domain of stability

Quite often, the objective of control system design is not merely to stabilize a given plant P but to place the closed loop poles in some pre-specified region of stability. In continuous time applications, it is often desirable to replace the open left half-plane by a more restricted domain of stability $\mathbf{D}$ which is open and has the property of being symmetric with respect to the real axis. Similarly, in the case of discrete time systems, it may be desired to restrict the closed loop poles to a subset of the unit disk.

Thus, given P and a domain of stability $\mathbf{D}$, the problem is to parametrize all compensators such that the closed loop transfer matrix $H(P, C_1, C_2)$ has all its poles in the prescribed domain of stability. It can be shown [7] that Theorem 1.2 carries over in toto if $\mathbf{S}$, the ring of all proper stable functions, is replaced by $\mathbf{S_D}$, the ring of all proper transfer functions with poles in the domain of stability $\mathbf{D}$.

Theorem 3.1 *Let $P = N_P D_P^{-1}$ with $N_P, D_P \in \mathbf{S_D}$ and (N_P, D_P) coprime[10] in $\mathbf{S_D}$. Let (N_{C_2}, D_C) be any two elements of $\mathbf{S_D}$ such that the following Bezout equation holds*

$$N_P N_{C_2} + D_P D_C = 1. \tag{3.1}$$

Then the set (denoted $\mathcal{C}_\mathbf{D}(R,S)$) of all two parameter compensators such that the closed loop transfer function $H(P, C_1, C_2)$ has all its poles in $\mathbf{D}$ is given by

$$\mathcal{C}_\mathbf{D}(R,S) = \left\{ C_1 = \frac{R}{D_C + SN_P}, C_2 = \frac{N_{C_2} - SD_P}{D_C + SN_P} : R, S \in \mathbf{S_D},\ \textit{and}\ D_C + SN_P \not\equiv 0 \right\} \tag{3.2}$$

■

This parametrization provides interesting tools to compute an LQG controller that guarantees a prescribed degree of stability and/or a minimum damping ratio for the closed loop system. Typical domains of stability for continuous time applications are

$$\mathbf{D} = \{ s : \mathrm{Re}\, s < -\sigma,\ \ |\mathrm{Im}\, s| \leq \tan\theta \, |\mathrm{Re}\, s| \ , \ \sigma > 0 \}. \tag{3.3}$$

If the poles of the closed loop system lie in $\mathbf{D}$, the step response of the compensated system exhibits a settling time of no more than $\frac{4}{\sigma}$ and a maximum overshoot corresponding to the angle θ.

One case that is treated in the literature is the case of LQG control with a prescribed degree of stability [1, 3]. The domain of stability (for continuous time applications) is of the type

$$\mathbf{D} = \{ s : \mathrm{Re}\, s < -\alpha,\ \ \alpha > 0 \}, \tag{3.4}$$

i.e. it is required that the closed loop poles lie to the left of $-\alpha$. In [2] a solution to this problem was proposed by minimizing the following modified criterion (expressed here in continuous time[11]):

$$J_{\text{mod}} = \int_{t_0}^{\infty} e^{2\alpha t} \left([y(t) - r(t)]^2 + \lambda [u(t)]^2 \right) dt. \tag{3.5}$$

[10] The pair N, D is said to be coprime in $\mathbf{S_D}$ iff

- either there exist $X, Y \in \mathbf{S_D}$ such that $XD + YN$ is a unit in $\mathbf{S_D}$ (i.e. its inverse belongs to $\mathbf{S_D}$).
- or equivalently, they have no common zeros in the complement of $\mathbf{S_D}$ (extended to the point at infinity).

[11] The discrete time case can be tackled in a similar way.

This solution is also discussed in [1, 3]. The strategy that is adopted in solving this modified problem is to introduce transformations that convert the problem to a regulator problem of the type considered in Section 2, with signals that are redefined as follows:

$$\hat{y}(t) = e^{\alpha t}y(t),\ \ \hat{u}(t) = e^{\alpha t}u(t),\ \ \hat{r}(t) = e^{\alpha t}r(t)\ \text{ and }\ \hat{v}(t) = e^{\alpha t}v(t). \tag{3.6}$$

Using the properties of the Laplace transform[12], this corresponds to shifting the poles and zeros of N_P, D_P, N_{C_2}, D_C (defined as in Theorem 3.1), the reference and the noise model by α, i.e. replacing s by $s-\alpha$. The input-output relations (2.2) are thus replaced by

$$\begin{aligned} \hat{u} &= \hat{D}_P\hat{R}\hat{r} - \hat{D}_P(\hat{N}_{C_2} - \hat{D}_P\hat{S})\,\hat{v} \\ \hat{y} &= \hat{N}_P\hat{R}\hat{r} + \hat{D}_P(\hat{D}_C + \hat{N}_P\hat{S})\,\hat{v} \end{aligned} \tag{3.7}$$

where $\hat{X}(s) = X(s-\alpha)$. Let $\hat{R}_{\text{opt}}$ and $\hat{S}_{\text{opt}}$ denote the optimal solutions obtained by the procedures of Section 2 for this modified problem. We then have

$$\begin{aligned} u(t) &= e^{-\alpha t}\hat{u}(t) \quad \text{and} \quad u = D_P R_{\text{opt}}\, r - D_P(N_{C_2} - D_P S_{\text{opt}})\, v \\ y(t) &= e^{-\alpha t}\hat{y}(t) \quad \text{and} \quad y = N_P R_{\text{opt}}\, r + D_P(D_C + N_P S_{\text{opt}})\, v \end{aligned} \tag{3.8}$$

where $R_{\text{opt}} = \hat{R}_{\text{opt}}(s+\alpha)$ and $S_{\text{opt}} = \hat{S}_{\text{opt}}(s+\alpha)$.

It can easily be seen that the minimization of (3.5) is performed by use of the transformation (i.e. the translation) that maps the domain of stability **D** onto the open left half-plane. By performing this transformation on the plant, the controller and the reference and noise model, we can use the minimization method of Section 2 with respect to all stable S and R, where stability is now understood in the classical sense. The optimal controller for the control design problem associated with (3.5) is then obtained by performing the inverse transformation on the optimal solution of the converted minimization problem.

The method proposed above, as well as the ones proposed in [1]- [3], make it possible to compute a controller which achieves a closed-loop system with a prescribed degree of stability α by minimizing the modified criterion (3.5). The resulting controller will have the same order as the optimal controller corresponding to the minimization of the original "unconstrained" control index:

$$J_{LQG} = \int_{t_0}^{\infty} \left([y(t) - r(t)]^2 + \lambda[u(t)]^2\right) dt. \tag{3.9}$$

However, the controller obtained by this method is not optimal with respect to control index (3.9) subject to the constraint of producing closed loop poles to the left of $-\alpha$, i.e. there will exist other controllers of the same order that achieve a lower cost and satisfy the closed loop constraints. The reason for this is that the formulation of the solution differs from:

$$\inf_{C_1,C_2} \left\{ \int_{t_0}^{\infty} \left([y(t) - r(t)]^2 + \lambda[u(t)]^2\right) dt. \right\} \tag{3.10}$$

subject to

$$\begin{cases} u = C_1(s)\, r + C_2(s)\, y \ \ \underline{\text{and}} \\ \text{closed loop poles are in } \mathbf{D}. \end{cases} \tag{3.11}$$

[12] $F(s-\alpha) \longleftrightarrow e^{\alpha t} f(t)$ where $F(s)$ is the Laplace transform of $f(t)$.

We will illustrate this by an example in Section 4.

In addition, the method described above can not be applied to an LQG control problem with a generalized domain of stability $\mathbf{D}$ which is open and symmetric w.r.t. the real axis. The reason for this is that the Schwarz-Christoffel transformation that maps such a generalized domain of stability $\mathbf{D}$ onto the open left half plane is not rational in most cases of interest.

We now show that the computational procedure described in this paper can be used to solve the constrained optimization problem (3.10)-(3.11). Thus, we consider a general domain of stability $\mathbf{D}$, open and symmetric w.r.t. the real axis, and the ring $\mathbf{S_D}$ of all proper rational transfer functions with poles in $\mathbf{D}$. With N_P, D_P, N_{C_2} and D_C defined as in Theorem 3.1, we minimize the LQG index (2.3) directly w.r.t. all R, $S \in \mathbf{S_D}$. The minimization of J_{dr} w.r.t. all $S \in \mathbf{S_D}$ is again considered first, and we treat the continuous time case; the discrete time case is very similar.
If $\mathcal{A}$ and $\mathcal{B}$ are defined as in (2.4) and (2.5), it is shown in subsection 2.1 that an equivalent minimization problem is given by

$$\inf_{S \in \mathbf{S_D}} \frac{1}{2\pi} \int \mathrm{d}\omega \left\{ [\mathcal{A}^\star S^\star + \mathcal{A}^{-1}\mathcal{B}^\star][\mathcal{A}S + \mathcal{A}^{-\star}\mathcal{B}] \right\} \tag{3.12}$$

$\mathcal{A}$ having all its poles and zeros in the open left half plane, it can always be factored as a product $\mathcal{A}_1\mathcal{A}_2$ where $\mathcal{A}_1$ has relative degree zero and has all its zeros in $\mathbf{D}$. The problem (3.12) is then equivalent with the minimization problems:

$$\inf_{S \in \mathbf{S_D}} \frac{1}{2\pi} \int \mathrm{d}\omega \left\{ [\mathcal{A}_1 S + \mathcal{A}_2^{-1}(\mathcal{A}_1\mathcal{A}_2)^{-\star}\mathcal{B}]^\star \mathcal{A}_2\mathcal{A}_2^\star [\mathcal{A}_1 S + \mathcal{A}_2^{-1}(\mathcal{A}_1\mathcal{A}_2)^{-\star}\mathcal{B}] \right\} \tag{3.13}$$

$$\inf_{S \in \mathbf{S_D}} \|\mathcal{A}_1 S - [-(\mathcal{A}_2\mathcal{A}_2^\star)^{-1}\mathcal{A}_1^{-\star}\mathcal{B}]\|^2_{L_2(\mathcal{A}_2\mathcal{A}_2^\star \mathrm{d}\omega)} \tag{3.14}$$

Thus, the minimization problem corresponds to finding the best approximant in the $L_2(\mathcal{A}_2\mathcal{A}_2^\star\mathrm{d}\omega)$ sense of $[-(\mathcal{A}_2\mathcal{A}_2^\star)^{-1}\mathcal{A}_1^{-\star}\mathcal{B}] \in L_2(\mathcal{A}_2\mathcal{A}_2^\star\mathrm{d}\omega)$ by a rational function $\mathcal{A}_1 S$ with S constrained to be in $\mathbf{S_D}$.
It turns out, however, that $\mathbf{S_D}$ is not closed in H_2 and a fortiori not in $L_2(\mathcal{A}_2\mathcal{A}_2^\star\mathrm{d}\omega)$[13]. As a result there might not exist a closest point to $[-(\mathcal{A}_2\mathcal{A}_2^\star)^{-1}\mathcal{A}_1^{-\star}\mathcal{B}]$ in $\mathbf{S_D}$ in the $L_2(\mathcal{A}_2\mathcal{A}_2^\star\mathrm{d}\omega)$ sense. It can nevertheless be shown using Runge's Theorem [6] that $\mathbf{S_D}$ is dense in H_2 for the $L_2(\mathrm{d}\omega)$ norm, thus also for $L_2(\mathcal{A}_2\mathcal{A}_2^\star\mathrm{d}\omega)$ norm. In other words, S_{opt}, which in general does not belong to $\mathbf{S_D}$, can be approximated arbitrarily well in the $L_2(\mathcal{A}_2\mathcal{A}_2^\star\mathrm{d}\omega)$ norm by some sequence $\{S_n \in \mathbf{S_D}\}$ such that $J_{LQG}(S_n)$ converges to the infimum $J_{LQG}^{\inf}$ of $J_{LQG}(S)$ on $\mathbf{S}$ as $n \longrightarrow \infty$.

[13] In other words, it can be shown that a sequence of functions $f_n \in \mathbf{S_D}$ that converges in the $L_2(\mathcal{A}_2\mathcal{A}_2^\star\mathrm{d}\omega)$ sense, does not necessarily converge to a function $f \in \mathbf{S_D}$.
Rephrasing this question mathematically, we obtain:
$\exists f_n \in \mathbf{S_D}$ such that $\|f_n - f\|_{L_2(\mathcal{A}_2\mathcal{A}_2^\star\mathrm{d}\omega)} \longrightarrow 0$, with $f_n \in L_2(\mathcal{A}_2\mathcal{A}_2^\star\mathrm{d}\omega)$ and $f \notin \mathbf{S_D}$.

3.1 A computational procedure for the LQG problem with a prescribed domain of stability

In the light of the previous considerations, we propose the following solution for the suboptimal LQG control problem with a prescribed domain of stability.

Find a coprime factorization (N_P, D_P) of P in $\mathbf{S_D}$. Select (N_{C_2}, D_C), two elements in $\mathbf{S_D}$, such that Bezout identity (3.1) holds. The set of all two-degree of freedom controllers that stabilize P with closed loop poles in $\mathbf{D}$ is given by $\mathcal{C}_{\mathbf{D}}(R,S)$ defined in (3.2).

By the analysis of subsection 2.1 and the results of Theorem 3.1, the minimization of the LQG criterion (3.9) under the constraint of closed loop poles in $\mathbf{D}$ is equivalent with the minimization of the expression (2.3) over all R, $S \in \mathbf{S_D}$. We first consider the minimization of (2.3) over all R, $S \in \mathbf{S}$ and denote the optimal solution R_{opt}, S_{opt}. We have shown in subsection 2.1 that R_{opt} and S_{opt} are the solution of the following minimization problem:

$$\min_{S,R\in\mathbf{S}} \overline{J}_{LQG}(R,S) \tag{3.15}$$

where

$$\overline{J}_{LQG}(R,S) = \overline{J}_{\text{tr}}(R) + \overline{J}_{\text{dr}}(S) = \|\mathcal{A}S - [-\mathcal{A}^{-\star}\mathcal{B}]\|_2^2 + \|\mathcal{D}R - z^{-d}\mathcal{D}^{-\star}N_P^\star\phi_r\|_2^2 \tag{3.16}$$

R_{opt}, S_{opt} and $\overline{J}_{LQG}^{\text{inf}}$, the minimizing value resulting from (3.15), are given by

$$\begin{aligned} R_{\text{opt}} &= \mathcal{D}^{-1}[z^{-d}\mathcal{D}^{-\star}N_P^\star\phi_r]_{\text{st}}, & (3.17)\\ S_{\text{opt}} &= -\mathcal{A}^{-1}[\mathcal{A}^{-\star}\mathcal{B}]_{\text{st}} \text{ and} & (3.18)\\ \overline{J}_{LQG}^{\text{inf}} &= \overline{J}_{LQG}^{\text{opt}} = \|[\mathcal{A}^{-\star}\mathcal{B}]_{\text{unst}}\|_2^2 + \|[z^{-d}\mathcal{D}^{-\star}N_P^\star\phi_r]_{\text{unst}}\|_2^2, & (3.19)\end{aligned}$$

where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ are defined as in Section 2 (in particular $\mathcal{A}$, $\mathcal{B}$, $\mathcal{D} \in \mathbf{S}$ and $N_P \in \mathbf{S_D}$), while $[\,]_{\text{st}}$ and $[\,]_{\text{unst}}$ refer to the usual definition of stable and unstable part used in Section 2. The LQG criterion (3.16) can be rewritten in the following form:

$$\begin{aligned} \overline{J}_{LQG}(R,S) = {} & \|[\mathcal{A}^{-\star}\mathcal{B}]_{\text{unst}}\|_2^2 + \|\mathcal{A}S - [-\mathcal{A}^{-\star}\mathcal{B}]_{\text{st}}\|_2^2 \\ & + \|[z^{-d}\mathcal{D}^{-\star}N_P^\star\phi_r]_{\text{unst}}\|_2^2 + \|\mathcal{D}R - [z^{-d}\mathcal{D}^{-\star}N_P^\star\phi_r]_{\text{st}}\|_2^2 \end{aligned} \tag{3.20}$$

We now consider the minimization of $\overline{J}_{LQG}$, as expressed in (3.20), w.r.t. all $R, S \in \mathbf{S_D}$. Since $\mathbf{S_D}$ is dense in H_2 for the $L_2(\mathrm{d}\omega)$ norm, it is possible to contruct two families of transfer functions $\{R_n \in \mathbf{S_D}\}$ and $\{S_n \in \mathbf{S_D}\}$ that converge, respectively, to R_{opt} and S_{opt} in the $L_2(\mathrm{d}\omega)$ sense when $n \longrightarrow \infty$. It can easily be seen from (3.20) that $\overline{J}_{LQG}(R_n, S_n)$ will converge to $\overline{J}_{LQG}^{\text{inf}}$ when $n \longrightarrow \infty$.

We have thus shown that one can construct a family of two-degree of freedom controllers $\{(C_{1_n}, C_{2_n})\}$ that all stabilize P in the sense that all closed loop poles lie in $\mathbf{D}$. The achieved cost $\overline{J}_{LQG}(R_n, S_n)$ corresponding to (C_{1_n}, C_{2_n}), converges to $\overline{J}_{LQG}^{\text{inf}}$ given by (3.19), the value of the cost that is achieved by $(C_1^{\text{opt}}, C_2^{\text{opt}})$ obtained with R_{opt} and S_{opt}.

A constructive procedure: It may very well happen that R_{opt} and S_{opt} belong to $\mathbf{S_D}$. In such case, the two-degree of freedom controller $(C_1^{\text{opt}}, C_2^{\text{opt}})$ is the optimal solution to the constrained problem. In the general case where R_{opt} and S_{opt} do not belong to $\mathbf{S_D}$, it is necessary to factorize them in a factor that belongs to $\mathbf{S_D}$ and one that belongs to $\mathbf{S}\backslash\mathbf{S_D}$. One can always take the numerator of the latter factor to be unity. This part is then approximated by a function in $\mathbf{S_D}$. This is done in the following way. Denote $S_{\text{opt}}^{\text{unst}}$ the factor of S_{opt} with poles in $\mathcal{C}_-\backslash\mathbf{D}$, i.e. the complement of $\mathbf{D}$ in the left half plane. Then $S_{\text{opt}}^{\text{unst}}$ can be expanded as follows:

$$S_{\text{opt}}^{\text{unst}}(s) = \frac{1}{a_n s^n + \cdots + a_1 s + a_0} = \sum_{j=n}^{\infty} \frac{p_j}{(s+p)^j}. \tag{3.21}$$

where $-p$ is a pole in $\mathbf{D}$. To perform this expansion, use the change of variable $z = \frac{1}{s+p}$, expand $S_{\text{opt}}^{\text{unst}}(z)$ around $z = 0$, and do the inverse transformation. By truncating this series, one obtains an approximation of $S_{\text{opt}}^{\text{unst}}(z)$ at any level of accuracy. The approximant for S_{opt} is then found by multiplication of the factor in $\mathbf{S_D}$ and the approximant of the factor in $\mathbf{S}\backslash\mathbf{S_D}$. The degree of the controller that results from this procedure depends on the number of terms that are needed to obtain a reasonable fit in the approximation of the "**D**-unstable" part and is typically high.

In practice, we are always interested in a low order controller, i.e. we are looking for low order "$\mathbf{S_D}$-approximants" of the optimal Youla parameters. The low order "$\mathbf{S_D}$ approximants" obtained by truncation of (3.21) are in general far from optimal w.r.t. the constrained LQG problem (3.10)-(3.11). It is therefore better to compute low order "$\mathbf{S_D}$ approximants" of the optimal Youla parameters by solving a constrained minimization of the following type:

$$\inf_{\alpha} \|\mathcal{A}\left(S_{\text{opt}} - S_{\text{approx}}(\alpha)\right)\|_2^2 \tag{3.22}$$

under the constraint that all the poles of $S_{\text{approx}}(\alpha)$ lie in the domain of stability $\mathbf{D}$. $S_{\text{approx}}(\alpha)$ has the desired McMillan degree and structure and α is a parameter vector (zeros, poles, $\cdots$). One can add additional constraints such as imposing an identical static or high frequency gain.

Note: The preceding constructice procedure can also be used to tackle the problem of H_∞ optimal control in a prescribed domain of stability $\mathbf{D}$. Let $K(s)$ be a stabilizing H_∞ controller and let $L(s)$ be any controller that achieves the pole contraints but does not necessarilly meet the H_∞ constraint. Using a Youla-parametrization based on fractional representations of the plant and of $L(s)$, which are "**D**-stable", $K(s)$ can be described in terms of a Youla parameter $S(s)$ ($\notin \mathbf{S_D}$). Then $S(s)$ is approximated (arbitrarily close) by $S_{\text{approx}}(s)$ which has domain of stability $\mathbf{D}$.

4 Numerical example

To illustrate the methods proposed earlier, let us take a system described by the s-domain transfer function

$$P(s) = \frac{1}{s-2}. \tag{4.1}$$

We propose to compute an LQG controller for this plant with as design parameters $\lambda = 0.001$, $\phi_v = 1$ (i.e. a flat noise spectrum) and $\phi_r = \frac{\omega^2+1}{\omega^2+10^{-8}}$. This corresponds to using as reference model

$$R(s) = \frac{s+1}{s+0.0001}. \tag{4.2}$$

In a first step, we compute the optimal controller without any constraint on the domain of stability using our method of Section 2 and then we show how the method proposed in Section 3 can be used to solve the problem with the following domain of stability

$$\mathbf{D} = \{s : \text{Re } s \leq -2\}. \tag{4.3}$$

Consider the following coprime factorization of P:

$$N_P = \frac{1}{s+3} \qquad D_P = \frac{s-2}{s+3}. \tag{4.4}$$

Note that (N_P, D_P) are also coprime in $\mathbf{S_D}$. By solving the Bezout identity (1.6), we obtain the following stabilizing controller:

$$N_C = \frac{25}{s+3} \qquad D_C = \frac{s+8}{s+3}. \tag{4.5}$$

This controller is stabilizing in the classical sense and in the restricted sense since the closed loop pole is -3 with a multiplicity of two.

We first solve the unconstrained LQG problem using the formulas of Section 2. This yields:

$$S_{\text{opt}} = \frac{-109.74\,(s+1.69)}{(s+2)\,(s+31.69)} \quad \text{and} \quad R_{\text{opt}} = \frac{31.56\,(s+3)}{(s+1)\,(s+31.69)}. \tag{4.6}$$

We note that $R_{\text{opt}} \notin \mathbf{S_D}$ because it contains an "unstable" pole at -1. The transfer function of the corresponding closed loop system is:

$$T_{yr}(s) = N_P R_{\text{opt}} = \frac{31.56}{(s+1)\,(s+31.69)}. \tag{4.7}$$

The upper curves (with unbroken line) in Figure 4.1 and Figure 4.2 show, respectively, the magnitude of the closed transfer function and the closed loop step response. We now turn to the solution of the constrained problem. The optimal control cost $\overline{J}^{\text{inf}}_{LQG}$ that corresponds to the infimum value of the cost in the constrained case can be computed using expression (3.19); its value is 37.11. The solution of the constrained problem is now found by approximating the "unstable" factor of R_{opt} by a function in $\mathbf{S_D}$.

$$R^{\text{unst}}_{\text{opt}} = \frac{1}{s+1} = \sum_{j=1}^{\infty} \frac{p_j}{(s+2)^j} \quad \text{where } p_j = 1\ \forall j. \tag{4.8}$$

By truncating this series, we obtain the following "$\mathbf{S_D}$ approximants" of R_{opt}:

$$R^1 = \frac{31.56\,(s+3)}{(s+2)\,(s+31.69)},\quad R^2 = \frac{31.56\,(s+3)^2}{(s+2)^2\,(s+31.69)} \text{ and } R^3 = \frac{31.56\,(s+3)\,(s^2+5s+7)}{(s+2)^3\,(s+31.69)}.$$

n	1	2	3	4	5	6	7	8
$\mathcal{J}^n_{LQG}$	1282	348.3	114.9	56.56	41.97	38.32	37.41	37.19

Table 4.1: LQG controls costs achieved by the controllers corresponding to an "$\mathbf{S_D}$ approximants of $R_{\rm opt}$ truncated at the n*th* term.

If we plug the expressions of R^1, R^2 and R^3 in the controller, we obtain the following expressions for the closed loop transfer function:

$$T^1_{yr}(s) = \frac{31.56}{(s+2)\,(s+31.69)},\ T^2_{yr}(s) = \frac{31.56\,(s+3)}{(s+2)^2\,(s+31.69)} \text{ and } T^3_{yr}(s) = \frac{31.56\,(s^2+5s+7)}{(s+2)^3\,(s+31.69)}.$$

The dashed-dotted lines in Figure 4.1 and Figure 4.2 show Bode plots and step responses of the closed loop system corresponding to different approximants of $R_{\rm opt}$. By taking a sufficient number of terms in the approximation of $R_{\rm opt}$, we approach the behaviour of the controller that achieves the infimum control cost: the controlled system has a time constant corresponding to a pole at -1, but has all its poles in $\mathbf{D}$. The corresponding control costs can be computed using expression (3.16); see Table 4.1 and recall that $\mathcal{J}^{\rm inf}_{LQG} = 37.11$. If we now look at the minimization of the modified criterion (3.5) with $\alpha = 2$, we obtain the following Youla parameters:

$$S^{\rm mod}_{\rm opt} = \frac{-261.99\,(s+4.77)}{(s+6)\,(s+33.87)} \quad \text{and} \quad R^{\rm mod}_{\rm opt} = \frac{29.52}{(s+33.87)} \tag{4.9}$$

and the corresponding closed loop transfer function:

$$T^{\rm mod}_{yr}(s) = N_P R^{\rm mod}_{\rm opt} = \frac{29.52}{(s+3)\,(s+33.87)}. \tag{4.10}$$

The dotted line in Figure 4.3 shows the corresponding closed loop step response. The LQG control cost (3.16) for this controller is 2536.9. We note from Table 4.1 that even with a first order approximant in $\mathbf{S_D}$ of the unconstrained $R_{\rm opt}$, our new technique achieves a better cost than the classical approach using the modified criterion (3.5). If we take $\alpha = 1.5$ in (3.5), we still achieve a degree of stability of 2 and the LQG control cost (3.16) decreases to 1531.8.

It is clear that the controller based on the modified cost (3.5) is far from optimal w.r.t. the constrained LQG problem (3.10)-(3.11) since it is possible to compute a controller with a degree of stability of 2 that achieves a lower cost by computing a first order approximant of $R_{\rm opt}$; see Table 4.1. Is it possible to do any better with a controller of the same McMillan degree ? Yes, if we take as Youla parameter

$$R_{\rm approx} = \frac{31.56\,(s+6)}{(s+2)\,(s+31.69)}, \tag{4.11}$$

corresponding to

$$T^{\rm approx}_{yr} = \frac{31.56\,(s+6)}{(s+2)\,(s+3)\,(s+31.69)}, \tag{4.12}$$

the LQG cost (3.16) drops drastically to 37.18, which is very close to $\overline{J}_{LQG}^{\text{inf}} = 37.11$. Note that R_{opt} and R_{approx} have the same static gain and that

$$\lim_{s \longrightarrow \infty} s(R_{\text{opt}} - R_{\text{approx}}) \longrightarrow 0. \tag{4.13}$$

The dashed line in Figure 4.3 shows the step response of T_{yr}^{approx}. This value of the LQG cost $\overline{J}_{LQG}$ can only be achieved by taking the controller corresponding to a nine term approximation in (4.8) and is much lower than the cost achieved by the controller computed using the modified criterion (3.5).

5 Conclusions

We have presented a method allowing the computation of a two-degree of freedom LQG controller that does not require the solution of a Riccati equation, thus allowing an easy implementation on a computer. The method is based on the Youla parametrization of all stabilizing two-degree of freedom controllers, and consists in the separate minimization of the disturbance and the tracking contribution to the LQG cost. We have shown that this method can easily handle the design of frequency weighted LQG controllers and LQG controllers in a prescribed domain of stability. A numerical example shows that, in the case of LQG control with a prescribed degree of stability, the results obtained by our method are far less conservative than the ones based on the classical procedure of exponential weighting of the integrand of the LQG criterion.

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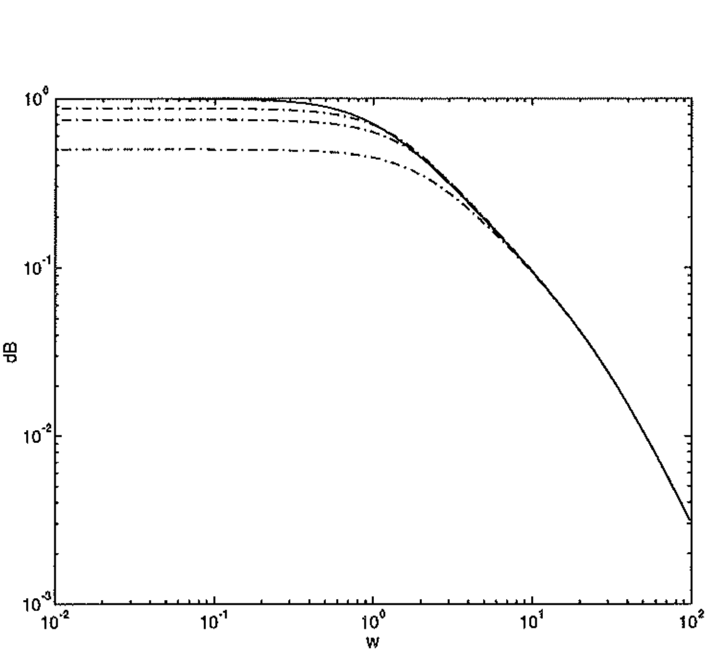

Figure 4.1: Frequency response of the optimal closed loop transfer function $T_{yr}(s)$ (—) and of the closed loops transfer functions $T^1_{yr}(s)$, $T^2_{yr}(s)$ and $T^3_{yr}(s)$ (-·-·), from bottom to top.

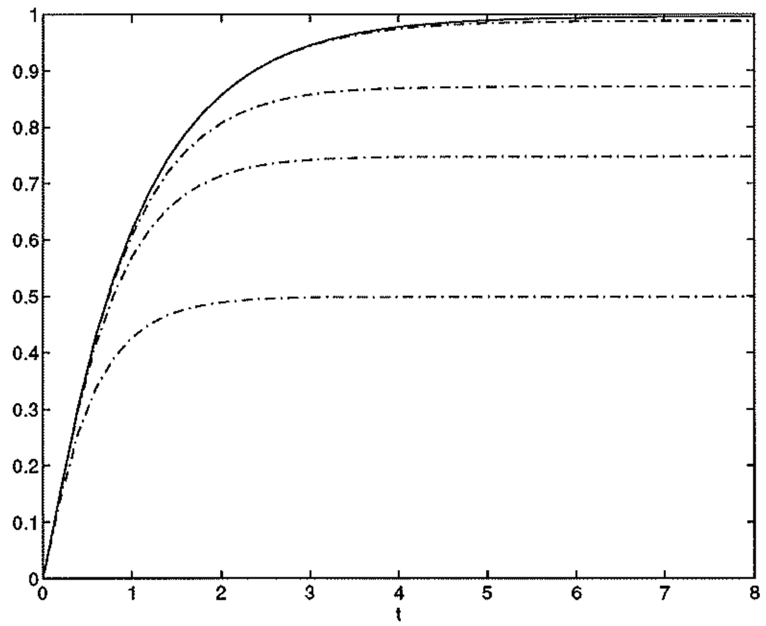

Figure 4.2: Step responses of the optimal closed loop transfer function $T_{yr}(s)$ (—) and of the closed loop transfer functions $T^1_{yr}(s)$, $T^2_{yr}(s)$, $T^3_{yr}(s)$ and $T^7_{yr}(s)$ (-·-·), from bottom to top.

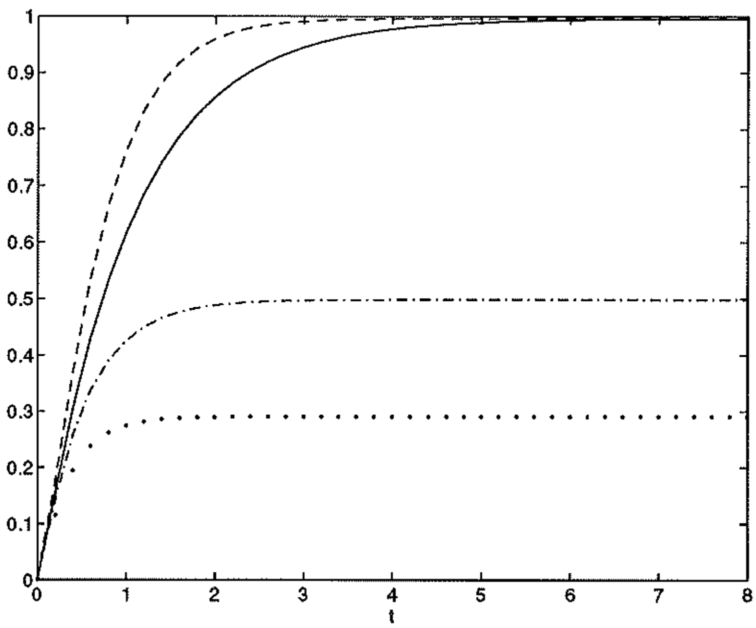

Figure 4.3: Step response of the optimal closed loop transfer function $T_{yr}(s)$ (—) and of the closed loop transfer functions $T^1_{yr}(s)$ (-·-·), $T^{\mathrm{mod}}_{yr}(s)$ ($\cdots$) and $T^{\mathrm{approx}}_{yr}(s)$ (- - -).