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When is a model good for control design ?[1]

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Abstract

The use of an identified model for the design of a feedback controller for an actual plant introduces strictures on the quality of the model which are different from those pertaining in open loop identification. For example, a model $\hat{P}$ is admissible for the design of a controller for the actual plant P only if the pair $(\hat{P}, P)$ is simultaneously stabilizable. This paper addresses the question of the quality of a model to be used for control design, by analysing the interplay between the plant P, the designed closed loop system T and the set of admissible models $\{\hat{P}\}$. For given P and T we characterize the set of admissible models $\{\hat{P}\}$, where admissible means that a controller designed from $\hat{P}$ and T yields a stable closed loop. Necessary conditions on (P, T) are derived for this set to be nonempty.

1 Introduction

It has been the conventional wisdom that a model need not necessarily be a very accurate description of the true system for it to deliver a high performance controller. The important feature is that the model should describe with high precision the dynamical characteristics that are essential for control design.

This observation naturally leads one to ask the question: *What is a 'good' model for control design?* It is commonly acknowledged that the model should be accurate around the cross-over frequency of the closed loop system to be designed. To go beyond such common sense rules and to obtain a precise characterization of all models that are 'good for control design' turns out to be a difficult question.

One reasonable definition of a 'good' model is if:

(i) the controller derived from that model stabilizes the actual plant;

(ii) this controller achieves a performance on the actual plant that is close to the performance achieved on the model.

The qualification of a 'good' model depends on two ingredients: the unknown plant and the control design criterion. Thus, the question of qualifying 'good' models for control design is a question that involves three players: the unknown plant, the control design criterion, and the set of models available for consideration. Understanding the connections between these three players is fundamental in addressing the question "When is a model good for control design?" In addition, the solution to the question "What are the good models $\{\hat{P}\}$ for a given P and a given criterion?" forces one to examine the compatibility of the design criterion with the known features of the plant P. As we shall reveal, there are combinations of plants and control criteria for which the set of models that satisfy the 'goodness' criterion (i) above is empty. Before we proceed further, we introduce some notations.

Notations

$T(s)$ is the designed complementary sensitivity function, or reference model, defined by $T = \frac{\hat{P}C}{1+\hat{P}C}$.[1]

$S(s)$ is the associated designed sensitivity function $S = 1 - T = \frac{1}{1+\hat{P}C}$.

$\mathcal{R}H_\infty$ is the set of stable, proper transfer functions.

$R(s)$ is the set of all real transfer functions in s.

$R_{+\infty}$ is the extended positive real axis in the s-plane.

$C_{+\infty}$ is the extended right half plane $\{s : Re(s) \geq 0\} \cup \{\infty\}$.

Stable, stabilizing loops refer to the internal stability of the loop, i.e. the stability of all four transfer

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[1]For ease of notation we will from now on drop the argument s from all transfer functions except when we won't.

functions in

$$H(P,C) = \frac{1}{1+PC}\begin{pmatrix} PC & P \\ C & 1 \end{pmatrix}.$$

Unstable zero of a transfer function $G(s)$ is a value α with $Re(\alpha) \geq 0$ where $G(\alpha) = 0$.

Unstable one of transfer function $G(s)$ is a value α with $Re(\alpha) \geq 0$ where $G(\alpha) = 1$.

$\delta(G)$ represents the relative degree of transfer function $G(s)$, i.e. the denominator degree minus the numerator degree.

Proper transfer function $G(s)$ is one where $\delta(G) \geq 0$.

Biproper transfer function $G(s)$ is one where both G and G^{-1} are proper, i.e. $\delta(G) = 0$.

Bistable transfer function $G(s)$ where both G and G^{-1} are stable.

The problem of assessing the quality of models for control design is hard. To make it revealing and tractable we will adopt the simplest possible framework.

- The system is noise-free and is single-input/single-output;
- A model reference control design is used.

In addition, our contribution will for the most part be limited here to the robust stability problem (property (i) above). We shall offer some tools for the study of the performance question (property (ii)). Thus, we consider the situation where there is a 'true' system with transfer function P, a stable reference model T and a model reference control design criterion which, for the model $\hat{P}$, computes a corresponding controller $C(\hat{P},T)$ from $\frac{\hat{P}C}{1+\hat{P}C} = T$. Thus,

$$C(\hat{P},T) = \frac{T}{(1-T)\hat{P}} = \frac{T}{S}\frac{1}{\hat{P}}. \qquad (1)$$

There are of course compatibility constraints between $\hat{P}$ and the admissible T, because the nominal closed loop system $H(\hat{P},C)$ must be stable. This means that the product $\hat{P}C$ cannot contain any unstable pole-zero cancellations. It requires that T must be zero at the unstable zeros of $\hat{P}$, and that T must be one at the unstable poles of $\hat{P}$.[2] If these two interpolation constraints are satisfied, then the controller $C(\hat{P},T)$ defined by (1) is stabilizing. For such a controller to be proper, the relative degree of T must be larger than or equal to that of $\hat{P}S$. We then ask the question: "What is the set of models $\mathcal{P} \triangleq \{\hat{P}\}$ for which the corresponding controllers $C(\hat{P},T)$ stabilize the true system P ?" We shall call such models **stabilizing models**.

In [GBB97] we showed that the existence of stabilizing models imposes necessary conditions on the connection between the pole-zero pattern of P and the zero-one pattern of T on the extended positive real axis $R_{+\infty}$. With some simplification, our results of [GBB97] can be summarized as follows.

- The problem of characterizing a controller that stabilizes the plant P and achieves a designed closed loop transfer function T is equivalent to the problem of stabilizing the plant $\frac{1}{PT}$ by a controller that has no unstable poles and zeros except at finitely many possible specific right half plane locations. The difficulty of this problem is essentially equivalent to that of stabilizing a plant by a *bistable controller*, for which no tractable necessary and sufficient conditions are known: see [BG93], [Blon94]. However, useful necessary conditions are known under which a plant is stabilizable by a bistable controller. These led us to show the following.
- The solution set $\{\hat{P}\}$ of stabilizing models is non empty only if:
  - ⋆ T has an unstable $R_{+\infty}$ -zero between any pair of $R_{+\infty}$ -poles of P between which P has an uneven number of $R_{+\infty}$ -zeros, and
  - ⋆ T has an unstable $R_{+\infty}$ -one between any pair of $R_{+\infty}$ -zeros of P between which P has an uneven number of $R_{+\infty}$ -poles.
- These necessary conditions for the existence of stabilizing models are automatically satisfied, and are also sufficient, in the following practically relevant cases:
  - ⋆ P has no unstable poles (i.e. P is stable);
  - ⋆ P has no unstable zeros;
  - ⋆ P has at most one unstable zero and one unstable pole.

These results show that 'difficult plants' P put constraints on the set of admissible nominal closed loop systems T for there to exist a nonempty set of stabilizing models $\hat{P}$. It is well known that plants with right half plane poles and zeros pose specific constraints on the *achievable* closed loop performance. However, we do not know of any results connecting unstable poles and zeros of P with the *designed* closed loop performance in the context of modelling for control.

The question addressed here is that of characterizing the plant models $\{\hat{P}\}$ that are stabilizing for a given plant P, in that the controllers designed on the basis of a $\hat{P}$ stabilize the true P. It is related to, but significantly harder than, the question of characterizing the set of plants $\{P\}$ that are stabilized by a controller $C(\hat{P},T)$ designed on the basis of a model $\hat{P}$ and a reference model T. This last question is easily solved using the dual Youla parametrization [HGK89].

Our new contributions in this paper are twofold. We provide a much more direct derivation of the results

[2] In this paper poles, zeros and ones are always to be taken with their multiplicity included.

of [GBB97]. In addition, we provide two alternative parametrizations of all stabilizing models that are much simpler than those derived in [GBB97] based on Youla and dual Youla parametrizations. One of these is given explicitly in terms of the designed and achieved closed loop transfer functions, thereby providing a grip on the robust performance question raised above.

Our exposé will unveil as follows. In Section 2 we state the problem of characterizing all stabilizing plant models for a given plant and a given model reference control design. Section 3 contains our main result: two alternative parametrizations of all stabilizing models. In Section 4 we present necessary conditions on the plant and on the chosen reference model T for the solution set of stabilizing models to be non-empty. Section 5 illustrates our results with some examples, and we show in Section 6 how our parametrization gives some handle on the performance question.

2 Statement of the problem

We present our results in the continuous-time domain, but they can be transposed without difficulty to discrete-time systems. Throughout the paper we shall make the following assumption which is generically satisfied.

Genericity Assumption:
The unstable poles of P are not zeros of the designed T, and the unstable zeros of P are not zeros of $S \triangleq 1-T$.

The problem addressed in this paper can then be formulated as follows. We shall call it 'boxed problem #1'[3], and give equivalent, more technical formulations, later.

Boxed problem #1
Given a proper plant P and a stable, proper reference model T satisfying the genericity assumption, characterize the set $\mathcal{P} = \{\hat{P}\}$ of all plant models for which there exists a controller $C(\hat{P},T)$ such that the following three conditions hold:

(A) $C(\hat{P},T)$ satisfies $\frac{\hat{P}C}{1+\hat{P}C} = T$;

(B) $C(\hat{P},T)$ stabilizes $\hat{P}$;

(C) $C(\hat{P},T)$ stabilizes P.

We shall call $\mathcal{P}$ the set of all *stabilizing models.*

Comments

1. Condition (A) above determines the diagonal elements of $H(\hat{P},C)$, while (B) implies that the off-diagonal elements must also be stable.
2. The problem statement involves three players, P, $\{\hat{P}\}$ and T, with $\{C(\hat{P},T)\}$ being just a function of the latter two.
3. Conditions (B) and (C) indicate that our problem can be viewed in the framework of simultaneous stabilization of two plants, a problem for which tractable necessary and sufficient conditions exist (see e.g. [YBJ76], [Vid85]). However, condition (A) complicates things considerably, as we shall discover.

[3] One denotes the problem as '#1' rather than as '1' because one is careful not to confuse it with a 'one' as in 'unstable one,' which one might equally write 'one.'

3 Parametrizations of all 'stabilizing models'

Our first theorem gives a parametrization of all solutions of boxed problem # 1.

Theorem 1 *Let $N, D \in \mathcal{R}H_\infty$ be factors of a coprime factorisation of P in $\mathcal{R}H_\infty$, i.e. $P = ND^{-1}$ with N and D proper, stable and having no common unstable zeros in $\boldsymbol{C}_{+\infty}$. The set of stabilizing models $\{\hat{P}\}$ of Boxed Problem # 1 is given by*

$$\mathcal{P} = \{\hat{P} : \hat{P} = \frac{T_1}{S_1}\} \tag{2}$$

where $T = T_1T_2$ and $S = S_1S_2$ are any factorizations of T and S with $T_1, T_2, S_1, S_2 \in \mathcal{R}H_\infty$ for which

$$NT_2 + DS_2 = 1. \tag{3}$$

With these notations we then have

$$C(\hat{P},T) = \frac{T_2}{S_2}. \tag{4}$$

Proof:
Since $S = 1-T$, and S and T are stable, they are necessarily coprime in $\mathcal{R}H_\infty$. By condition (A) of Boxed Problem # 1 any solution pair $(\hat{P}, C(\hat{P},T))$ must satisfy the following relationship:

$$\hat{P}C(\hat{P},T) = \frac{T}{1-T} = \frac{T}{S}. \tag{5}$$

Therefore $\hat{P}$ and $C(\hat{P},T)$ are necessarily of the form $\hat{P} = \frac{T_1}{S_1}$ and $C(\hat{P},T) = \frac{T_2}{S_2}$, where $\frac{T_1T_2}{S_1S_2}$ is a decomposition of $\frac{T}{S}$ with $T_1, T_2, S_1, S_2 \in \mathcal{R}H_\infty$. In addition, by condition (B) this factorization must be coprime in $\mathcal{R}H_\infty$; indeed, if T_1T_2 and S_1S_2 contained a common unstable zero, then $T_1T_2 + S_1S_2$ would contain an unstable zero, and the nominal loop $H(\hat{P}, C(\hat{P},T))$ would be unstable. Since S and T are coprime in $\mathcal{R}H_\infty$, conditions (A) and (B) imply that $\hat{P}$ and $C(\hat{P},T)$ are given by (2) and (4), where $T = T_1T_2$ and $S = S_1S_2$ are any factorizations of T and S with $T_1, T_2, S_1, S_2 \in \mathcal{R}H_\infty$. The stability of the closed loop system $H(P, C(\hat{P},T))$ additionally imposes that the Bezout equation (3) be satisfied (see e.g. [Vid85]). ■

The following theorem gives a characterization of the set of stabilizing models that relates more clearly the designed and achieved closed loop transfer functions.

Theorem 2 *The set of stabilizing models $\{\hat{P}\}$ of Boxed Problem # 1 is given by*

$$\mathcal{P} = \{\hat{P} : \hat{P} = \frac{T}{1-T} \times \frac{1-Q}{Q} \times P\} \qquad (6)$$

where Q is any proper stable transfer function satisfying the following constraints: for any $s \in C_{+\infty}$,

$$\begin{array}{l} (a)\ P(s) = 0 \Rightarrow Q(s) = 0 \Rightarrow \{P(s) = 0 \text{ or } T(s) = 0\} \\ (b)\ P(s) = \infty \Rightarrow Q(s) = 1 \Rightarrow \{P(s) = \infty \text{ or } T(s) = 1\}. \end{array} \qquad (7)$$

Proof:
Let $\hat{P}$ be a stabilizing model. Then the associated controller is $C(\hat{P},T) = \frac{T}{1-T} \times \frac{1}{\hat{P}}$. Let $P = \frac{N}{D}$, with N and D in $\mathcal{R}H_\infty$ be a coprime factorization of P. Since C stabilizes P, there exists a coprime factorization $C = \frac{N_c}{D_c}$, with N_c and D_c in $\mathcal{R}H_\infty$, such that $N_cN + D_cD = 1$ [Vid85]. Now define $Q = N_cN$ and observe that $\frac{Q}{1-Q} = \frac{N_cN}{D_cD} = PC$. Therefore $\hat{P} = \frac{T}{1-T} \times \frac{1-Q}{Q} \times P$, where Q is proper and stable and satisfies (a) and (b).

Conversely, suppose that Q is a proper stable transfer function that satisfies (a) and (b), and let $P = \frac{N}{D}$ be a coprime factorisation of P with N and D in $\mathcal{R}H_\infty$. By (a) we have $Q = NT_2$ with $T = T_1T_2$ and by (b) we have $1 - Q = DS_2$ with $1 - T = S_1S_2$ for some T_1, T_2, S_1, S_2 in $\mathcal{R}H_\infty$. Define $C \triangleq \frac{T_2}{S_2}$ and $\hat{P} \triangleq \frac{T_1}{S_1}$, and observe that $\hat{P}C = \frac{T}{S}$. Therefore condition (A) of Boxed Problem # 1 is satisfied. Note that $NT_2 + DS_2 = 1$, which shows that the controller C thus defined stabilizes P [Vid85], and hence condition (C) is also satisfied. Condition (B) is trivially satisfied because $S_1S_2 + T_1T_2 = 1$. Finally, it follows from $\hat{P} = \frac{T_1}{S_1} = \frac{T}{1-T}\frac{S_2}{T_2}$, $Q = NT_2$ and $1 - Q = DS_2$ that $\hat{P}$ can also be expressed as in

$$\hat{P} = \frac{T}{1-T} \times \frac{1-Q}{Q} \times P. \qquad (8)$$

■

Comments

1. Denoting the unstable zeros of a transfer function P by *uz(P)* and the unstable ones by *uo(P)* for brevity, we note that an alternative characterization of conditions (a) and (b) is as follows:

$$\begin{array}{ll} (a) & \{\mathrm{uz}(P)\} \subseteq \{\mathrm{uz}(Q)\} \subseteq \{\mathrm{uz}(P)\} \cup \{\mathrm{uz}(T)\} \\ (b) & \{\mathrm{uo}(P)\} \subseteq \{\mathrm{uo}(Q)\} \subseteq \{\mathrm{uo}(P)\} \cup \{\mathrm{uo}(T)\}. \end{array}$$

2. The parametrization of (8) shows that for given P and T any choice of a stable proper Q that satisfies the interpolation constraints (a) and (b) yields a solution $\hat{P}$. The set of solutions is empty when no stable proper Q exists that satisfies these interpolation constraints: see examples in Section 5.

3. An important advantage of this new parametrization is that the free parameter Q is precisely the achieved complementary sensitivity function:

$$\frac{PC(\hat{P},T)}{1+PC(\hat{P},T)} = NT_2 = Q. \qquad (9)$$

Therefore the distance between the designed and achieved closed loop transfer function can be written as $Q - T$.

The following corollary is an immediate consequence.

Corollary 1 *Assume that T has no unstable zero and no unstable one. Then there exists a stabilizing model $\hat{P}$ for the pair (P,T) if and only if there exists a decomposition $P = \frac{N}{D}$ with $N, D \in \mathcal{R}H_\infty$ such that*

$$N(s) = 1 \text{ for } s \in C_{+\infty} \Longleftrightarrow D(s) = 0.$$

4 Necessary conditions on the reference model

The conditions of Theorems 1 and 2 are necessary and sufficient. However, in both cases the interpolation conditions (3) and (7) make the solution to our problem very hard. Consider, for example, condition (3). By the genericity assumption, we can regard (D, NT_2) as a coprime factorization of the plant $\frac{1}{PT_2}$. The interpolation condition then translates into a stabilization problem of $\frac{1}{PT_2}$ by a stable controller, S_2, whose unstable zeros (if any) may only be unstable ones of T. This is essentially a 'bistable stabilization problem' for $\frac{1}{PT_2}$ for which there exist necessary conditions for the existence of a solution S_2, which we now develop for some special cases.

The reference model T has no unstable zeros

In this case Theorem 1 states that the system $\frac{D}{N} = \frac{1}{P}$ must be stabilizable by a *stable controller* $\frac{S_2}{T_2}$ whose zeros can only be at specific and isolated locations (i.e. the unstable ones of T). A necessary condition is that $\frac{1}{P}$ obeys the *parity interlacing property*[4], i.e. P must have an even number of $R_{+\infty}$-zeros between any two poles on $R_{+\infty}$ (see e.g. [Blon94]). Since there are constraints on the unstable zeros of $\frac{S_2}{T_2}$, the parity interlacing property of $\frac{1}{P}$ is necessary but not sufficient for the existence of a solution to (3).

[4] Well known in control engineering circles as the PIP.

The reference model T has no unstable zeros and no unstable ones

In this case $\frac{1}{P}$ must be stabilizable by a stable and inverse stable controller $\frac{S_2}{T_2}$. This is known as the *bistable stabilization* problem. A necessary condition for a plant P to be stabilized by a bistable controller is that P has the *even interlacing property* [BG93], i.e. P must have an even number of $R_{+\infty}$ -poles between any pair of $R_{+\infty}$ -zeros of P, and an even number of $R_{+\infty}$ -zeros between any pair of $R_{+\infty}$ -poles.

Analysis of the general case

In the situation where the designed closed loop transfer function T can have both unstable zeros and unstable ones, the following necessary conditions can be derived.

Theorem 3 *[GBB97] The following are necessary conditions for Boxed Problem #1 to have a solution.*

1. *If P has 2 or more poles on $R_{+\infty}$, then T must have a $R_{+\infty}$ -zero between any pair of $R_{+\infty}$ -poles of P between which P has an odd number of zeros.*
2. *If P has 2 or more zeros on $R_{+\infty}$, then T must have a $R_{+\infty}$ -one between any pair of $R_{+\infty}$ -zeros of P between which P has an odd number of poles.*

Proof: see [GBB97].

Special cases of interest

The following are special cases of practical interest for which the necessary conditions are automatically satisfied: (i) P has no unstable poles; (ii) P has no unstable zeros; (iii) P has at most one unstable zero and one unstable pole. It is shown in [Blon94] that these conditions are then also sufficient for the existence of a bistable stabilizing controller $\frac{S_2}{T_2}$ of $\frac{1}{P}$, and therefore the solution set $\mathcal{P}$ is nonempty.

5 Examples

We present a series of examples that illustrate the main results of our paper. We start with a 'difficult' plant, namely one that has more than one unstable pole and more than one unstable zero.

Example 1
Consider the 'plant' $P = \frac{s-1}{(s-2)(s+1)}$, and a model reference control design based on a nominal model, with reference model $T = \frac{s^2+6}{(s+2)(s+3)}$. Observe that the plant has two $R_{+\infty}$ -zeros (at $s = 1$ and $s = +\infty$) with only one $R_{+\infty}$ -pole in between them. Therefore, by Theorem 3 the solution set $\mathcal{P} = \{\hat{P}\}$ is empty.

Example 2
We consider the same plant P as in Example 1, but now we modify $T(s)$ so that it contains an unstable 'one' on $(1, +\infty)$: $T = \frac{10s+6}{(s+2)(s+3)}$. Observe that $T(5) = 1$. We now use the result of Theorem 2 to construct a stabilizing model $\hat{P}$ for P that will yield a nominal closed loop transfer function T. We need to construct a $Q(s)$ such that $Q(1) = Q(\infty) = 0$, with the added constraint that $Q(2) = 1$. If $Q(s) = 1$ elsewhere on $C_{+\infty}$, then $P(s) = \infty$ or $T(s) = 1$ there. Observe that $Q(s) = \frac{9(s-1)}{(s+1)^2}$ obeys all these constraints; in addition $Q(5) = 1 = T(5)$. This choice of $Q(s)$ yields the following solutions for $\hat{P}$ and $C(\hat{P}, T)$: $\hat{P} = \frac{(10s+6)}{9s(s+1)}$ and $C = \frac{9(s+1)}{s-5}$. Note that $\frac{\hat{P}C}{1+\hat{P}C} = T$ and $\frac{PC}{1+PC} = Q$, and that there are no unstable pole-zero cancellations in forming the products $\hat{P}C$ and PC.

Example 3
Finally, we present an example of a 'well-behaved plant'. Take $P = \frac{1}{s-2}$, $T = \frac{1}{s+1}$. Here P has an unstable zero at $+\infty$ and an unstable pole at $s = 2$, while T has an unstable zero at $+\infty$ and an unstable one at $s = 0$. By Theorem 2 Q must be stable with a zero at $+\infty$ (with possible multiplicity two), and a one at $s = 2$ and possibly at $s = 0$, but nowhere else in $C_{+\infty}$. $Q = \frac{3}{s+1}$ satisfies these requirements and leads to a solution $\hat{P} = \frac{1}{3s}$ and $C = 3$. For $\alpha > 2$, $Q = \frac{\alpha}{s+(\alpha-2)}$ also satisfies the requirements. This leads to a set $\{\frac{1}{\alpha s} : \alpha > 2\}$ of stabilizing models. Note that the controller $C(\hat{P}, T)$ associated to $\hat{P} = \frac{1}{\alpha s}$ is given by $C = \alpha$. The set $\mathcal{P} = \{\frac{1}{\alpha s} : \alpha > 2\}$ is the set of stabilizing models leading to a proportional controller.

6 Performance considerations

Our analysis of 'good models' for control has so far concentrated solely on the robust stability question. A reasonable request for a 'good model' for control is that it should also possess some robust performance qualities. We could formally state this as the following modification of the earlier stability problem.

Boxed problem #2
Given a proper plant P and a stable, proper reference model T satisfying the genericity assumption, and given some positive number ϵ, characterize the set $\mathcal{P} = \{\hat{P}\}$ of all plant models for which there exists a controller $C(\hat{P}, T)$ such that the following four conditions hold:

(A) $C(\hat{P}, T)$ satisfies $\frac{\hat{P}C}{1+\hat{P}C} = T$;

(B) $C(\hat{P}, T)$ stabilizes $\hat{P}$;

(C) $C(\hat{P}, T)$ stabilizes P.

(D) $||\frac{PC}{1+PC} - T|| < \epsilon$

Any norm can be used in (D). We shall call a model $\hat{P}$ that satisfies properties (A) to (D) an *ϵ-stabilizing model.* The questions we want to address are:

- For given ϵ, under what conditions does a pair (P, T) have ϵ-stabiling models?
- Can we parametrize all such models?
- What is the smallest ϵ, if any, for which this set is non-empty?

The following result is an immediate consequence of the formulation of Boxed Problem # 2, and of Theorem 2.

Theorem 4 *The set of ϵ-stabilizing models $\{\hat{P}\}$ of Boxed Problem # 2 is given by (6), where Q is any proper stable transfer function satisfying the following constraints:*
(1) for any $s \in C_{+\infty}$

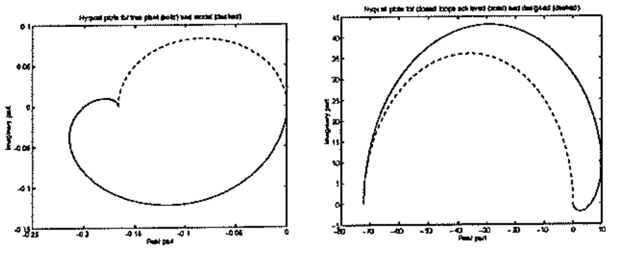
$$\begin{array}{ll} (a) & P(s) = 0 \Rightarrow Q(s) = 0 \Rightarrow \{P(s) = 0 \ \ or \ T(s) = 0\} \\ (b) & P(s) = \infty \Rightarrow Q(s) = 1 \Rightarrow \{P(s) = \infty \, or \ T(s) = 1\}. \end{array} \quad (10)$$

(2) $||Q - T|| < \epsilon$.

We conclude this section by illustrating through an example that a model that is 'good for control design' may be a very poor 'open loop' model of P.

Example 4
Consider $P = \frac{s-1}{(s-2)(s-3)}$ and $T = \frac{-7.22}{s+0.1}$. Note that P has no unstable zeros between its two unstable poles, and it has two unstable zeros, one at $s = 1$ and one at $s = \infty$. It follows from our discussion in Section 4 that there are no constraints on T for $\mathcal{P}$ to be non-empty. One stabilizing model is $\hat{P} = \frac{-1.22}{s+7.32}$, yielding the proportional controller $C(\hat{P}, T) = 5.918$. This controller produces an achieved complementary sensitivity function

$$\frac{PC}{1+PC} = \frac{5.92(s-1)}{(s+0.1)(s+0.82)}.$$



The figure shows the Nyquist plots of P (full line) and $\hat{P}$ (dashed line) on the left, and of the achieved (full line) and designed (dashed line) complementary sensitivity functions on the right. Clearly the two closed loop systems are close, but $\hat{P}$ is by anybody's definition a bad model of P.

7 Conclusions

We have addressed the problem of model quality for control by stating that a model used for control design is 'good' if it stabilizes the actual plant and if it achieves on that plant a performance that is not too different from the designed performance.

For the robust stability property and for model reference design, the outcome has been a sequence of algebraic constraints on the reference model without which no model could yield a stabilizing controller for the real plant. The technicalities are more complex than, but an outgrowth of, the questions of simultaneous, strong and bistable stabilization. These are unfamiliar but fundamental requirements in the analysis of the validity of a model set for control design. The necessary conditions relating the unstable pole-zero pattern of the plant to demands on the reference model are clear and simple.

One of our parametrizations of the class of all stabilizing models gives some handle on the robust performance question. However, the characterization of all models that are ϵ-stabilizing, or the computation of the model (and hence the controller) that would yield an achieved performance that is as close as possible to the designed performance is still beyond reach. Stay tuned.

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