Closed-loop Optimal Experiment Design: Solution via Moment Extension

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***Abstract*—We consider optimal experiment design for parametric prediction error system identification of linear time-invariant multiple-input multiple-output (MIMO) systems in closed-loop when the true system is in the model set. The optimization is performed jointly over the controller and the spectrum of the external excitation, which can be reparametrized as a joint spectral density matrix. We have shown in [16] that the optimal solution consists of first computing a finite set of generalized moments of this spectrum as the solution of a semi-definite program. A second step then consists of constructing a spectrum that matches this finite set of optimal moments and satisfies some constraints due to the particular closed-loop nature of the optimization problem. This problem can be seen as a moment extension problem under constraints. Here we first show that the so-called *central extension* always satisfies these constraints, leading to a constructive procedure for the o! ptimal controller and excitation spectrum. We then show that, using this central extension, one can construct a broader set of parametrized optimal solutions that also satisfy the constraints; the additional degrees of freedom can then be used to achieve additional objectives. Finally, our new solution method for the MIMO case allows us to considerably simplify the proofs given in [16] for the single-input single-output case.**

I. Introduction

Optimal experiment design for system identification has seen an intense development in the last decade. This advance was initiated by the appearance of modern convex optimisation methods in the nineties, most notably semi-definite programming. Accordingly, most of the recent work focusses on casting different experiment design problems as semi-definite programs, for which commercial or free solvers are available. One of the pioneering contributions introducing semi-definite programming into optimal input design for open loop identification was [23]. For further motivation and an extensive reference list we refer to [18].

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In this paper we provide a solution via semi-definite programming to a general class of optimal experiment design problems for the identification of parametric stable linear time-invariant (LTI) systems operating in closed loop. We work in the frequency domain and assume that the data set is sufficiently large such that formulas that are asymptotic in the number of data are valid. The degrees of freedom which are relevant for closed-loop experiment design problems are the power spectrum of the external excitation signal fed into the system and the feedback controller transfer function. Both can easily be converted into a joint power spectrum of some signals present in the loop. These spectra are frequency-dependent functions and as such infinite-dimensional objects. Their infinitely many degrees of freedom have to be condensed into a finite-dimensional vector of design variables. A semi-definite description of optimal experiment design problems in this class has for years b! een elusive.

Two basic approaches to the choice of the design variables can be distinguished in the literature. The first is based on a *finite dimensional approximation* of the joint spectrum, the second, often called *partial correlation approach*, is based on expressing the criterion and the constraints as a function of a finite number of linear functionals of the joint spectrum, called *generalized moments*. In both cases, the optimal experiment design problem is then transformed into a semi-definite program expressed in terms of the parameters of the finite dimensional approximation for the first approach, and the generalized moments for the second approach.

In [17] the finite dimensional approximation approach was used. A solution was obtained by first parametrizing the joint spectrum mentioned above using a Youla-Kucera parametrization to constrain the solution set to deliver a stabilizing closed loop controller, and then using a finite dimensional approximation of this joint spectrum. The optimal design problem is then reduced to a convex optimization problem under *Linear Matrix Inequality* (LMI) constraints over the coefficients of this finite dimensional approximation. Given that the solution space is restricted by the finite dimensional approximation, it leads to a *suboptimal* solution.

In [16] we provided an *optimal* solution based on the partial correlation approach. Our solution applies to a wide class of optimal design problems in which the criterion and the constraints are expressed as integral functions over the frequency range.

In this framework the criterion and the constraints can be expressed as linear functions of *a finite set of* $n+1$ *generalized moments*, which are linear functionals of the joint power spectrum. They become the design variables of the optimal design problem. The conditions on the vector of design variables to correspond to a realizable experiment design are then shown to be equivalent to the satisfaction of an LMI, possibly involving additional auxiliary variables. The optimal moment sequence is then obtained by solving a standard semi-definite program. Geometrically, the optimization is performed over a finite-dimensional *projection* of the infinite-dimensional cone of possible joint power spectra. The optimal finite moment sequence will then in general correspond to an infinite set of spectral density matrices rather than a single spectrum, and every possible spectrum is represented by some point in the cone generated by the finite set of optimal moments, thus re! sulting in a truly optimal solution.

The construction of a spectrum or a set of spectra whose first $n+1$ generalized moments coincide with the optimal moments that solve the semi-definite program is known as the *Carathéodory extension problem*. The case of scalar-valued moments has been well studied in the last century [6], [28], [2], [22], [19], [1]. The scalar theory can be generalized to the case of matrix-valued moments [25], [26], [3], [21], [9], [10]. The key result for solving the Carathéodory extension problem is the Carathéodory-Fejer theorem. This theorem implies that a given finite sequence of moments is indeed generated by a positive power spectrum if and only if it satisfies a certain LMI [20, Chapter VI, Theorem 4.1]. Such a spectrum can be represented in a number of equivalent ways. Thi! s includes the representation as a matrix-valued positive semi-definite measure on the unit circle, as an infinite sequence of moments, or as a Carathéodory function, i.e., a matrix-valued holomorphic function defined on the open unit disc whose Hermitian part is positive semi-definite. The representations can easily be transformed in one another [25, Section II].

The set of all possible infinite extensions of a finite moment sequence may be parametrized by an infinite sequence of complex contractive matrices [9, Theorem 1]. The first k contractive matrices in the sequence define the first k undetermined moments of the extension, i.e., the first k moments which follow the $n+1$ moments given by the solution of the semi-definite program. In this way, fixing the contractive matrices one by one, the user can consecutively construct all moments of the extension. These matrices hence represent a *choice sequence*. The contractive matrices can be defined in different ways and carry different names, e.g., *Schur parameters*, *Szegö parameters*, *reflection parameters*, *canonical moments*, or *Verblunsky coefficients* [8], [1], [26], [25], [4]; see [7] and [27, p.30] for a discussion.

The particular extension corresponding to the case when all Verblunsky coefficients vanish is called *central extension* [9], [10], [29, Section 3.6], and the measure on the unit circle which defines the corresponding positive semi-definite spectrum is called *central measure* [4, Remark 8.4, p.104]. If a non-degeneracy condition is satisfied, then the power spectrum defined by the central measure can be expressed in closed-form as a rational function with coefficients depending in an explicit manner on the optimal truncated moment sequence [25], [29].

A more compact way to parametrize the set of all possible extensions of a given finite moment sequence is via the representation of the extensions as Carathéodory functions. The set of all such functions which can be obtained from the finite moment sequence is given by a linear-fractional transformation (LFT) of a single parameter. This parameter takes values in the *Schur class*, i.e., the set of all holomorphic matrix-valued functions on the open unit disc which are contractive. The coefficients of the LFT depend explicitly on the original finite moment sequence [5, Theorem 1.1]. The central extension then corresponds to the case when the Schur function is identically zero. The Carathéodory function corresponding to the central measure, called *central Carathéodory function*, is hence a rational function with coefficients depending explicitly on the finite set of optimal moments [11], [5, Theorem 1.3].

The classical Carathéodory-Fejer theorem, which establishes conditions under which an infinite extension of the finite set of optimal moments exists, holds *only* if no restrictions are imposed on the spectrum other than to produce the truncated sequence of moments under consideration, and positivity. In other words, a finite sequence of moments can be extended to an infinite sequence of moments of a positive spectrum if and only if it satisfies the LMI condition, but no additional constraint on the moments of this extension can be guaranteed to be satisfied. However, in closed-loop optimal experiment design, where the controller is part of the design variables, constraints have to be imposed on the matrix-valued joint power spectrum under consideration, and hence on the infinite moment extension. These constraints reflect the fact that the controller must produce a stable closed loop, and that some elements of a joint spectrum defined by signals in the loop are fixed. The constraints on this joint power spectrum translate into additional constraints on the infinite moment extensions in order for these extensions to define an admissible spectrum.

In [16] we have shown that the Carathéodory-Fejer theorem *also* holds for the type of structured generalized moment problem arising in closed-loop optimal optimal experiment design. Namely, if a finite sequence of moments satisfies the additional stability constraints, then the LMI condition given by the Carathéodory-Fejer theorem not only insures the existence of a general extension of this moment sequence, but the existence of an extension which also satisfies the constraints.

The proof of this main result in [16] had several drawbacks. First it was written for single-input single-output (SISO) systems, even though an extension to multiple-input multiple-output (MIMO) is easily obtained. More importantly, it proved the *existence* of an extension that satisfies the constraints on the joint spectrum, but it was not constructive. Finally, the proof was very long and complicated, as it relied on the partial positive definite matrix completion theorem from [14], which itself required to appeal to graph-theoretical properties of the Toeplitz matrix made up of the generalized moments.

The present paper makes progress in several directions with respect to [16]. First we allow the system to have multiple inputs and outputs. Our main contribution is to show that the stability constraints are satisfied by the so-called *central extension*, which under a non-degeneracy condition can be explicitly computed from the set of $n+1$ optimal moments. The central extension defines a unique power spectrum, which solves the optimal experiment design problem. Thus once the optimal truncated moment sequence has been obtained by solving the semi-definite program, an optimal joint power spectrum can be immediately written down in closed form, shortcutting the somewhat ad hoc and complicated recovery step in [16].

Our second main contribution is to show that the set of all extensions which satisfy the additional constraints on the joint power spectrum can also be parametrized by a choice sequence of contractive matrices. These matrices have a smaller size than the Verblunsky coefficients, because at each step, a part of the degrees of freedom given by the Verblunsky coefficient is fixed by the additional constraint on the corresponding moment. We call these contractive matrices *restricted Verblunsky coefficients*. The central extension corresponds to the case when all restricted Verblunsky coefficients vanish. This result allows one to generate a finite-dimensional, explicitly parametrized family of optimal solutions by first fixing a finite number of restricted Verblunsky coefficients, constructing the corresponding finite moment extension, and then using the central extension of this already finitely extended moment sequence. The additional degrees of freedom embodied by the restricted Verblunsky coefficients can be used to satisfy additional performance criteria, constraints, or robustness properties that the user may want to inject into the problem.

Feasibility of the central extension actually implies the validity of the Carathéodory-Fejer theorem for the structured generalized moment problem. This allows us to significantly shorten the proof of this result given in [16]. For this reason, and in order to make the present contribution self-contained, we also provide the new proof of the structured Carathéodory-Fejer theorem here.

The remainder of the paper is organized as follows. In the next section we define the class of input design problems to be solved. In Section III we introduce the concepts of central extensions, central measures, Carathéodory functions and Verblunsky coefficients. Our main result is in Section IV, where we show the feasibility of the central extension for optimal closed-loop experiment design and parametrize the set of all feasible solutions by the choice sequence of restricted Verblunsky coefficients. In Section V we present a complete solution algorithm for the proposed class of problems, including a semi-definite description of the feasible set of truncated moment sequences. In Section VI we illustrate via an example that even in the case where the Toeplitz matrix made up of the $n+1$ optimal moments is singular, the central extension may produce an optimal spectrum that remains finite. In the Appendix we provide au! xiliary results on a special case of the partial positive matrix completion problem.

II. Problem formulation

In this section we define the class of optimal experiment design problems treated in this paper. We intend to perform parametric prediction error identification of a stable MIMO LTI system in closed loop. The system dynamics is given by the relation

$$y = G_0(q)u + H_0(q)e, \tag{1}$$

where the signal u is of dimension m, and e, y are of dimension p. Here G_0 is the plant transfer function matrix, H_0 is the noise transfer function matrix, q is the forward-shift operator, e is a vector-valued zero mean white noise with (co-)variance $\lambda_0 I_p$ where I_k is the $k \times k$ identity matrix, u is the input vector, and y is the output vector of the system. The transfer function matrices $G_0(z), H_0(z)$ are embedded in a model structure $G(z;\theta), H(z;\theta)$ and correspond to some true parameter value θ_0, $G_0(z) = G(z;\theta_0)$, $H_0(z) = H(z;\theta_0)$. We assume that the plant transfer function G_0 is stable, and the noise model H_0 is stable and inversely stable.

The parameter vector θ_0 is to be identified by an experiment, which consists in collecting a set of input-output data u, y on the system, which is possibly under closed-loop control according to the relation

$$u = -K(q)y + r, \tag{2}$$

where r is a quasistationary process of dimension m, and $K(q)$ is a $m \times p$ matrix-valued feedback controller. The configuration of the identification experiment is schematically depicted in Fig. 1. The estimator $\hat{\theta}$ of the true parameter value θ_0 is then evaluated as the minimizer of some prediction error criterion.

The design variables at our disposal for this identification experiment are thus the power spectrum $\Phi_r(\omega)$ of the external vector-valued input signal r and the controller $K(q)$. The experiment design problem studied in this paper consists of choosing $\Phi_r(\omega)$ and $K(q)$ such that some cost function $\Gamma(\Phi_r, K)$ is minimized and some constraints $\Gamma_k(\Phi_r, K) \leq \gamma_k$ on the pair (Φ_r, K) are satisfied.

Following [17], we first move from the quantities Φ_r, K to the spectra Φ_u, Φ_{ue}, which yield an equivalent description of the experimental conditions. The power spectrum Φ_r of r and the controller K determine Φ_u, Φ_{ue} by the formulas

$$\begin{aligned} \Phi_u(\omega) &= \lambda_0(I_m+KG_0)^{-1}KH_0H_0^*K^*(I_m+KG_0)^{-*} \\ &\quad +(I_m+KG_0)^{-1}\Phi_r(\omega)(I_m+KG_0)^{-*}, \quad (3) \\ \Phi_{ue}(\omega) &= -\lambda_0(I_m+KG_0)^{-1}KH_0, \quad (4) \end{aligned}$$

where the transfer functions on the right-hand side are evaluated at $z = e^{j\omega}$. By A^* we denote the complex conjugate transpose of the matrix A and by A^{-*} the inverse of A^*. On the other hand, Φ_r and K can be recovered from Φ_u, Φ_{ue} by the formulas

$$\begin{aligned} \Phi_r &= (I_m+KG_0)(\Phi_u - \lambda_0^{-1}\Phi_{ue}\Phi_{ue}^*)(I_m+KG_0)^*, \\ K &= -\Phi_{ue}(\lambda_0 H_0 + G_0\Phi_{ue})^{-1}. \quad (5) \end{aligned}$$

Thus there is a one-to-one relationship between (Φ_r, K) and (Φ_u, Φ_{ue}). Note that the matrix inverses in (3)–(5) exist by the stability of the loop.

Parametrizing the experimental conditions by the joint power spectrum

$$\Phi_{\chi_0} = \begin{pmatrix} \Phi_u & \Phi_{ue} \\ \Phi_{ue}^* & \lambda_0 I_p \end{pmatrix} \tag{6}$$

of the signals u, e instead of the quantities Φ_r, K has the advantage that the feasible set becomes convex, which is a prerequisite for a semi-definite representation [17]. The matrix Φ_{χ_0} is of size $(m+p) \times (m+p)$.

Within the framework of the partial correlation approach, the ultimate design variables are a finite set of moments of the joint power spectrum Φ_{χ_0}. Accordingly, the cost criterion and the constraints of the optimal experiment design problem have to be expressible in a tractable manner in terms of these moments. Apart from this compatibility requirement, we do not impose any condition on the cost criterion and the constraints.

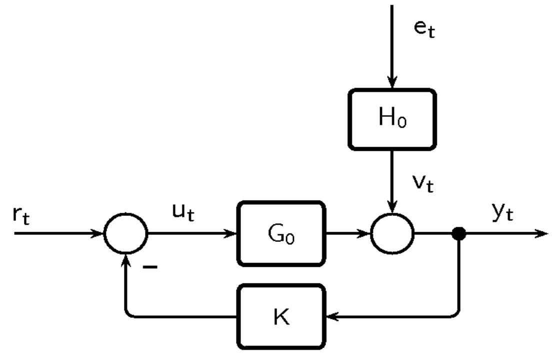

Fig. 1. Experimental setup

Assumption 1. *There exist integers $n \geq s \geq 0$ and a polynomial $d(z) = \sum_{l=0}^{s} d_l z^l$ of degree s with the following properties. The coefficients d_l are real, obey $d_0 \neq 0$, $d_s \neq 0$, and the polynomial $d(z)$ has all roots outside the closed unit disk. Define $(m+p) \times (m+p)$ matrices*

$$m_k = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{1}{|d(e^{j\omega})|^2}\Phi_{\chi_0}(\omega)e^{jk\omega}\, d\omega \tag{7}$$

for integral k. Then the constraints of the experiment design problem can be written as a linear matrix inequality

$$\mathcal{A}(m_0, m_1, \dots, m_n, x_1, x_2, \dots, x_N) \succeq 0 \tag{8}$$

and the cost function of the experiment design problem can be expressed as a tractable convex function

$$f_0(m_0, \dots, m_n, x_1, \dots, x_N), \tag{9}$$

where $(m_0, \dots, m_n)$ are defined in (7) *and where $\mathcal{A}$ and f_0 may depend on N additional auxiliary variables $(x_1, \dots, x_N)$.*

The matrices m_k defined by (7) are called the generalized moments of the spectrum Φ_{χ_0}. Note that the moments m_k are real and obey the relation $m_k = m_{-k}^T$.

In [15], [16] we presented a semi-definite description of the set of finite moment sequences $(m_0, \dots, m_n)$ corresponding to valid experiment designs. A sequence $(m_0, \dots, m_n)$ can be realized by a closed-loop experiment design if and only if a certain LMI $\mathcal{B}(m_0, \dots, m_n) \succeq 0$ holds. Under Assumption 1, the optimal experiment design problem stated above

$$\min_{\Phi_r,K} \Gamma(\Phi_r,K) \quad \text{subject to} \quad \Gamma_k(\Phi_r,K) \leq \gamma_k \qquad (10)$$

can thus be recast as a convex program with semi-definite constraints:

$$\min_{\{m_k,x_l\}} f_0 \quad \text{subject to} \quad \mathcal{A} \succeq 0,\ \mathcal{B} \succeq 0. \qquad (11)$$

The solution of this optimization problem yields the optimal truncated moment sequence $(m_0,\dots,m_n)$.

Under some mild assumptions the asymptotic in the number of data average per data sample information matrix of the experiment is given by [24]

$$\overline{M} = \frac{1}{2\pi\lambda_0}\sum_{k=1}^{p}\int_{-\pi}^{+\pi} F_k(e^{j\omega})\Phi_{\chi_0}(\omega)F_k^*(e^{j\omega})\,d\omega, \qquad (12)$$

where $p = dim(y)$, and the l-th row of the matrix F_k is given by the k-th row of the matrix $[H_0^{-1}G'_{\theta^l}(\theta_0),\ H_0^{-1}H'_{\theta^l}(\theta_0)]$. Here G'_{θ^l}, H'_{θ^l} denote the gradients of $G(z;\theta), H(z;\theta)$ with respect to the l-th entry of the parameter vector θ. If the model structure is rational, then (12) is affine in the moment matrices $m_0, m_1, \dots, m_n$ for a suitably chosen polynomial $d(z)$. In addition, most experiment design criteria are formulated as scalar functions of $\overline{M}$. Therefore, Assumption 1 covers a wide variety of problem formulations in closed-loop optimal experiment design, see also [23], [18], [17]. In particular, all classical designs (D-optimal, A-optimal, L-optimal etc.) subject to variance constraints on the signals fall within the framework of Assumption 1.

Example: We now demonstrate by an example how to construct the polynomial $d(z)$, the constraints $\mathcal{A}$, and the cost function f_0 for a concrete closed-loop experiment design problem. Let us consider an ARX model structure $G = \frac{bz^{-1}}{1+az^{-1}}$, $H = \frac{1}{1+az^{-1}}$ with true parameters b_0, a_0, where $|a_0| < 1$ to ensure that the plant G is stable. We wish to identify the system in closed-loop under a constraint on the output power, $\overline{E}y^2 \leq c$, where $c > \lambda_0$, such that the determinant of the information matrix is maximized (D-optimality). Equivalently, we minimize the negative logarithm of the determinant.

Since the plant and noise model are scalar, the system dimensions are given by $m = p = 1$. The joint power spectrum Φ_{χ_0} is thus of size 2×2. The matrix F_1 from (12) is given by $\begin{pmatrix} H^{-1}G'_b & H^{-1}H'_b \\ H^{-1}G'_a & H^{-1}H'_a \end{pmatrix} = \begin{pmatrix} z^{-1} & 0 \\ -\frac{bz^{-2}}{1+az^{-1}} & -\frac{z^{-1}}{1+az^{-1}} \end{pmatrix}$, where b, a are evaluated at their true values b_0, a_0.

Our goal is now to express the information matrix (12) as a linear function of the generalized moments m_k, or equivalently, as a convolution of a polynomial in $z = e^{j\omega}$ with the ratio $\frac{\Phi_{\chi_0}}{|d(z)|^2}$. To this end, we set $d(z)$ equal to the common denominator $1 + a_0 z$ of the elements of F_1^*. Then the elements of the information matrix and the output power can be expressed by the generalized moments as

$$\begin{aligned} \overline{M}_{11} &= \lambda_0^{-1}((1+a_0^2)m_{0,11} + 2a_0 m_{1,11}), \qquad (13)\\ \overline{M}_{12} &= \lambda_0^{-1}(-b_0(a_0 m_{0,11}+m_{1,11}) - a_0 m_{1,21} - m_{0,12}),\\ \overline{M}_{22} &= \lambda_0^{-1}(b_0^2 m_{0,11} + 2b_0 m_{1,21} + m_{0,22}),\\ \overline{E}y^2 &= b_0^2 m_{0,11} + 2b_0 m_{1,21} + m_{0,22}. \end{aligned}$$

The constraint $\overline{E}y^2 \leq c$ and the cost function $f_0 = -\log\det\overline{M}$ involve only the moments m_0, m_1, and we can set $n = 1$. Moreover, we do not need auxiliary variables and can set $N = 0$. We then get

$$\mathcal{A}(m_0,m_1) = c - \overline{E}y^2 \geq 0,$$

$$f_0(m_0,m_1) = -\log(\overline{M}_{11}\overline{M}_{22} - \overline{M}_{12}^2),$$

where for $\overline{E}y^2$ and the elements of $\overline{M}$ we have to insert expressions (13).

III. Central extensions

In this section we introduce the concept of moment extensions, and in particular, central extensions. Before we focus on the generalized moments (7) of the structured power spectrum (6), we will first consider the case of moment sequences of general power spectra. First we shall consider different ways to represent a positive semi-definite power spectrum in Subsection III-A. Then the set of all possible moment extensions and its parametrizations is considered in Subsection III-B. In Subsection III-C we introduce the central extension, which is a particular moment extension. Finally, we consider the central extension under the assumption of a certain non-degeneracy condition in Subsection III-D.

A. Representations of power spectra

Let $\Phi(\omega)$ be an integrable 2π-periodic matrix-valued complex-Hermitian positive semi-definite function of size $l \times l$, possibly containing a singular part consisting of Dirac δ-functions. The *moments* of Φ are defined by

$$m_k = \frac{1}{2\pi}\int_{-\pi}^{+\pi}\Phi(\omega)e^{jk\omega}\,d\omega. \qquad (14)$$

Note that $m_{-k} = m_k^*$. Then the block-Toeplitz matrices

$$T_k = \begin{pmatrix} m_0 & m_1^* & \ddots & m_{k-1}^* & m_k^* \\ m_1 & m_0 & \ddots & m_{k-2}^* & m_{k-1}^* \\ \ddots & \ddots & \ddots & & \\ m_k & m_{k-1} & \ddots & m_1 & m_0 \end{pmatrix} \qquad (15)$$

are positive semi-definite for all $k \geq 0$. On the other hand, given an infinite sequence of matrices m_k, $k \in \mathbb{Z}$, satisfying $m_{-k} = m_k^*$ and such that all block-Toeplitz matrices T_k, $k \geq 0$, are positive semi-definite, there exists a unique Hermitian positive semi-definite function $\Phi(\omega)$ producing the matrices m_k as in (14) [25, Theorem 1]. Note that if $\Phi(-\omega) = \Phi(\omega)^T$, then all moments m_k are real, and the complex conjugate transpose in (15) becomes the ordinary transpose.

There exist other representations of the function $\Phi(\omega)$ than by its infinite moment sequence. One of these is the *Carathéodory function*

$$F(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{j\omega} + z}{e^{j\omega} - z} \Phi(\omega) d\omega, \tag{16}$$

which is an analytic function defined on the open unit disc such that its Hermitian part $\frac{1}{2}(F(z) + F^*(z))$ is positive semi-definite and $F(0)$ is Hermitian. The spectrum can be recovered from F as the limit

$$\Phi(\omega) = \lim_{r \to 1-} \frac{1}{2}(F(re^{j\omega}) + F^*(re^{j\omega})). \tag{17}$$

If Φ has a singular part, then the limit has to be understood in the sense of a distribution [25, Section II]. The Carathéodory function $F(z)$ can be also determined from the moment sequence by the Taylor expansion [25, p.151]

$$F(z) = m_0 + 2\sum_{k=1}^{\infty} m_{-k} z^k. \tag{18}$$

B. General moment extensions

We have shown in Section II that the optimal experiment design problem can be reformulated as a convex program with semi-definite constraints whose solution takes the form of an optimal finite moment sequence $m_0, \dots, m_n$. In order for this finite moment sequence to deliver an optimal spectrum Φ_{χ_0}, which in turn will deliver an optimal pair (Φ_r, K), it needs to be extended to an infinite sequence $m_0, \dots, m_n, m_{n+1}, \dots$ which must define a valid positive semi-definite Hermitian function via the formula (14). Such extension is by no means unique (and hence the corresponding spectrum is by no means unique); the problem of generating such extension is called the *moment extension problem.*

An obvious necessary condition for a *finite* sequence $m_0, \dots, m_n$ of $l \times l$ matrices to be extendable to an infinite sequence $m_0, \dots, m_n, m_{n+1}, \dots$ which can be obtained from some positive semi-definite function Φ by formula (14) is that the block-Toeplitz matrix T_n is positive semi-definite, $T_n \succeq 0$. The Carathéodory-Fejer theorem (see, e.g., [20, Chapter VI, Theorem 4.1]) states that this is also a sufficient condition.

We call such infinite sequence $m_0, \dots, m_n, m_{n+1}, \dots$ an (infinite) *extension* of the finite sequence $m_0, \dots, m_n$. Recall that these infinite extensions are by no means unique (actually there are infinitely many extensions), but they must all obey the property that $T_k \succeq 0$ for all k. Our goal in this subsection is to parametrize this set of infinite extensions. We present two ways of parametrizing all infinite extensions of a finite moment sequence. The first is by parametrizing the successive moments $m_{n+1}, m_{n+2}, \dots$ as a function of $m_0, \dots, m_n$ and of a free parameter; the second is by parametrizing the Carathéodory functions $F(z)$ of all spectra $\Phi(\omega)$ whose first moments coincide with $m_0, \dots, m_n$.

Parametrization of the successive moments

Since the condition $T_k \succeq 0$ implies $T_{k'} \succeq 0$ for all $k' \leq k$, it makes sense to first consider extensions by a finite number $m_{n+1}, \dots, m_{n'}$ of matrices and to parametrize these. The sequence $m_0, \dots, m_n, m_{n+1}, \dots, m_{n'}$ is a finite extension of the sequence $m_0, \dots, m_n$ if and only if $T_{n'} \succeq 0$. We first parameterize all extensions of the finite sequence $m_0, \dots, m_n$ by *one* additional matrix m_{n+1}.

Theorem 1. *[29, Theorem 3.4.1],[4, Theorem 2.11b] Let $m_0, \dots, m_n$ be a sequence of $l \times l$ matrices such that the block-Toeplitz matrix T_n defined by* (15) *is positive semi-definite. Then the $l \times l$ matrix m_{n+1} extends the sequence $m_0, \dots, m_n$ in such a way that $T_{n+1} \succeq 0$ if and only if it can be written as*

$$m_{n+1} = \left(\begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}^* T_{n-1}^\dagger \begin{pmatrix} m_n^* \\ \vdots \\ m_1^* \end{pmatrix} + L_{n+1}\Delta_{n+1}R_{n+1} \right)^* \tag{19}$$

with

$$\begin{aligned} L_{n+1} &= \left(m_0 - \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix}^* T_{n-1}^\dagger \begin{pmatrix} m_1 \\ \vdots \\ m_n \end{pmatrix} \right)^{1/2}, \\ R_{n+1} &= \left(m_0 - \begin{pmatrix} m_n^* \\ \vdots \\ m_1^* \end{pmatrix}^* T_{n-1}^\dagger \begin{pmatrix} m_n^* \\ \vdots \\ m_1^* \end{pmatrix} \right)^{1/2}, \end{aligned} \tag{20}$$

and Δ_{n+1} a $l \times l$ contractive matrix, i.e., a matrix satisfying $\sigma_{\max}(\Delta_{n+1}) \leq 1$. Here $T_{n-1}^\dagger$ denotes the pseudo-inverse of T_{n-1}.

If the moments $m_0, \dots, m_n$ are real and the extending moment m_{n+1} is also required to be real, then the set of all extensions is parametrized by a real contractive matrix Δ_{n+1}.

The contractive matrix Δ_{n+1} is called *Verblunsky coefficient* [7]. It has been shown in [8] that up to a possible sign change it is equal to the Schur or Szegö parameters, which are contractive matrices defined in a different way [1], [25], [26].

A longer extension $m_0,\ldots,m_{n'}$ of the sequence $m_0,\ldots,m_n$ can be obtained step by step. We proceed by first choosing a contractive matrix Δ_{n+1} and calculating the next moment m_{n+1} from it. Then we choose a matrix Δ_{n+2} and compute m_{n+2}. Note that m_{n+2} then depends also on Δ_{n+1} via its dependence on m_{n+1}. Then we choose Δ_{n+3} and so on, until the final choice of $\Delta_{n'}$ which determines the last moment matrix $m_{n'}$ of the extension. In this way, all extensions $m_0,\ldots,m_{n'}$ can be parametrized by $n'-n$ contractive $l\times l$ matrices Δ_k, $k=n+1,\ldots,n'$. In the same way, an infinite extension is determined by an infinite sequence of matrices $\Delta_{n+1},\Delta_{n+2},\ldots$, and the set of all such extensions is parametrized by the set of all such sequences.

Parametrization of all Carathéodory functions

A more compact way to parametrize the set of all extensions of a finite sequence $m_0,\ldots,m_n$ is via the Carathéodory function (16). In order to formulate this result, we need a couple of definitions. For every $0\le k<n$, we define from the sequence $m_0,\ldots,m_k$ the positive semi-definite $l\times l$ matrices L_{k+1} and R_{k+1} as in (20). For a polynomial $f(z)$ which is formally of degree k, define the reciprocal polynomial $\tilde{f}^{[k]}(z)=z^k f^*(1/\bar{z})$. Let the matrix-valued polynomials $\mathbf{a}_k,\mathbf{b}_k,\mathbf{c}_k,\mathbf{d}_k$, formally of degree k in z, be recursively defined by

$$\mathbf{a}_0=\mathbf{c}_0=m_0,\ \mathbf{b}_0=\mathbf{d}_0=I_l,$$

$$\begin{aligned}
\mathbf{a}_{k+1}(z) &= \mathbf{a}_k + z\tilde{\mathbf{c}}_k^{[k]}(z)L_{k+1}^\dagger\Delta_{k+1}R_{k+1},\\
\mathbf{b}_{k+1}(z) &= \mathbf{b}_k - z\tilde{\mathbf{d}}_k^{[k]}(z)L_{k+1}^\dagger\Delta_{k+1}R_{k+1},\\
\mathbf{c}_{k+1}(z) &= \mathbf{c}_k + zL_{k+1}\Delta_{k+1}R_{k+1}^\dagger\tilde{\mathbf{a}}_k^{[k]}(z),\\
\mathbf{d}_{k+1}(z) &= \mathbf{d}_k - zL_{k+1}\Delta_{k+1}R_{k+1}^\dagger\tilde{\mathbf{b}}_k^{[k]}(z), \quad (21)
\end{aligned}$$

where Δ_{k+1} is any contractive matrix satisfying (19) with n replaced by k. Its existence is guaranteed by Theorem 1 and the polynomials (21) do not depend on the choice of Δ_{k+1}.

The following result then provides a parametrization of the Carathéodory functions $F(z)$ of all spectra $\Phi(\omega)$ whose first $n+1$ moments coincide with $m_0,\ldots,m_n$. They are parametrized by a Schur function $\phi(z)$.

Proposition 1. *[5, Theorem 4.1] Let $m_0,\ldots,m_n$ be a finite sequence of $l\times l$ matrices such that the block-Toeplitz matrix* (15) *satisfies $T_n\succeq 0$. Then the Carathéodory function* (16) *obtained from an infinite extension of the sequence $m_0,\ldots,m_n$ has the general form*

$$\begin{aligned}
F(z) &= \Big(\mathbf{a}_n(z) - z\tilde{\mathbf{c}}_n^{[n]}(z)L_{n+1}^\dagger\phi(z)R_{n+1}\Big)\cdot\\
&\quad\cdot\Big(\mathbf{b}_n(z) + z\tilde{\mathbf{d}}_n^{[n]}(z)L^\dagger\phi(z)R_{n+1}\Big)^{-1}\\
&= \Big(\mathbf{d}_n(z) + zL_{n+1}\phi(z)R_{n+1}^\dagger\tilde{\mathbf{b}}_n^{[n]}(z)\Big)^{-1}\cdot\\
&\quad\cdot\Big(\mathbf{c}_n(z) - zL_{n+1}\phi(z)R_{n+1}^\dagger\tilde{\mathbf{a}}_n^{[n]}(z)\Big),
\end{aligned}$$

where $\phi(z)$ is an arbitrary Schur function of size $l\times l$, i.e., an analytic function on the open unit disc which is contractive. Moreover, the denominator matrices are invertible. □

The function $F(z)$ is hence a matrix-valued LFT of the Schur function $\phi(z)$, with coefficients given by polynomials which are explicit functions of the moments $m_0,\ldots,m_n$. For a given Schur function $\phi(z)$, the spectrum $\Phi(\omega)$ can be recovered from F by the limit (17).

C. Central extensions

In Subsection III-B we have seen that every extension of a finite sequence $m_0,\ldots,m_n$ is determined by the choice of a sequence of contractive $l\times l$ matrices $\Delta_{n+1},\Delta_{n+2},\ldots$ or, alternatively, by the choice of a Schur function $\phi(z)$ of size $l\times l$. In this subsection we introduce a special moment extension, called the *central extension*.

Definition 1. *[4, Def. 2.12] Let $m_0,\ldots,m_n$ be a finite sequence of $l\times l$ matrices such that the block-Toeplitz matrix* (15) *satisfies $T_n\succeq 0$. The* central extension *of $m_0,\ldots,m_n$ is the extension determined by the specific choice $\Delta_k=0$, $k\ge n+1$. The corresponding measure $\Phi(\omega)$ is called the* central measure*.*

The next result shows that the central extension can also be obtained as a special case of the parametrization of moment extensions given by Proposition 1, providing a simple closed-form expression for the Carathéodory function (16) defined by the central measure.

Proposition 2. *[11, Prop. 2.2, Theorem 2.3], [5, Theorem 1.3] Let $m_0,\ldots,m_n$ be a finite sequence of $l\times l$ matrices such that the block-Toeplitz matrix* (15) *satisfies $T_n\succeq 0$. Then the Carathéodory function* (16) *obtained from the central extension of the sequence $m_0,\ldots,m_n$ is given by the rational functions*

$$F(z)=\mathbf{a}_n(z)\mathbf{b}_n^{-1}(z)=\mathbf{d}_n^{-1}(z)\mathbf{c}_n(z),$$

where $\mathbf{a}_n,\mathbf{b}_n,\mathbf{c}_n,\mathbf{d}_n$ are the polynomials defined in (21)*. In other words, it corresponds to the choice $\phi\equiv 0$ of the contractive Schur function.* □

The function $F(z)$ in Proposition 2 is called the *central Carathéodory function*. The central measure can then be recovered from the central Carathéodory function by the limit (17). If the rational function F has poles on the unit circle, then the corresponding spectrum Φ might have a singular part, and the limit is to be considered in the sense of a distribution. Otherwise Φ is just the restriction of the Hermitian part of F on the unit circle and is also rational. If the matrices $m_0, \dots, m_n$ are real, then the central Carathéodory function $F(z)$ is real-rational. In this case all moments of the central extension will be real, and $\Phi(-\omega) = \Phi(\omega)^T$.

Let $m_{n+1}, m_{n+2}, \dots$ denote the subsequent moments of the central extension of the moment sequence $m_0, \dots, m_n$. They relate to the central measure Φ as in (14). From the construction in [5, p.256] it follows that the central Carathéodory function $F(z)$ produced by the extended moment sequence $m_0, \dots, m_k$ for $k \geq n$ as in Proposition 2 is the same as that produced by the sequence $m_0, \dots, m_n$. Hence the central extension of the sequence $m_0, \dots, m_n, m_{n+1}, \dots, m_k$ coincides with the central extension of $m_0, m_1, \dots, m_n$ for every $k \geq n$, and they all have the same central measure and the same central Carathéodory function. The polynomials $\mathbf{a}_k, \mathbf{b}_k, \mathbf{c}_k, \mathbf{d}_k$ defined by the extended sequence $m_0, \dots, m_k$ for $k > n$ coincide with the polynomials $\mathbf{a}_n, \mathbf{b}_n, \mathbf{c}_n, \mathbf{d}_n$ defined by the original sequence $m_0, \dots, m_n$, respectively [5, Prop. 4.4]. Although the central Carathéodory function $F(z)$ produced by the sequence $m_0, \dots, m_k$ is formally rational of degree k, $k \geq n$, it is then effectively rational of degree n due to a pole-zero cancellation at $z = \infty$.

D. Central extension in the regular case

In this subsection we consider the case when the matrix T_n constructed from a finite sequence $m_0, \dots, m_n$ of $l \times l$ matrices is positive definite, $T_n \succ 0$. Then the central measure can be written in a closed form as a function of the moments $m_0, \dots, m_n$. Following [25], define the $l \times l$ matrix-valued polynomial

$$A_n(z) = U_n(z) T_n^{-1} U_n^T(0) = \sum_{k=0}^{n} A_n^k z^k.$$

where $U_k(z)$ is a $l \times (k+1)l$ matrix-valued polynomial defined by[1]

$$U_k(z) = \begin{pmatrix} z^k I_l & z^{k-1} I_l & \cdots & I_l \end{pmatrix} \tag{22}$$

The matrix coefficient A_n^k of z^k is given by the $(n+1-k, n+1)$-th $l \times l$ block of the inverse T_n^{-1}. Note also that $A_n(0) = A_n^0$ is positive definite. By [25, Theorem 6] the polynomial $A_n(z)$ has no zeros in the closed unit disk. By [29, Section 3.6] the central measure is then given by the $l \times l$ matrix-valued function

$$\Phi(\omega) = A_n(e^{j\omega})^{-*} A_n(0) A_n(e^{j\omega})^{-1}. \tag{23}$$

By [25, Theorem 3] the function Φ is positive definite at all ω. Note that Φ is rational when considered as a function of $z = e^{j\omega}$ on the unit circle.

IV. Moment extensions for closed-loop experiment design

In this section we return to our optimal closed-loop experiment design problem described in Assumption 1. In Subsection IV-A we describe the constraints on the infinite generalized moment sequence $m_0, \dots, m_n, \dots$ which result from the particular structure (6) of the joint spectrum and the constraint (4) on Φ_{ue}. We show that these constraints impose linear relations between s successive moments, where s is the degree of $d(z)$. In Subsection IV-B we determine necessary and sufficient conditions such that a finite moment sequence $m_0, \dots, m_n$ is extendable to an infinite moment sequence satisfying these specific constraints; we call such extension a *feasible extension*. We do this by showing that the central extension is a feasible infinite extension. In particular, we can use the central extension of the truncated moment sequence $(m_0, \dots, m_n)$ to recover the jo! int power spectrum (6) which realizes the sequence according to formula (7). In Subsection IV-D we parameterize all infinite extensions corresponding to valid experiment designs by a choice sequence of *restricted Verblunsky coefficients*. The central extension corresponds to the case when all restricted Verblunsky coefficients are zero.

Throughout this section, the moments $m_0, \dots, m_n, \dots$ are defined by formula (7). This means that the m_k are the *generalized* moments of the joint power spectrum Φ_{χ_0}. Since in Section III the moments have been defined by formula (14), the power spectrum $\Phi(\omega)$ from this section has to be identified with the quotient $\frac{1}{|d(e^{j\omega})|^2} \Phi_{\chi_0}(\omega)$.

A. Structure of the infinite moment sequence

In this subsection we deduce linear relations between the moments $m_0 = m_0^T, m_1, \dots, m_n, \dots$ from the particular structure of the power spectrum Φ_{χ_0} in (7). Set $m_{-k} = m_k^T$ and partition the $l \times l$ matrix moments m_k into 4 blocks $m_{k,11}, m_{k,12}, m_{k,21}, m_{k,22}$, according to the partition of $\mathbb{R}^l$ into a sum $\mathbb{R}^m \oplus \mathbb{R}^p$. The moment matrices m_k depend on the spectra Φ_u, Φ_{ue}, which in turn determine the experimental conditions. However, as a result of the constraints (3), (4) and (6),

[1] The order of the components in the matrix U_n in (22) differs from that in [25, eq. (9)] because the definition (14) is different from [25, eq. (7)].

not all pairs (Φ_u, Φ_{ue}), and hence not all sequences $(m_0, \dots, m_n, \dots)$, correspond to valid experiment designs.

From (7) it follows that

$$m_{k,22} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{\lambda_0 I_p}{|d(e^{j\omega})|^2} e^{jk\omega}\, d\omega \qquad (24)$$

for all $k \in \mathbb{Z}$. The positivity of the joint power spectrum Φ_{χ_0} implies by the Carathéodory-Fejer theorem that the block-Toeplitz matrix

$$T_k = \begin{pmatrix} m_0 & m_1^T & \ddots & m_{k-1}^T & m_k^T \\ m_1 & m_0 & \ddots & m_{k-2}^T & m_{k-1}^T \\ \ddots & \ddots & \ddots & & \\ m_k & m_{k-1} & \ddots & m_1 & m_0 \end{pmatrix} \qquad (25)$$

is positive semi-definite for all $k \geq 0$. Further, the transfer functions from the signals r, e to the signals u, y are stable. Let $\mathbb{T} \subset \mathbb{C}$ be the unit circle. Then the function $f_{ue} : \mathbb{T} \to \mathbb{C}^{m\times p}$, defined by the cross spectrum Φ_{ue} by means of $f_{ue}(e^{j\omega}) = \Phi_{ue}(\omega)$, can be extended to a holomorphic function outside of the unit disc, including the point at infinity (compare also [17]). From

$$m_{k,12} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{1}{d(e^{j\omega})}\frac{\Phi_{ue}(\omega)}{d(e^{-j\omega})} e^{jk\omega}\, d\omega$$

it follows that

$$\sum_{i=0}^{s} d_i m_{k+i,12} = \frac{1}{2\pi j}\int_{\mathbb{T}} \frac{f_{ue}(z)}{d(z^{-1})} z^{k-1}\, dz.$$

Since all zeros of $d(z^{-1})$ are in the open unit disc, the ratio $f_{ue}(z)/d(z^{-1})$ is also holomorphic outside of the unit disc. It follows that $\sum_{i=0}^{s} d_i m_{k+i,12} = 0$ for all $k < 0$, and hence

$$\sum_{i=0}^{s} d_i m_{k-i,21} = 0 \qquad (26)$$

for all $k > 0$. Similarly it follows that the matrices (24) satisfy

$$\sum_{i=0}^{s} d_i m_{k-i,22} = 0 \qquad (27)$$

for all $k > 0$. The next result shows that these relations are also sufficient.

Theorem 2. *Let $m_0 = m_0^T, \dots, m_n, \dots$ be an infinite sequence of real $l \times l$ matrices, and set $m_{-k} = m_k^T$, $k > 0$. Then the sequence $m_0, \dots, m_n, \dots$ is generated by formula* (7) *from a joint power spectrum Φ_{χ_0} as in* (3),(4),(6) *if and only if $T_k \succeq 0$ for all $k \geq 0$, and relations* (24),(26) *hold for all $k \in \mathbb{Z}$ and $k > 0$, respectively.*

Proof: The only if part has been demonstrated above. Let us show the if part.

Assume that $T_k \succeq 0$ for all $k \geq 0$, and relations (24),(26) hold. We have to show that the moment sequence $m_0, \dots, m_n, \dots$ is generated by some joint power spectrum Φ_{χ_0} such that its lower right $p \times p$ subblock is given by $\lambda_0 I_p$, as required in (6), and its upper right $m \times p$ subblock is a stable transfer function. This allows to construct the controller and external input spectrum K, Φ_r in (3),(4) by virtue of (5), obtaining a stable control loop.

By [25, Theorem 1] there exists a unique positive semi-definite power spectrum $\Phi(\omega)$ which produces the moment sequence $m_0, \dots, m_n, \dots$ as in (14). Set $\Phi_{\chi_0}(\omega) = |d(e^{j\omega})|^2\Phi(\omega)$. Then (7) holds.

Let $\Phi_{\chi_0,22}$ be the $p \times p$ lower right subblock of Φ_{χ_0}. Relations (7) and (24) imply that $\int_{-\pi}^{+\pi} \frac{e^{jk\omega}}{|d(e^{j\omega})|^2}(\Phi_{\chi_0,22}(\omega) - \lambda_0 I_p) d\omega = 0$ for all k. Again from [25, Theorem 1] it then follows that $\Phi_{\chi_0,22}(\omega) = \lambda_0 I_p$.

Denote the upper right $m \times p$ subblock of Φ_{χ_0} by Φ_{ue}. Relation (26) implies $\sum_{i=0}^{s} d_i m_{k+i,12} = 0$ for all $k < 0$. Writing this out, we obtain $\int_{-\pi}^{+\pi} \frac{\Phi_{ue}(\omega)}{d(e^{-j\omega})} e^{jk\omega}\, d\omega = 0$ for all $k < 0$. It follows that the function $\tilde{f}_{ue} : \mathbb{T} \to \mathbb{C}^{m\times p}$ defined by $\tilde{f}_{ue}(e^{j\omega}) = \frac{\Phi_{ue}(\omega)}{d(e^{-j\omega})}$ can be extended to a holomorphic function outside of the unit disc, including the point at infinity. The product $f_{ue}(z) = \tilde{f}_{ue}(z)d(z^{-1})$ is then a holomorphic extension of the function $f_{ue} : \mathbb{T} \to \mathbb{C}^{m\times p}$ defined by $f_{ue}(e^{j\omega}) = \Phi_{ue}(\omega)$. Thus Φ_{ue} represents a stable transfer function, which concludes the proof. ∎

B. Feasibility of the central extension

In this subsection we consider *finite* sequences $m_0 = m_0^T, m_1, \dots, m_n$ of real $l \times l$ matrices and their central extensions in relation to Theorem 2. Set $m_{-k} = m_k^T$ for $k = 1, \dots, n$.

In order for the finite sequence $(m_0, \dots, m_n)$ to be extendable to an infinite sequence $m_0, \dots, m_n, \dots$ satisfying the conditions of Theorem 2, it must clearly satisfy the following necessary conditions:

$$\begin{aligned} T_n &\succeq 0, & (28)\\ m_{k,22} &= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{\lambda_0 I_p}{|d(e^{j\omega})|^2} e^{jk\omega}\, d\omega, & \\ & \quad k = 0, \dots, n, & (29)\\ \sum_{i=0}^{s} d_i m_{k-i,21} &= 0, \qquad k = 1, \dots, n. & (30) \end{aligned}$$

In [16, Theorem 1] we have shown for the SISO case that conditions (28)—(30) are also sufficient to guarantee the existence of a positive semi-definite joint power spectrum (6), satisfying $\Phi_{\chi_0}(\omega) = \Phi_{\chi_0}(-\omega)^T$, such that Φ_{ue} represents a stable transfer function, which reproduces the truncated moment sequence $(m_0, \dots, m_n)$ by formula (7). This proof extends without modifications also to the MIMO case considered here. The result [16, Theorem 1] is, however, non-constructive, because it does not yield an explicit power spectrum Φ_{χ_0}, but merely proves its existence.

We will now give a constructive proof by showing that the explicit power spectrum obtained by virtue of the central extension yields a feasible optimal experiment.

Theorem 3. *Let $m_0 = m_0^T, m_1, \dots, m_n$ be a finite sequence of real $l \times l$ matrices, and set $m_{-k} = m_k^T$ for $k = 1, \dots, n$. Assume that conditions (28)—(30) hold. Then the central extension of the sequence $(m_0, \dots, m_n)$ satisfies the conditions of Theorem 2.*

Proof: The condition $T_k \succeq 0$ is fulfilled for all $k \geq 0$ because the central extension is by definition a positive semi-definite moment extension. It remains to show the equality conditions (24),(26) for $k > n$.

This can be done by induction over k. Indeed, the central extension $m_0, \dots, m_n, m_{n+1}, \dots$ of the finite sequence $(m_0, \dots, m_n)$ coincides with the central extension of the finite sequence $(m_0, \dots, m_n, m_{n+1})$. Suppose we are able to show that the moment matrix m_{n+1} satisfies the conditions (24),(26) for $k = n + 1$. Incrementing n by one and repeating the reasoning will then prove the conditions for $k = n + 2$. Repeating the process, we prove the conditions for all $k > n$.

We shall hence consider the case $k = n+1$. Note that

$$\sum_{i=0}^{s} d_i \left(\frac{1}{2\pi} \int_{-\pi}^{+\pi} \frac{\lambda_0 I_p}{|d(e^{j\omega})|^2} e^{j(n+1-i)\omega} \, d\omega \right) = \frac{1}{2\pi j} \int_{\mathbb{T}} \frac{\lambda_0 I_p}{d(e^{j\omega})} e^{jn\omega} \, de^{j\omega} = 0,$$

because the integrand in the second integral can be extended to a function which is holomorphic inside the unit disc. It follows that (24) is valid for $k = n + 1$ if and only if (27) is valid for $k = n + 1$.

Now let the blocks of the matrix M in Lemma 2 have the values $A = m_0$, $B = \begin{pmatrix} m_1^T & \dots & m_n^T \end{pmatrix}$, $C = T_{n-1}$, $D^T = \begin{pmatrix} m_{n,21} & m_{n,22} & \dots & m_{1,21} & m_{1,22} \end{pmatrix}$, $E = m_{0,22}$. By virtue of the condition $T_n \succeq 0$ the matrix M in this lemma is partial positive semi-definite. Let now the block X from Lemma 3 be defined by $X^T = \begin{pmatrix} m_{n+1,21} & m_{n+1,22} \end{pmatrix}$. The relation $X = BC^\dagger D$ then follows from Definition 1. Let further F^T consist of the last p rows of the $l \times (n + 1)l$ matrix $\begin{pmatrix} 0 & 0 & \cdots & 0 & d_s I_l & d_{s-1} I_l & \cdots & d_0 I_l \end{pmatrix}$. Then the relation $\begin{pmatrix} C & D \end{pmatrix} F = 0$ follows from (26),(27) for $k = 1, \dots, n$. It then follows from Lemma 3 that $\begin{pmatrix} B & X \end{pmatrix} F = 0$ which is! equivalent to (26),(27) for $k = n + 1$.

Thus both conditions (24),(26) are valid for $k = n+1$, which completes the proof. ∎

Theorem 4. *Let $m_0 = m_0^T, m_1, \dots, m_n$ be a finite sequence of real $l \times l$ matrices, and set $m_{-k} = m_k^T$ for $k = 1, \dots, n$. Then $(m_0, \dots, m_n)$ is extendable to an infinite sequence $m_0, \dots, m_n, \dots$ satisfying the conditions of Theorem 2 if and only if conditions (28)—(30) hold.*

Proof: The only if part follows from the fact that the conditions in Theorem 2 imply (28)-(30). The if part follows from Theorem 3. ∎

Theorem 4 identifies (28)-(30) as the conditions on a finite sequence $m_0 = m_0^T, m_1, \dots, m_n$ of real $l \times l$ matrices to be realizable as a truncated sequence of generalized moments as in formula (7), with the joint power spectrum Φ_{χ_0} defining valid experimental conditions by virtue of (5),(6). This allows us to rewrite experiment design problems satisfying Assumption 1 as a convex program with the linear and semi-definite constraints (28)—(30). This will be accomplished in Section V.

Theorem 5. *Let $(m_0, \dots, m_n)$ be a $(n + 1)$-tuple of real $l \times l$ matrices satisfying $m_0 = m_0^T$, and define $m_{-k} = m_k^T$ for all $k = 1, \dots, n$. Suppose that these matrices satisfy conditions (28)-(30). Let $F(z)$ be the central Carathéodory function defined in Proposition 2. Then the power spectrum $\Phi_{\chi_0}(\omega) = |d(e^{j\omega})|^2 \cdot \Phi(\omega)$, where $\Phi(\omega)$ is given by the limit (17), satisfies the following properties: it is of the form (6), positive semi-definite, satisfies $\Phi_{\chi_0}(\omega) = \Phi_{\chi_0}(-\omega)^T$, its upper right block Φ_{ue} represents a stable transfer function, and it reproduces the truncated moment sequence $(m_0, \dots, m_n)$ by formula (7).*

Proof: The theorem follows from Proposition 2, Theorem 2, Theorem 3, and formula (17). ∎

C. Description of all feasible extensions

In Proposition 1 we have given the general form of the Carathéodory function of an infinite moment extension of a given finite moment sequence $m_0, \dots, m_n$. However, not every extension corresponds to a valid closed-loop experiment design. As we established in Section IV-A, a necessary and sufficient condition for this are the constraints (24), (26). We shall now express these conditions in terms of the Carathéodory function itself.

Theorem 6. *Let $(m_0, \dots, m_n)$ be a $(n + 1)$-tuple of real $l \times l$ matrices satisfying $m_0 = m_0^T$, and define $m_{-k} = m_k^T$ for all $k = 1, \dots, n$. Suppose that these matrices satisfy conditions (28)-(30). Let*

$F_c(z)$ be the central Carathéodory function of the finite moment sequence $(m_0,\dots,m_n)$, and let $F(z)$ be the Carathéodory function of an arbitrary infinite moment extension $m_0,\dots,m_n,m_{n+1},\dots$. Let $\Phi_{\chi_0}(\omega) = |d(e^{j\omega})|^2\cdot\Phi(\omega)$, where $\Phi(\omega)$ is the power spectrum constructed from $F(z)$ by the limit (17)*. Then $\Phi_{\chi_0}(\omega)$ is the joint power spectrum corresponding to a valid closed-loop experiment design if and only if the last p columns of the matrix-valued functions $F(z)$ and $F_c(z)$ coincide.*

Proof: Partition the matrix-valued functions $F = (F_1\ F_2)$, $F_c = (F_{c1}\ F_{c2})$ into submatrices with m and p columns, respectively. From (18) we obtain

$$\begin{aligned} F(z)d(z) &= m_0 d(z) + 2\sum_{i=0}^{s}\sum_{k=1}^{\infty} d_i m_k^T z^{k+i} \\ &= m_0 d(z) + 2\sum_{i=0}^{s}\sum_{k=i+1}^{\infty} d_i m_{k-i}^T z^k \\ &= m_0 d(z) - 2\sum_{i=0}^{s}\sum_{k=1}^{i} d_i m_{k-i}^T z^k \\ &\quad +2\sum_{k=1}^{\infty} z^k \sum_{i=0}^{s} d_i m_{k-i}^T. \end{aligned} \tag{31}$$

Suppose now that $\Phi_{\chi_0}(\omega)$ corresponds to a valid closed-loop experiment design. Then relations (26), (27) hold for all $k>0$. From (30) and (31) we then obtain that the product $F_2(z)d(z)$ is given by the last p columns of the expression $m_0d(z) - 2\sum_{i=0}^{s}\sum_{k=1}^{i} d_i m_{k-i}^T z^k$. Thus $F_2(z)$ is a rational function of degree s which depends only on the moments $m_0,\dots,m_{s-1}$. Since $s\le n$ by Assumption 1, it follows that *every* joint power spectrum $\Phi_{\chi_0}(\omega)$ corresponding to a valid closed-loop experiment design produces *the same* rational function $F_2(z)$. By Theorem 5 this holds in particular for the power spectrum $\Phi_{\chi_0}(\omega)$ coming from the central extension. Therefore we have $F_2(z) = F_{c2}(z)$.

Suppose now that $F_2(z) = F_{c2}(z)$. Note that by (17) the lower right $p\times p$ subblock of $\Phi_{\chi_0}(\omega)$ depends only on F_2. It has thus to be the same as that coming from the central extension, namely $\lambda_0 I_p$ by Theorem 5. This proves (24) for all $k\in\mathbb{Z}$. By the preceding paragraph, the product $F_{c2}(z)d(z)$ is given by the last p columns of the expression $m_0d(z) - 2\sum_{i=0}^{s}\sum_{k=1}^{i} d_i m_{k-i}^T z^k$. The product $F_2(z)d(z)$ is given by the same expression. But then from (31) it follows that the last p columns of the sum $\sum_{k=1}^{\infty} z^k\sum_{i=0}^{s} d_i m_{k-i}^T$ are identically zero. Therefore the moment extension satisfies (26),(27) for all $k>0$. Finally, we have $T_k\succeq 0$ for all $k\ge 0$ because $m_0,\dots,m_n,m_{n+1},\dots$ is a moment extension. Thus by Theorem 2 the power spectrum $\Phi_{\chi_0}(\omega)!$ corresponds to a valid closed-loop experiment design. ■

Theorem 6 allows to deduce a linear necessary and sufficient condition on the Schur function $\phi(z)$ in Proposition 1 guaranteeing that the Carathéodory function $F(z)$ defined by the parameter $\phi(z)$ defines a feasible extension.

Corollary 1. *Let $(m_0,\dots,m_n)$ be a $(n+1)$-tuple of real $l\times l$ matrices satisfying $m_0 = m_0^T$, and define $m_{-k} = m_k^T$ for all $k=1,\dots,n$. Suppose that these matrices satisfy conditions (28)-(30). Let $\phi(z)$ be a Schur function of size $l\times l$, and let $F(z)$ be the Carathéodory function defined by $\phi(z)$ as in Proposition 1. Let $\Phi_{\chi_0}(\omega) = |d(e^{j\omega})|^2\cdot\Phi(\omega)$, where $\Phi(\omega)$ is the power spectrum constructed from $F(z)$ by the limit* (17)*. Then $\Phi_{\chi_0}(\omega)$ is the joint power spectrum corresponding to a valid closed-loop experiment design if and only if the function $\phi(z)$ satisfies the condition*

$$\left(\mathbf{c}_n(z) - zL_{n+1}\phi(z)R_{n+1}^{\dagger}\tilde{\mathbf{a}}_n^{[n]}(z)\right)\begin{pmatrix}0\\ I_p\end{pmatrix} = \left(\mathbf{d}_n(z) + zL_{n+1}\phi(z)R_{n+1}^{\dagger}\tilde{\mathbf{b}}_n^{[n]}(z)\right)\mathbf{d}_n^{-1}(z)\mathbf{c}_n(z)\begin{pmatrix}0\\ I_p\end{pmatrix}.$$

Proof: The corollary follows from the expression for $F(z)$ in Proposition 1, Theorem 6, and the fact that $F_c(z) = \mathbf{d}_n^{-1}(z)\mathbf{c}_n(z)$. ■

D. Parametrization of all feasible extensions

In Section III we have considered finite and infinite extensions of a finite moment sequence $m_0,\dots,m_n$. Not every matrix m_{n+1} defines an extension of the sequence $m_0,\dots,m_n$. In Theorem 1 of Subsection III-B we have parametrized the set of such matrices m_{n+1} by a contractive matrix Δ_{n+1}, the Verblunsky parameter. The moment m_{n+1} is an explicit function of the parameter Δ_{n+1} and the preceding moments $m_0,\dots,m_n$, and the dependence on Δ_{n+1} is affine. Once the moment m_{n+1} is specified by the choice of a contractive matrix Δ_{n+1}, it becomes available for defining the coefficients of the parametrization of the next moment m_{n+2} by the next Verblunsky coefficient Δ_{n+2}. In this manner, one can construct step by step an extension of the sequence $m_0,\dots,m_n$ of arbitrary length, by choosing subsequently contractive matrices $\Delta_{n+1},\Delta_{n+2}!,\dots$. An infinite extension is then determined by the choice of an infinite number of Verblunsky coefficients. The central extension is determined by a special choice of these coefficients, namely $\Delta_k = 0$ for $k>n$.

However, an arbitrary choice of contractive matrices $\Delta_{n+1},\Delta_{n+2},\dots$ does not necessarily lead to a feasible extension as defined in the introduction to Section IV, i.e., an infinite moment sequence which may be realized by a valid closed-loop experiment design setup. In

Theorem 2 we have shown that in order to be feasible in this sense, the moments of the extension have to satisfy the constraints (24), (26). An arbitrary choice of Δ_{n+1} may lead to a moment m_{n+1} violating these conditions and precluding the possibility to achieve a feasible extension. In Subsection IV-B we have shown that the central extension is a feasible extension, and hence the choice $\Delta_{n+1} = 0$ leads to a moment m_{n+1} such that the sequence $m_0, \dots, m_n, m_{n+1}$ still possesses a feasible extension.

The purpose of this subsection is to deduce an analog of Theorem 1 providing a parametrization of the set of moments m_{n+1} such that the sequence $m_0, \dots, m_{n+1}$ possesses a feasible extension. The parameter will also be a contractive matrix $\hat{\Delta}_{n+1}$, which we call *restricted Verblunsky coefficient*. It will have smaller size and hence less degrees of freedom than the Verblunsky coefficient Δ_{n+1}, reflecting the stronger requirements on m_{n+1}. The moment m_{n+1} will also affinely depend on the parameter $\hat{\Delta}_{n+1}$, and the choice $\hat{\Delta}_{n+1} = 0$ will lead to the moment m_{n+1} defined by the central extension.

We shall determine all real $l \times l$ matrices m_{n+1} such that the block-Toeplitz matrix T_{n+1} is positive semi-definite and relations (24),(26) hold for $k = n + 1$. By virtue of $d_0 \neq 0$ the lower p rows of m_{n+1} are uniquely determined by the equivalent relations (26),(27) for $k = n + 1$. Namely, we have

$$m_{n+1,2\alpha} = -d_0^{-1} \sum_{i=1}^{s} d_i m_{n+1-i,2\alpha}, \qquad \alpha = 1, 2.$$

These lower p rows must in particular coincide with the lower p rows of the moment m_{n+1}^c defined by the central extension.

Now consider the positive semi-definite block-Toeplitz matrix T_{n+1}. The entries of this matrix which are determined by the moments $m_0, \dots, m_n$ and the lower p rows of m_{n+1} are specified, while the entries determined by the upper m rows of m_{n+1} remain unspecified. We shall now use Lemma 4 of the Appendix to describe this partially specified matrix. By setting $A = E = m_0$, $B = \begin{pmatrix} m_1^T & \dots & m_n^T \end{pmatrix}$, $C = T_{n-1}$, $D^T = \begin{pmatrix} m_n & \dots & m_1 \end{pmatrix}$, $X_2^T = \begin{pmatrix} m_{n+1,21} & m_{n+1,22} \end{pmatrix}$ in this lemma, the partially specified matrix $\hat{M}$ becomes equal to T_{n+1}. In order to describe the remaining unspecified entries, we set $X_1^T = \begin{pmatrix} m_{n+1,11} & m_{n+1,12} \end{pmatrix}$. Let the matrices D, E be partitioned as in Lemma 4 into submatrices of m and p columns. The relation $X_2 = BC^\dagger D_2$ required by the! conditions of the lemma then follows from the definition of the central extension in Subsection III-C. By Lemma 4, the matrix X_1 containing the elements of the upper m rows of m_{n+1} is parametrized as in (33) of that lemma by a contractive $l \times m$ matrix $\hat{\Delta}$.

This yields us the required parametrization of the set of moments m_{n+1} which may appear in a feasible extension of the sequence $m_0, \dots, m_n$. We will denote the parameter $\hat{\Delta}_{n+1}$ and call it *restricted Verblunsky parameter*.

Having determined the moment m_{n+1} by the choice of the restricted Verblunsky parameter $\hat{\Delta}_{n+1}$, we may proceed in an analogous manner to the definition of the next moment m_{n+2} by the choice of the restricted Verblunsky parameter $\hat{\Delta}_{n+2}$. In this way, all the infinite moment extensions of the sequence $(m_0, \dots, m_n)$ which satisfy the conditions of Theorem 2 can be parametrized by the infinite choice sequence $\hat{\Delta}_{n+1}, \hat{\Delta}_{n+2}, \dots$ of contractive $l \times m$ matrices.

By Lemma 5 in the Appendix, the choice $\hat{\Delta}_k = 0$ for all $k > n$ leads to the central extension of the sequence $(m_0, \dots, m_n)$. In the same way, the choice $\hat{\Delta}_{k'} = 0$ for all $k' > n+k$ leads to the central extension of the sequence $(m_0, \dots, m_n, m_{n+1}, \dots, m_{n+k})$. Here the moments $m_{n+1}, \dots, m_{n+k}$ are parameterized by the remaining k free restricted Verblunsky parameters $\hat{\Delta}_{n+1}, \dots, \hat{\Delta}_{n+k}$. In this way, we obtain a set of infinite moment extensions which is parameterized algebraically by the klm elements of these matrices.

Note that m_{n+1} is affine in $\hat{\Delta}_{n+1}$. The Verblunsky parameter Δ_{n+1} in (19) and hence the polynomials $\mathbf{a}_{n+1}, \mathbf{b}_{n+1}, \mathbf{c}_{n+1}, \mathbf{d}_{n+1}$ can then also be written as affine functions in $\hat{\Delta}_{n+1}$. If only the first parameter $\hat{\Delta}_{n+1}$ is free, while the other parameters are fixed to zero, then by Proposition 2 the Carathéodory function associated to the joint power spectrum Φ_{χ_0} is rational in $\hat{\Delta}_{n+1}$. The parameter $\hat{\Delta}_{n+1}$ can then be chosen in order to achieve additional goals of the experiment.

V. Solution algorithm

In this section we outline a general scheme for the solution of optimal experiment design problems satisfying Assumption 1. The scheme consists of two steps. First we find the optimal truncated moment sequence by solving a convex program with semi-definite constraints, and then we recover the experimental conditions (i.e., the power spectrum Φ_r of the external input and the controller K from this moment sequence) from the central extension of this finite moment sequence.

Apart from the constraints following from the formulation of the particular problem instance under consideration, the moment sequence $(m_0, \dots, m_n)$ has to satisfy conditions (28)—(30). Condition (28) amounts to a linear matrix inequality. Condition (29) determines the blocks $m_{k,22}$ explicitly, while condition (30) yields linear relations on the blocks $m_{k,21}$. The optimal experiment design problem defined in Assumption 1 is thus turned into the following convex program.

$$\min_{m_k, x_l} f_0(m_0, \ldots, m_n, x_1, \ldots, x_N) \tag{32}$$

subject to the constraints

$$\mathcal{A}(m_0, m_1, \ldots, m_n, x_1, x_2, \ldots, x_N) \succeq 0,$$

$$m_{k,22} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{\lambda_0 I_p}{|d(e^{j\omega})|^2} e^{jk\omega}\, d\omega, \quad k = 0, \ldots, n,$$

$$\sum_{i=0}^{s} d_i m_{k-i,21} = 0, \quad k = 1, \ldots, n,$$

$$T_n = \begin{pmatrix} m_0 & m_1^T & \ddots & m_n^T \\ m_1 & m_0 & \ddots & m_{n-1}^T \\ \ddots & \ddots & \ddots & \ddots \\ m_n & m_{n-1} & \ddots & m_0 \end{pmatrix} \succeq 0,$$

where $m_{-k} = m_k^T$. By solving this convex program, the user obtains the optimal truncated moment sequence $(m_0, \ldots, m_n)$ and the optimal value of the cost function. Note that if the cost function f_0 is linear, then the convex program is a semi-definite program.

The power spectrum (6) can be obtained as a rational function with possibly a singular part by the formula $\Phi_{\chi_0}(\omega) = |d(e^{j\omega})|^2 \cdot \Phi(\omega)$, where $\Phi(\omega)$ is given by the limit (17) and $F(z)$ is the central Carathéodory function given in Proposition 2. Note that this solution works even if T_n is singular. We shall give an example in the next section when the singular part is absent even if T_n is singular.

The power spectrum Φ_r and the controller K can then be recovered from Φ_{ue} and the upper left $m \times m$ block Φ_u by formulas (5).

If the matrix T_n corresponding to the solution happens to be positive definite, then (23) allows to recover the joint power spectrum (6) in a more straightforward manner by the explicit formula

$$\Phi_{\chi_0}(\omega) = |d(e^{j\omega})|^2 \cdot A(e^{j\omega})^{-*} A(0) A(e^{j\omega})^{-1},$$

where $A(z) = U(z) T_n^{-1} U^T(0)$ and $U(z) = \begin{pmatrix} z^n I_l & z^{n-1} I_l & \cdots & I_l \end{pmatrix}$.

As is often the case in optimal experiment design, the calculation of the optimal experimental conditions requires knowledge of the transfer functions G_0, H_0 to be identified. This obstacle can be circumvented by performing a preliminary identification experiment and/or applying an iterative procedure, using the estimates from the previous iteration for the design of the experimental conditions in the current one.

VI. Examples

Example 1

In this first example, we illustrate the construction by virtue of the Carathódory function (16) of the spectrum Φ corresponding to the central extension of a singular moment matrix. In this example the spectrum defined by this central extension remains finite. Let $n = 1$, $l = 2$, and consider the moment matrix

$$T_1 = \begin{pmatrix} m_0 & m_1^T \\ m_1 & m_0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \frac{2}{3} & \frac{2}{3} \\ 0 & 1 & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & -\frac{2}{3} & 1 & 0 \\ \frac{2}{3} & \frac{1}{3} & 0 & 1 \end{pmatrix}.$$

This matrix is positive semi-definite and singular. From (19), (20) we get $L_1 = R_1 = m_0^{1/2} = I_2$, $\Delta_1 = m_1^T$. The polynomials (21) are then given by $\mathbf{a}_1(z) = \mathbf{c}_1(z) = m_0 + z m_1^T$, $\mathbf{b}_1(z) = \mathbf{d}_1(z) = m_0 - z m_1^T$. By Proposition 2, the Carathéodory function (16) obtained from the central extension of the sequence (m_0, m_1) is given by $F(z) = (m_0 + z m_1^T)(m_0 - z m_1^T)^{-1} = (m_0 - z m_1^T)^{-1}(m_0 + z m_1^T) = \frac{1}{1 - z + \frac{2}{3}z^2} \begin{pmatrix} 1 + \frac{1}{3}z - \frac{2}{3}z^2 & \frac{4}{3}z \\ -\frac{4}{3}z & 1 - \frac{1}{3}z - \frac{2}{3}z^2 \end{pmatrix}$. The polynomial in the denominator has all roots outside of the unit circle, and hence $F(z)$ can be analytically extended to a neighbourhood of the unit disk. From (17) we then obtain

$$\Phi(\omega) = \frac{1}{2}(F(e^{j\omega}) + F^*(e^{j\omega}))$$
$$= \begin{pmatrix} \frac{2 + e^{j\omega} + e^{-j\omega}}{9|1 - e^{j\omega} + \frac{2}{3}e^{2j\omega}|^2} & \frac{2(e^{j\omega} - e^{-j\omega})}{9|1 - e^{j\omega} + \frac{2}{3}e^{2j\omega}|^2} \\ \frac{2(e^{-j\omega} - e^{j\omega})}{9|1 - e^{j\omega} + \frac{2}{3}e^{2j\omega}|^2} & \frac{8 - 4(e^{j\omega} + e^{-j\omega})}{9|1 - e^{j\omega} + \frac{2}{3}e^{2j\omega}|^2} \end{pmatrix}.$$

The spectrum is not positive definite, but still finite.

Example 2

Here we shall solve the problem given in the example in Section II. Following Section V, we obtain the maxdet problem

$$\min_{m_0, m_1} \left(-\log(\overline{M}_{11}\overline{M}_{22} - \overline{M}_{12}^2) \right)$$

under the constraints

$$c - \overline{E}y^2 \geq 0,$$

$$m_{k,22} = \frac{1}{2\pi}\int_{-\pi}^{+\pi} \frac{\lambda_0 I_p}{|d(e^{j\omega})|^2} e^{jk\omega}\, d\omega, \quad k = 0, 1,$$

$$m_{1,21} + a_0 m_{0,21} = 0,$$

$$T_n = \begin{pmatrix} m_0 & m_1^T \\ m_1 & m_0 \end{pmatrix} \succeq 0,$$

where for $\overline{E}y^2$ and the elements of $\overline{M}$ we have to insert expressions (13). The integrals evaluate to $m_{0,22} = \frac{\lambda_0}{1-a_0^2}$, $m_{1,22} = -\frac{\lambda_0 a_0}{1-a_0^2}$. The other elements of the moment matrices are given by the explicit solution

$$
\begin{aligned}
m_{0,11} &= \frac{(c(1-a_0^2)+\lambda_0 a_0^2)(ca_0^2+c-\lambda_0)}{b_0^2(1-a_0^2)(c+(c-\lambda_0)a_0^2)}, \\
m_{0,12} &= \frac{\lambda_0 a_0(2c-\lambda_0)}{b_0(1-a_0^2)(c+(c-\lambda_0)a_0^2)}, \\
m_{1,11} &= -\frac{\lambda_0 a_0(ca_0^2+c-\lambda_0)}{b_0^2(1-a_0^2)(c+(c-\lambda_0)a_0^2)}, \\
m_{1,12} &= -\frac{m_{0,12}a_0(\Xi+(c-\lambda_0)\lambda_0(1-a_0^2)^2)}{\Xi}, \\
m_{1,21} &= -a_0 m_{0,12},
\end{aligned}
$$

where we denoted $\Xi = c^2(1+a_0^2)^2 - c\lambda_0(2a_0^4+a_0^2+1)+\lambda_0^2 a_0^4$. This solution gives rise to a positive definite block-Toeplitz matrix T_1. The controller K and power spectrum Φ_r resulting from the central extension of T_1 are given by

$$
\begin{aligned}
K &= -\frac{a_0(2c-\lambda_0)(ca_0^2+c-\lambda_0)(1+a_0z^{-1})}{b_0(\Xi+a_0(2c(c-\lambda_0)(1+a_0^2)+\lambda_0^2a_0^2)z^{-1})}, \\
\Phi_r &= \frac{(c-\lambda_0)(ca_0^2+c-\lambda_0)(c+(c-\lambda_0)a_0^2)\Xi|e^{j\omega}+a_0}{b_0^2|\Xi e^{j\omega}+a_0(2c(c-\lambda_0)(1+a_0^2)+\lambda_0^2a_0^2)|^2}
\end{aligned}
$$

Let us compare this solution to the optimal open-loop experiment. In this case $\Phi_{ue} = 0$, and the off-diagonal elements of the moment matrices are fixed to zero. Moreover, for an open-loop experiment to be feasible the condition $c \geq \frac{\lambda_0}{1-a_0^2}$ is necessary. Under this condition the remaining moments have the optimal values $m_{0,11} = b_0^{-2}(c - \frac{\lambda_0}{1-a_0^2})$,

$$
m_{1,11} = \begin{cases} \operatorname{sgn}(a_0)m_{0,11}, & c \leq \frac{\lambda_0}{1-|a_0|}, \\ \frac{\lambda_0 a_0}{b_0^2(1-a_0^2)}, & c > \frac{\lambda_0}{1-|a_0|}. \end{cases}
$$

Thus for $c \leq \frac{\lambda_0}{1-|a_0|}$ the input power spectrum for the optimal open-loop experiment is discrete and given by $\Phi_u(\omega) = 2\pi(1+|a_0|)^2 m_{0,11}\delta(\omega-\omega_0)$, with $\omega_0 = 0$ if $a \geq 0$, and $\omega_0 = \pi$ if $a_0 < 0$. For $c > \frac{\lambda_0}{1-|a_0|}$ the continuous input power spectrum given by $\Phi_u(\omega) = m_{0,11}|1+a_0e^{j\omega}|^2\frac{m_{0,11}^2-m_{1,11}^2}{|m_{0,11}-m_{1,11}e^{j\omega}|^2}$ is optimal. Incidentally, for this latter case the determinants of the information matrices obtained from the optimal open-loop and closed-loop experiment designs have the same value. For $c < \frac{\lambda_0}{1-|a_0|}$ the optimal closed-loop experiment design leads to a strictly smaller value of the cost function than the optimal open-loop design.

Simulation results. For the values $\lambda_0 = 1$, $c = 1.4$, $b_0 = 0.5$, $a_0 = 0.4$ we first identify the system with an open-loop experiment using white noise with variance $\sigma^2 = 1$ as input. From the identified parameters two experimental configurations are computed, namely the optimal open-loop input, and the optimal closed-loop input-controller pair. An optimal open-loop and an optimal closed-loop experiment are then performed and the parameter vector identified. The data length in each of the experiments is $N = 1000$. The identified parameter vectors for 500 runs are plotted in Fig. 2. The empirical covariance matrices of the 500 identified parameter vectors have determinant $0.49736N^{-2}$ and $0.38796N^{-2}$ for the open-loop and the closed-loop experiments, respectively. We see that the empirical covariance matrix has a 28% smaller determinant for the closed-loop experiments.

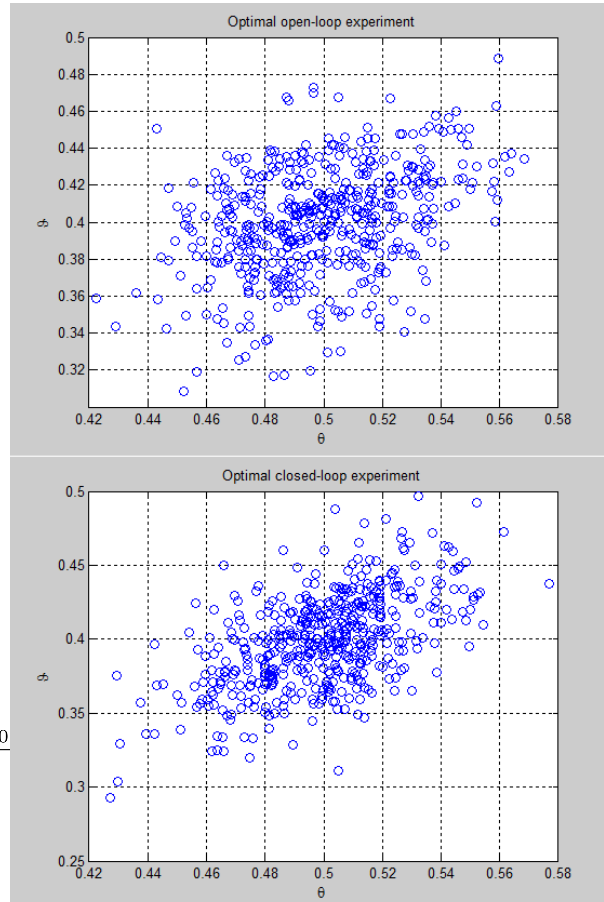

Fig. 2. Identified parameter vectors for optimal open-loop (top) and closed-loop (bottom) experiments

VII. Conclusions

We have provided a solution to the closed loop optimal experiment design for MIMO systems. The solution uses the so-called *partial correlation approach* in which the criterion and the constraints are expressed as a function of a finite set of generalized moments. The optimal moments are then obtained as the solution of a semi-definite program. The key difficulty of this approach, which had been a stumbling block so far, is to extend the finite set of optimal moments into an infinite set, or equivalently into a spectrum, because the spectrum must obey some constraints which are due to the closed loop setup. Thus, the classical Carathéodory-Fejer theorem cannot be used to produce a feasible extension.

Our main contribution has been to show that the so-called *central extension* is a feasible extension, which satisfies these constraints. In addition, using properties of the central extension, as well as results on the *positive matrix completion theorem*, we have shown how to construct families of parametrized optimal extensions which also obey the constraints of the optimal experiment design problem.

One of the key advantages of the solution method developed in the present paper is that it allows one to explicitly compute an optimal solution for the spectrum Φ_r of the external excitation signal and the feedback controller K. They can be computed straightforwardly from the optimal moments that result from the solution of the semi-definite program. This is a significant progress over our previous result [16] which only proved the existence of an optimal spectrum, but without an explicit computational procedure.

Appendix

In this Appendix we provide auxiliary results related to the *positive matrix completion problem*. This is the problem of completing a real symmetric matrix, only part of whose entries are specified, to a full positive semi-definite matrix. A partially specified matrix M is said to be *partial positive semi-definite* if all diagonal entries of M are specified, and every principal submatrix of M which is fully specified is positive semi-definite. A partially specified matrix M is said to be *positive semi-definite completable* if there exists a specification of the unspecified entries of M such that the resulting fully specified matrix is positive semi-definite. Clearly partial positive semi-definiteness is a necessary condition for positive semi-definite completability. There exist specification patterns for which this condition is also sufficient. These patterns have been completely described in [14] by graph-theoretic! means. We shall need only a special case of such specification patterns, namely when the unspecified entries can be arranged in a rectangular block by a suitable permutation of the row and column indices of M. In this case the set of all completions has a closed-form description as an affine image of a matrix ball. This fact has been brought to our attention by Keith Glover.

The results in this Appendix, and in particular Lemma 2 and Lemma 4, are required to prove that the moment extension in Theorem 1 is an admissible extension in that it produces $T_{n+1} \succeq 0$.

Lemma 1. *[12, Theorem 16.1, p.435] A real symmetric matrix* $M = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ *is positive semi-definite if and only if* $C \succeq 0$, $(I - CC^\dagger)B^T = 0$, *and* $A - BC^\dagger B^T \succeq 0$. *In this case we have the factorization*

$$M = \begin{pmatrix} I & BC^\dagger \\ 0 & I \end{pmatrix} \begin{pmatrix} A - BC^\dagger B^T & 0 \\ 0 & C \end{pmatrix} \begin{pmatrix} I & 0 \\ C^\dagger B^T & I \end{pmatrix}.$$

Here $C^\dagger$ *denotes the pseudo-inverse of* C, *and* I *denote identity matrices of appropriate size.* □

Lemma 2. *[13] Let* $M = \begin{pmatrix} A & B & * \\ B^T & C & D \\ * & D^T & E \end{pmatrix}$ *be a real partial positive semi-definite matrix, where* A, B, C, D, E *are blocks of compatible sizes. Then the matrix* $M_X = \begin{pmatrix} A & B & X \\ B^T & C & D \\ X^T & D^T & E \end{pmatrix}$ *is a positive semi-definite completion of* M *if and only if the block* X *can be written as* $X = BC^\dagger D + (A - BC^\dagger B^T)^{1/2} \Delta (E - D^T C^\dagger D)^{1/2}$, *where* Δ *is a real matrix satisfying the condition* $\sigma_{\max}(\Delta) \leq 1$. *Here* $\sigma_{\max}$ *denotes the maximal singular value and* $W^{1/2}$ *the positive semi-definite matrix square root of the positive semi-definite matrix* W.

Proof: Since M is partial positive semi-definite, the matrices $\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$ and $\begin{pmatrix} E & D^T \\ D & C \end{pmatrix}$ are positive semi-definite. Applying Lemma 1 to these matrices, we obtain that $C \succeq 0$, $(I - CC^\dagger)B^T = 0$, $(I - CC^\dagger)D = 0$, $A - BC^\dagger B^T \succeq 0$, $E - D^T C^\dagger D \succeq 0$. Applying Lemma 1 to the matrix $\left(\begin{array}{cc|c} A & X & B \\ X^T & E & D^T \\ \hline B^T & D & C \end{array}\right)$, we obtain that $M_X \succeq 0$ if and only if

$$\begin{aligned} &\begin{pmatrix} A & X \\ X^T & E \end{pmatrix} - \begin{pmatrix} B \\ D^T \end{pmatrix} C^\dagger \begin{pmatrix} B^T & D \end{pmatrix} \\ &= \begin{pmatrix} A - BC^\dagger B^T & X - BC^\dagger D \\ (X - BC^\dagger D)^T & E - D^T C^\dagger D \end{pmatrix} \succeq 0. \end{aligned}$$

The claim of the lemma now easily follows. ■

The next result deals with the specific choice $\Delta = 0$.

Lemma 3. *Assume the conditions of Lemma 2, and set* $X = BC^\dagger D$. *Assume that there exists a matrix* F *of appropriate size such that* $\begin{pmatrix} C & D \end{pmatrix} F = 0$. *Then we have also* $\begin{pmatrix} B & X \end{pmatrix} F = 0$.

Proof: Partition $F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$ into subblocks of appropriate size. We have $CF_1 + DF_2 = 0$, and hence

$BF_1 + XF_2 = B(F_1 + C^\dagger DF_2) = B(I - C^\dagger C)F_1 = 0$. Here the last equality follows from Lemma 1. ∎

Lemma 2 permits to obtain a parametrization of all positive semi-definite matrix completions not only in the case when the unspecified elements form a rectangular block in the upper right corner, but also when such a situation can be achieved by a suitable permutation of the row and column indices.

Lemma 4. *Assume the conditions of Lemma 2, but let the unknown block be partitioned as* $X = \begin{pmatrix} X_1 & X_2 \end{pmatrix}$. *Let the blocks* $D = \begin{pmatrix} D_1 & D_2 \end{pmatrix}$, $E = \begin{pmatrix} E_{11} & E_{12} \\ E_{12}^T & E_{22} \end{pmatrix}$ *be partitioned in a compatible manner.*

Then the partially specified matrix $\hat{M} = \begin{pmatrix} A & B & * & X_2 \\ B^T & C & D_1 & D_2 \\ * & D_1^T & E_{11} & E_{12} \\ X_2^T & D_2^T & E_{12}^T & E_{22} \end{pmatrix}$, *where* $X_2 = BC^\dagger D_2$, *is partial positive semi-definite. The general form of a positive semi-definite completion* X_1 *of* $\hat{M}$ *is given by*

$$\begin{aligned} &\begin{pmatrix} B & X_2 \end{pmatrix} \begin{pmatrix} C & D_2 \\ D_2^T & E_{22} \end{pmatrix}^\dagger \begin{pmatrix} D_1 \\ E_{12}^T \end{pmatrix} + \\ &+ \left(A - \begin{pmatrix} B & X_2 \end{pmatrix} \begin{pmatrix} C & D_2 \\ D_2^T & E_{22} \end{pmatrix}^\dagger \begin{pmatrix} B^T \\ X_2^T \end{pmatrix} \right)^{1/2} \hat{\Delta} \\ &\times \left(E_{11} - \begin{pmatrix} D_1^T & E_{12} \end{pmatrix} \begin{pmatrix} C & D_2 \\ D_2^T & E_{22} \end{pmatrix}^\dagger \begin{pmatrix} D_1 \\ E_{12}^T \end{pmatrix} \right)^{1/2}, \end{aligned} \tag{33}$$

where $\hat{\Delta}$ *is any real matrix of size compatible with those of* A *and* E_{11} *such that* $\sigma_{\max}(\hat{\Delta}) \leq 1$.

Proof: The choice $\Delta = 0$ in Lemma 2 leads to $X_\alpha = BC^\dagger D_\alpha$, $\alpha = 1, 2$. Hence $\hat{M}$ is positive semi-definite completable. In particular, it must be partial positive semi-definite. The general form of its positive semi-definite completion X_1 follows by application of Lemma 2 to $\hat{M}$, after an appropriate permutation of rows and columns. ∎

Lemma 5. *Assume the conditions of Lemma 2 and Lemma 4. Completing the matrix* M *by* $X = BC^\dagger D$, *i.e., by the choice* $\Delta = 0$, *leads to the same result as first setting* $X_2 = BC^\dagger D_2$ *and then completing* $\hat{M}$ *by* $X_1 = \begin{pmatrix} B & X_2 \end{pmatrix} \begin{pmatrix} C & D_2 \\ D_2^T & E_{22} \end{pmatrix}^\dagger \begin{pmatrix} D_1 \\ E_{12}^T \end{pmatrix}$, *i.e., by the choice* $\hat{\Delta} = 0$.

Proof: We have to show that $BC^\dagger D_1 = \begin{pmatrix} B & BC^\dagger D_2 \end{pmatrix} \begin{pmatrix} C & D_2 \\ D_2^T & E_{22} \end{pmatrix}^\dagger \begin{pmatrix} D_1 \\ E_{12}^T \end{pmatrix}$. By Lemma 1 we have

$$\begin{aligned} \begin{pmatrix} C & D_2 \\ D_2^T & E_{22} \end{pmatrix} &= \begin{pmatrix} I & 0 \\ D_2^T C^\dagger & I \end{pmatrix} \cdot \\ &\cdot \begin{pmatrix} C & 0 \\ 0 & E_{22} - D_2^T C^\dagger D_2 \end{pmatrix} \begin{pmatrix} I & C^\dagger D_2 \\ 0 & I \end{pmatrix}, \end{aligned}$$

and hence

$$\begin{aligned} \begin{pmatrix} C & D_2 \\ D_2^T & E_{22} \end{pmatrix}^\dagger &= \begin{pmatrix} I & -C^\dagger D_2 \\ 0 & I \end{pmatrix} \cdot \\ &\cdot \begin{pmatrix} C^\dagger & 0 \\ 0 & (E_{22} - D_2^T C^\dagger D_2)^\dagger \end{pmatrix} \begin{pmatrix} I & 0 \\ -D_2^T C^\dagger & I \end{pmatrix}. \end{aligned}$$

It follows that $\begin{pmatrix} I & C^\dagger D_2 \end{pmatrix} \begin{pmatrix} C & D_2 \\ D_2^T & E_{22} \end{pmatrix}^\dagger = \begin{pmatrix} C^\dagger & 0 \end{pmatrix}$, which implies our claim. ∎

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