



For Model-based Control Design, Closed-loop Identification Gives Better Performance*

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In identification for control with no unmodelled dynamics, the best controller performance is achieved by performing the identification in closed loop.

Key Words—Identification for control; experiment design; closed-loop identification.

Abstract—We compare open loop versus closed loop identification when the identified model is used for control design, and when the system itself belongs to the model class, so that only variance errors are relevant. Our measure of controller performance (which is used as our design criterion for identification) is the variance of the error between the output of the ideal closed loop system (with the ideal controller) and that of the actual closed loop system (with the controller computed from the identified model). Under those conditions, we show that, when the controller is a smooth function of the input–output dynamics and the disturbance spectrum, the best controller performance is achieved by performing the identification in closed loop with an operating controller that we characterize. For minimum variance and model reference control design criteria, we show that this ‘optimal operating controller for identification’ is the ideal controller. This then leads to a suboptimal but feasible iterative scheme. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

Consider that a linear time-invariant system, perturbed by noise, is to be controlled and that a control design criterion has been selected. In this paper we shall focus particularly on model-based control design criteria for which the controller at any frequency is a smooth function of the plant input–output model and of the disturbance spectrum (i.e. the noise description) at that same frequency. We shall comment on the situation when the controller depends on the input–output dynamics only. In particular, we shall study in detail minimum-variance (MV) and model reference (MR) control design. If the

plant input–output transfer function $G(q)$ and its noise model $H(q)$ were known exactly, the solution of the control design problem would lead to the (one- or two-degree-of-freedom) controller,[§] $C(q) = c(G, H)$. Throughout this paper we shall denote this controller that would result from the design criterion applied to the true system $[G, H]$ as the *ideal controller*. The corresponding closed loop system is represented in Fig. 1, which has been drawn in all generality for a two-degree-of-freedom controller, and which we will call the *ideal closed-loop system*. We denote this ideal closed-loop system as $\mathcal{H}(G, H, C)$.

We now consider the situation where the ‘true system’ of Fig. 1 is unknown and the controller is computed on the basis of the same criterion as above from a model that is to be obtained by identification from a finite set of N input–output data collected on the true system. We further assume that the model set $\{G(q, \theta), H(q, \theta), \theta \in D_\theta\}$ used during the identification is sufficiently complex that this set contains the true system. Thus the identified model contains only variance errors (i.e. errors caused by noise on a finite data set). In modern day engineering-speak we say that ‘the system is in the model set’ or that ‘there is no bias error’. Because the identified model $[\hat{G}_N, \hat{H}_N] = [G(\hat{\theta}_N), H(\hat{\theta}_N)]$ is not identical to the true system $[G, H]$, the controller computed from $[\hat{G}_N, \hat{H}_N]$ using the control design criterion, denoted by $\hat{C}_N = c(\hat{G}_N, \hat{H}_N)$, will differ from the ideal one $C = c(G, H)$ described above. When this controller is applied to the actual system, the ‘ideal closed loop’ of Fig. 1 is replaced by the *actual (or achieved) closed-loop system* of Fig. 2.

* Received 14 October 1994; revised 7 August 1995; received in final form 4 June 1996. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor Paul Van den Hof under the direction of Editor Torsten Söderström. Corresponding author Professor Michel Gevers. Tel. +32 10 472590; Fax +32 10 472180; E-mail gevers@auto.ucl.ac.be.

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[§] In the case of a two-degree-of-freedom controller, $C(q)$ denotes the vector $[C_1(q) \ C_2(q)]$.

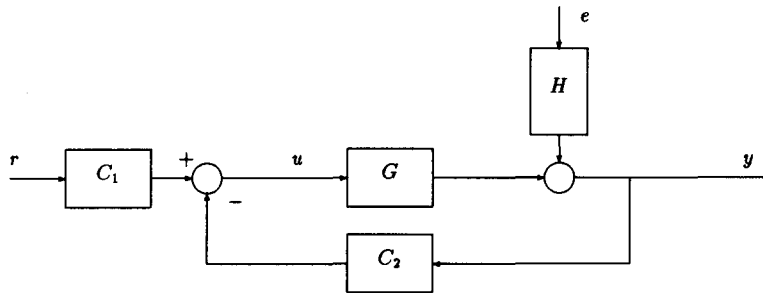


Fig. 1. The ideal two-degree-of-freedom control loop.

We denote this closed loop system as $\mathcal{H}(G, H, \hat{C})$.

Since the objective of the identification of $[\hat{G}_N, \hat{H}_N]$ is only to compute the controller (i.e. the model is only used as a vehicle to do control design), what really matters is not the error between the pairs $[G, H]$ and $[\hat{G}_N, \hat{H}_N]$, but the error between C and \hat{C}_N , or—more importantly—the difference between the behaviour of the ideal closed-loop system $\mathcal{H}(G, H, C)$ of Fig. 1 and that of the actual closed-loop system $\mathcal{H}(G, H, \hat{C}_N)$ of Fig. 2. There are of course many ways of measuring the mismatch between two closed-loop systems. Since we shall deal in this paper with H_2 identification criteria, we shall measure this error by the variance of the error between the output signals of these two loops, i.e. by $\lim_{T \rightarrow \infty} T^{-1} E\{\sum_{t=1}^T [y(t) - y_N(t)]^2\}$, when these two loops are driven by the same signals r and e : see Figs 1 and 2. This is a control performance measure: it is a measure of the degradation that results from applying the suboptimal controller $\hat{C}_N = c(\hat{G}_N, \hat{H}_N)$ to the plant $[G, H]$, instead of the optimal controller $C = c(G, H)$. Since \hat{C}_N results from the identification, this control performance measure is a ‘control-oriented measure’ of the quality of the identification. Thus we shall take this control performance measure as our identification design criterion; that is, we shall determine the identification design that minimizes this control performance measure. Our results carry over easily to other control performance measures, such as any

frequency-weighted measure of $E[|C(e^{i\omega}) - \hat{C}_N(e^{i\omega})|^2]$, or a measure of the form $\lim_{T \rightarrow \infty} T^{-1} E\{\sum_{t=1}^T [y(t) - y_N(t)]^2 + \lambda [u(t) - u_N(t)]^2\}$, more appropriate in the case of LQG design.

The problem addressed in this paper is that of *identification design*, in the context explained above where the identified model is just a tool for the computation of a controller (with an a priori chosen control design criterion), and where the error on $[\hat{G}_N, \hat{H}_N]$ is only due to the noise on the realization used for identification. The error on the controller, that is $C - \hat{C}_N$, is then also due to noise only, i.e. there is no bias error on \hat{C}_N . We observe that $\hat{C}_N = c(\hat{G}_N, \hat{H}_N)$ is a random variable. The identification design problem is then to design the identification experiment in such a way as to minimize the performance degradation between the ideal and the actual closed loops, as measured by the output error variance described above. We will assume here that the length of the experiment, i.e. the number of data N that can be collected, is fixed and that a direct least-squares prediction error method based on N input–output data is used (Ljung, 1987). The identification design problem then involves questions such as

- should one perform the identification in open or closed loop?
- if the identification should be performed in closed loop, which controller should one apply during identification?

Our contribution is threefold. First we show

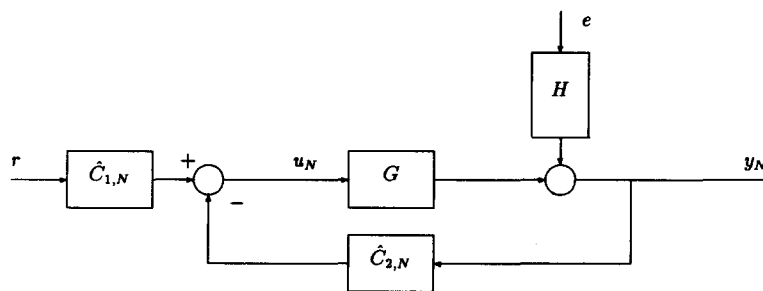


Fig. 2. The actual two-degree-of-freedom control loop.

that, when the system is in the model set, when both G and H are estimated and when the controller $C = c(G, H)$ at any frequency is a smooth function of both G and H at that particular frequency, the ideal experimental conditions are to perform the identification in closed loop, with some ideal controller C_{id}^{opt} operating on the plant. This result holds independently of the level of external excitation or of possible constraints on input or output power, as long as the classical closed-loop identifiability conditions are satisfied (see e.g. Söderström *et al.*, 1976).

Next we show that, for minimum-variance (MV) or model reference (MR) control, this controller C_{id}^{opt} that should be applied during closed-loop identification is the ideal controller $C = c(G, H)$. For MV control design, this result is well known (by those who know): Gevers and Ljung (1986) showed that for minimum-variance control design the optimal identification setup is to perform closed-loop identification with the optimal (MV) controller operating during data collection. The problem with these results, as with most optimal experiment design results, is that the optimal identification design depends on the unknown system that is being identified, since it requires knowledge of the ideal controller. An obvious question then is whether applying a controller that is close to this optimal controller is any good, or is at least better than just doing open-loop identification. Gevers and Ljung (1986) had nothing to say about this.

Our third contribution is to show that, under the conditions outlined above (a finite set of N noisy data, no bias error on the model, a control design such that the controller depends on both G and H) and under an additional stabilizability condition that is, for example satisfied for MV and for MR control design, one can obtain a better control performance on the actual system by computing the controller from a model that has been identified partly on closed-loop data, provided N is large enough. More precisely, we shall compare two identification designs.

1. Identify the model $[\hat{G}_N, \hat{H}_N]$ using N data collected in open loop; then compute the controller \hat{C}_N from this model.
2. Identify a model $[\hat{G}_{N_1}, \hat{H}_{N_1}]$ from open-loop data using a fraction N_1 of the total experiment length; compute the corresponding optimal controller for identification \hat{C}_{id, N_1}^{opt} ; apply it to the system and identify the model $[\hat{G}_N, \hat{H}_N]$ using the remaining $N - N_1$ data collected on the corresponding closed-loop system. Compute the controller \hat{C}_N from this model.

We show that one can always obtain a better performance (i.e. a smaller degradation *vis-à-vis* the optimal controller as measured by our output error variance criterion) with the second design than with the first, provided N is large enough. This new result is a practical one. It suggests a practically implementable identification design in which the experiment itself does not require knowledge of the system that is being identified. This design is not optimal (since the optimal design depends on the unknown system), but it is implementable.

A number of recent results tend to support the idea that if a model is identified for the purpose of designing a controller then closed-loop identification is to be preferred over open-loop identification: see Schrama (1992), and the survey papers by Gevers (1993) and Van den Hof and Schrama (1994, 1995) with the many references therein. Most of this recent work focuses on bias errors, and the arguments developed in these papers are heuristic and supported by simulations. However, see Hjalmarsson *et al.* (1994) for an analysis of the convergence/divergence properties of the iterative identification and control schemes that includes the case of unmodelled dynamics. Our results in this paper are focused on variance errors only, and in that sense they can be criticized for requiring high-order models. However, given this limitation, they are probably the first hard results that demonstrate that one can achieve better performance with a controller obtained from a model identified on closed-loop data than from a model identified on open-loop data, with a practically feasible experiment design scheme.

Finally, we note that all our results are derived for the situation where a 'direct' prediction error method is used for both open-loop and closed-loop identification, i.e. a prediction error method based on input-output data only. Alternative indirect methods for closed-loop identification that use knowledge of the controller or of the reference signal (see e.g. Van den Hof and Schrama, 1994, 1995) have not been examined here.

The organization of the paper is as follows. In Section 2 we introduce the problem and we define the error criterion that is relevant in this context of identification for control with high order models. In Section 3 we compute the optimal identification conditions for any control design criterion leading to a one-degree-of-freedom controller. We show that the optimal experimental condition is always to perform closed-loop identification, and we compute the optimal controller for identification, $C_{id}^{opt}(q)$, as a

function of controller sensitivity functions. In Section 4 we show that, given enough data, an iterative closed-loop identification design can always outperform open-loop identification in the sense of leading to more accurate controllers for the same number of data. We illustrate this iterative design by a simulation in Section 5; at the same time, our simulation gives an idea of the performance gains that are typically achieved by performing an optimal or iterative suboptimal identification design as compared to classical open loop identification. In Section 6 we show that, for MV and MR control design, the optimal controller to be applied during identification, $C_{id}^{opt}(q)$, is the ideal controller C . Conclusions are given in Section 7.

2. STATEMENT OF THE PROBLEM

The system

We consider that the task is to design a controller for some 'true' linear time-invariant system described by

$$\mathcal{S}: y(t) = G(q)u(t) + H(q)e(t), \quad (1)$$

where $G(q)$ and $H(q)$ are scalar rational transfer functions, with $G(q)$ strictly proper and $H(q)$ stable, proper and monic. Here q^{-1} is the backward shift operator ($q^{-1}u(t) = u(t-1)$), $u(t)$ is the control input signal, $y(t)$ is the observed output signal and $e(t)$ is white noise of zero mean with variance σ^2 and bounded fourth moments.

The identification

The controller is to be designed on the basis of a model of the plant identified using a finite set of N input and output data $\{y(t), u(t), t = 1, 2, \dots, N\}$ collected on the plant. A parameterized model set $\mathcal{M} = \{M(\theta) : \theta \in D_\theta \subset \mathbb{R}^d\}$ is used, where $M(\theta)$ is described by

$$M(\theta): y(t) = G(q, \theta)u(t) + H(q, \theta)e(t). \quad (2)$$

We shall use the following set of technical assumptions.

Assumptions A_1 .

1. The system \mathcal{S} is in the model set \mathcal{M} , i.e. there exists a $\theta_0 \in D_\theta$ such that

$$G(q) = G(q, \theta_0), \quad H(q) = H(q, \theta_0). \quad (3)$$

2. The model structure \mathcal{M} is uniformly stable, i.e. we assume that the predictor filters $H^{-1}(q, \theta)$ and $H^{-1}(q, \theta)G(q, \theta)$ that correspond to this model set along with their first- and second-order derivatives with respect to θ are uniformly stable for all $\theta \in D_\theta$ (Ljung, 1987),

3. A fixed time interval is available to perform the data collection for the identification. For any given sampling interval, this corresponds to assuming that N is fixed.

In addition we shall assume throughout that least squares prediction error identification is used. The data collection can be done in open loop or in closed loop. In the case of closed-loop identification, we shall denote by $C_{id} = [C_{1,id} \ C_{2,id}]$ the controller that operates during identification:

$$u(t) = C_{1,id}(q)r(t) - C_{2,id}(q)y(t), \quad (4)$$

where $r(t)$ is the reference excitation signal used during identification, which we assume to be quasi-stationary (Ljung, 1987).

The focus of this paper will be the design of the identification, given that the identified model is to be used for control design. The key design issues are

- open-loop or closed-loop identification;
- choice of $\phi_{u,id}(\omega)$ if open-loop identification is used;
- choice of $\phi_{r,id}(\omega)$ and $C_{id}(q)$ if closed-loop identification is used.

All these design issues will be handled in this paper. Since only variance errors are considered (i.e. $\mathcal{S} \in \mathcal{M}$), the use of a data prefilter is irrelevant (Ljung, 1987).

The least-squares prediction error method applied to N input-output data delivers an estimate $\hat{\theta}_N$ of θ :

$$\hat{\theta}_N = \arg \min_{\theta \in D_\theta} \sum_{t=1}^N \epsilon(t, \theta)^2, \quad (5)$$

$$\epsilon(t, \theta) = \frac{1}{H(q, \theta)} [y(t) - G(q, \theta)u(t)]. \quad (6)$$

In turn, this produces a model

$$\begin{aligned} \hat{G}_N &= \hat{G}_N(e^{i\omega}) = G(e^{i\omega}, \hat{\theta}_N), \\ \hat{H}_N &= \hat{H}_N(e^{i\omega}) = H(e^{i\omega}, \hat{\theta}_N). \end{aligned} \quad (7)$$

We shall sometimes use the vector notation

$$\hat{T}_N = [\hat{G}_N \ \hat{H}_N]^T, \quad T = [G \ H]^T. \quad (8)$$

The control design

We consider at this point that the controller is designed on the basis of one of several possible control design mechanisms for which the controller is a smooth function of both the plant

and the noise model, i.e. we shall make the following assumptions.

Assumptions A₂.

1. At any given frequency ω , $C(e^{i\omega})$ only depends on $G(e^{i\omega})$ and $H(e^{i\omega})$ at that particular frequency (and not on $G(\cdot)$ and $H(\cdot)$ at other frequencies), i.e. $C(e^{i\omega}) = c(G(e^{i\omega}), H(e^{i\omega}))$.
2. At any given frequency ω , i.e. at any given point $(G(e^{i\omega}), H(e^{i\omega}))$ in \mathbb{C}^2 , the controller $C(e^{i\omega})$ is a complex differentiable function of $G(e^{i\omega})$ and $H(e^{i\omega})$ (Henrici, 1974).

In addition, we assume that the certainty equivalence principle is used: the control design mechanism acts as a mapping $c(\cdot)$ that uniquely maps a plant model $[\hat{G}, \hat{H}]$ into a controller $\hat{C} = c(\hat{G}, \hat{H})$. For example, for a MR design criterion, the corresponding mapping is defined by the solution $\hat{C} = [\hat{C}_1, \hat{C}_2]$ of the following equations:

$$\frac{\hat{H}}{1 + \hat{G}\hat{C}_2} = G_{ye}, \quad \frac{\hat{G}\hat{C}_1}{1 + \hat{G}\hat{C}_2} = \hat{G}_{yr}. \quad (9)$$

Here G_{ye} is the designer's choice of the transfer function from disturbance input e to plant output y , while \hat{G}_{yr} is his/her choice of the transfer function from reference signal r to plant output y . Relation (9) takes into account that the reference model \hat{G}_{yr} can be a function of the estimated model, e.g. when \hat{G} has non-minimum-phase zeros. It is assumed, of course, that these design choices are feasible; in particular, they are compatible with the delay structure of the system and do not produce unstable pole-zero cancellations.

The first of Assumptions A₂ implies that, at any given frequency ω , we have

$$\begin{aligned} C(e^{i\omega}) &= c(G(e^{i\omega}), H(e^{i\omega})) \\ &= c_r(g_r(\omega), g_i(\omega), h_r(\omega), h_i(\omega)) \\ &\quad + ic_i(g_r(\omega), g_i(\omega), h_r(\omega), h_i(\omega)), \end{aligned}$$

where $g_r(\omega)$, $g_i(\omega)$, $h_r(\omega)$, $h_i(\omega)$, $c_r(\omega)$ and $c_i(\omega)$ are respectively the real and imaginary parts of $G(e^{i\omega})$, $H(e^{i\omega})$ and $C(e^{i\omega})$. The second of Assumptions A₂ implies that, at any given frequency ω , the Cauchy-Riemann conditions are satisfied at

$$(G(e^{i\omega}), H(e^{i\omega})) = (g_r(\omega) + ig_i(\omega), h_r(\omega) + ih_i(\omega)),$$

i.e.

$$\begin{aligned} \frac{\partial c_r}{\partial g_r} &= \frac{\partial c_i}{\partial g_i}, & \frac{\partial c_r}{\partial g_i} &= -\frac{\partial c_i}{\partial g_r}, \\ \frac{\partial c_r}{\partial h_r} &= \frac{\partial c_i}{\partial h_i}, & \frac{\partial c_r}{\partial h_i} &= -\frac{\partial c_i}{\partial h_r}. \end{aligned} \quad (10)$$

Therefore, by the properties of the differentials

$$F_G = \frac{\partial c_r}{\partial g_r} + i \frac{\partial c_i}{\partial g_r}, \quad F_H = \frac{\partial c_r}{\partial h_r} + i \frac{\partial c_i}{\partial h_r}, \quad (11)$$

we have

$$\begin{aligned} c(G(e^{i\omega}) + \Delta G(e^{i\omega}), H(e^{i\omega}) + \Delta H(e^{i\omega})) \\ = c(G(e^{i\omega}), H(e^{i\omega})) + [F_G \quad F_H] \begin{bmatrix} \Delta G(e^{i\omega}) \\ \Delta H(e^{i\omega}) \end{bmatrix} \\ + \psi(\Delta G(e^{i\omega}), \Delta H(e^{i\omega})), \end{aligned} \quad (12)$$

where

$$\psi(\Delta G(e^{i\omega}), \Delta H(e^{i\omega})) = o(\Delta G(e^{i\omega}), \Delta H(e^{i\omega})) \quad (13)$$

uniformly for every ω , i.e.

$$\begin{aligned} \lim_{\substack{\Delta G \rightarrow 0 \\ \Delta H \rightarrow 0}} \frac{\psi(\Delta G(e^{i\omega}), \Delta H(e^{i\omega}))}{\Delta G(e^{i\omega})} &= 0, \\ \lim_{\substack{\Delta G \rightarrow 0 \\ \Delta H \rightarrow 0}} \frac{\psi(\Delta G(e^{i\omega}), \Delta H(e^{i\omega}))}{\Delta H(e^{i\omega})} &= 0. \end{aligned} \quad (14)$$

Here $\Delta G(e^{i\omega})$ and $\Delta H(e^{i\omega})$ approach 0 in an arbitrary way (Henrici, 1974).

Notice that the second of Assumptions A₂ on the control design mechanism is not satisfied in general for H_∞ control, LQG control and, in the case of a non-minimum-phase system, for MV control.

The degradation measure

Consider that a control design criterion has been chosen. Suppose first that the plant $[G, H]$ was known exactly. Then the control design mapping would yield $C = c(G, H)$: this is the ideal controller, which is not computable. The corresponding ideal closed loop is that of Fig. 1: with the external signals r and e , it generates the ideal control signal u and output signal y .

With a model $[\hat{G}_N, \hat{H}_N]$ identified from N data pairs, the same control design mapping generates $\hat{C}_N = c(\hat{G}_N, \hat{H}_N)$. when this controller \hat{C}_N is applied to the actual plant, it produces the actual closed loop of Fig. 2. We denote by u_N and y_N the control and output signals in this actual closed loop when it is driven by the exact same external signals r and e that drive the loop of Fig. 1.

One measure of the degradation that results from using the suboptimal controller \hat{C}_N on the plant instead of the optimal C is the variance of the error between the outputs of the ideal and the actual closed-loop systems, when the two closed-loop systems are driven by the same signals, the reference r with spectrum $\phi_r(\omega)$ and

the white noise e with variance σ^2 . We denote this degradation measure by J_V :

$$J_V = \lim_{T \rightarrow \infty} \frac{1}{T} E \left\{ \sum_{t=1}^T [y(t) - y_N(t)]^2 \right\}. \quad (15)$$

In the case of a minimum-variance regulation criterion, J_V is a measure of the control performance degradation since the achieved performance $J(G, H, \hat{C}_N)$ can be written as

$$J(G, H, \hat{C}_N) = J(G, H, C) + J_V, \quad (16)$$

where $J(G, H, C)$ is the optimal performance. Thus if J_V can be made very small then the achieved cost $J(G, H, \hat{C}_N)$ is very close to the ideal cost $J(G, H, C)$.

Thus our objective from now on will be to derive experimental conditions for the identification of $[\hat{G}_N, \hat{H}_N]$ that make J_V as small as possible. We shall first compute the optimal but non-feasible experimental conditions. We shall then show that, for some control design criteria, a *feasible* iterative scheme can always lead to a smaller J_V than open-loop identification.

3. OPTIMAL SOLUTION FOR ONE-DEGREE-OF-FREEDOM CONTROLLERS

For pedagogical reasons, we start with a control design criterion that produces a one-degree-of-freedom controller, i.e. $C_1(q) = 1$ and $C(q) = C_2(q)$. As before, we denote by $\hat{C}_N = c(\hat{G}_N, \hat{H}_N)$ in the feedback controller that results from a model $[\hat{G}_N, \hat{H}_N]$ obtained from N noisy input-output data, and by C the ideal controller corresponding to $[G, H]$.

From Figs 1 and 2, we note that

$$y(t) = \frac{G(q)}{1 + G(q)C(q)} r(t) + \frac{H(q)}{1 + G(q)C(q)} e(t), \quad (17)$$

$$y_N(t) = \frac{G(q)}{1 + G(q)\hat{C}_N(q)} r(t) + \frac{H(q)}{1 + G(q)\hat{C}_N(q)} e(t). \quad (18)$$

At any given frequency ω , using our assumptions on the control design mechanism, we can write

$$\begin{aligned} \frac{1}{1 + G(e^{i\omega})\hat{C}_N(e^{i\omega})} &= \frac{1}{1 + G(e^{i\omega})C(e^{i\omega})} \\ &+ \frac{-G(e^{i\omega})}{[1 + G(e^{i\omega})C(e^{i\omega})]^2} \Delta C_N(\omega) + o(\Delta C_N(\omega)), \end{aligned} \quad (19)$$

where $\Delta C_N \triangleq \hat{C}_N - C$. Also,

$$\Delta C_N = [F_G \quad F_H] \begin{bmatrix} \Delta G_N(\omega) \\ \Delta H_N(\omega) \end{bmatrix} + o(\Delta G_N(\omega), \Delta H_N(\omega)), \quad (20)$$

where $\Delta G_N \triangleq \hat{G}_N - G$, $\Delta H_N \triangleq \hat{H}_N - H$ and F_G and F_H are defined in (11). The sensitivity functions F_G and F_H exist by our differentiability assumption on the control design mapping $C = c(G, H)$. We shall assume throughout that, whatever the identification and control design, the model error $\Delta T_N(\omega)$, and the controller error $\Delta C_N(\omega)$ at any given frequency ω are sufficiently small that the last terms in (19) and (20) can be neglected. Notice that this holds for sufficiently large N , provided the closed-loop identifiability conditions are satisfied, given our assumptions on the model set and the control design mechanism. Assuming that the reference signal $r(t)$ and the noise signal $e(t)$ are bounded, we can therefore write (dropping the operators q):

$$\begin{aligned} y(t) - y_N(t) &\approx \frac{G}{[1 + GC]^2} \Delta C_N [Gr(t) + He(t)], \\ &\approx \frac{G}{1 + GC} \Delta C_N y(t). \end{aligned} \quad (21)$$

Alternatively, note that, without approximation,

$$y(t) - y_N(t) = \frac{G}{1 + G\hat{C}_N} \Delta C_N y(t).$$

It is important for the subsequent argument to note that the right-hand side of (21) contains two *independent random quantities*: the first, ΔC_N , results from the signal realizations $\{y(t), u(t), t = 1, \dots, N\}$ during the *identification phase* that resulted in a model $[\hat{G}_N, \hat{H}_N]$, and hence a controller \hat{C}_N . The second, $y(t)$, is the output realization that would result from the application of the random signals $r(t)$ and $e(t)$ to the ideal closed-loop system through (17): these signals $r(t)$ and $e(t)$ are the same as those that are applied to the actual system of Fig. 2 during the *application phase*, i.e. after the identification and control design phase.

Using Parseval's relation, and the independence of ΔC_N and $y(t)$, we now get

$$\begin{aligned} J_V &= \lim_{T \rightarrow \infty} \frac{1}{T} E \left\{ \sum_{t=1}^T [y(t) - y_N(t)]^2 \right\}, \\ &\approx \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|G|^2 \phi_y}{|1 + GC|^2} E |\Delta C_N|^2 d\omega. \end{aligned} \quad (22)$$

Here the expected value is taken with respect to the probability distribution of the noise during the identification experiment, which produces the random variable $\hat{C}_N = c(\hat{G}_N, \hat{H}_N)$.

Using the vector notation $\Delta T_N \triangleq \hat{T}_N - T$ (see (8)), we can then write

$$E |\Delta C_N(e^{i\omega})|^2 \approx [F_G \quad F_H] E \begin{bmatrix} \Delta T_N & \Delta T_N^* \\ F_G^* & F_H^* \end{bmatrix}, \quad (23)$$

where the superscript * denotes the conjugate transpose. We note that all the expressions are frequency functions. For brevity of notation, the argument ω has been dropped in this and subsequent expressions.

We can now apply the covariance formula of Ljung (1987) that allows one to compute the covariance of the vector of transfer-function errors ΔT_N at a frequency ω for high-order models:

$$E[\Delta T_N \ \Delta T_N^*] \approx \frac{n}{N} |H|^2 \sigma^2 \begin{bmatrix} \phi_u & \phi_{ue}^* \\ \phi_{eu}^* & \sigma^2 \end{bmatrix}^{-1} \approx \frac{n}{N} \frac{|H|^2 \sigma^2}{\sigma^2 \phi_u - |\phi_{ue}|^2} \begin{bmatrix} \sigma^2 & -\phi_{ue}^* \\ -\phi_{eu}^* & \phi_u \end{bmatrix}. \tag{24}$$

Here n is the model order, N the number of data, H the noise description and σ^2 the white-noise variance of the true system (1), while ϕ_u is the spectrum of the input applied during identification and ϕ_{ue} the cross-spectrum between this input and the white-noise source. This formula has been derived under the assumption that the model order tends to infinity, but has been shown to produce good approximations even for low-order models. We refer the reader to Ljung (1985) for details as well as the full set of technical assumptions. These are essentially the assumptions that we have made on the system. The model set and the controller C_{id} at the beginning of Section 2. The formula applies to both open-loop identification ($\phi_{ue} = 0$) and closed-loop identification ($\phi_{ue} \neq 0$).

Combining (23) and (24) now yields

$$E |\Delta C_N(e^{i\omega})|^2 \approx \frac{n}{N} \frac{|H|^2 \sigma^2}{\sigma^2 \phi_u - |\phi_{ue}|^2} \times (\phi_u |F_H|^2 - F_G F_H^* \phi_{ue}^* - F_G^* F_H \phi_{eu}^* + \sigma^2 |F_G|^2). \tag{25}$$

If the identification is performed in closed loop, we have

$$u(t) = \frac{1}{1 + GC_{id}} r(t) - \frac{HC_{id}}{1 + GC_{id}} e(t), \tag{26}$$

where $C_{id}(q)$ is the controller that operates during data collection. Therefore

$$\phi_{ue} = \frac{-HC_{id}}{1 + GC_{id}} \sigma^2. \tag{27}$$

We now rewrite the formula (25) in a simpler

way by introducing two artificial signals that are filtered versions of the signals $r(t)$, $u(t)$ and $e(t)$:

$$z(t) \triangleq F_H u(t) - F_G e(t), \tag{28}$$

$$w(t) \triangleq \sigma u(t) + \frac{\sigma HC_{id}}{1 + GC_{id}} e(t) = \frac{\sigma}{1 + GC_{id}} r(t).$$

Observe that

$$z(t) = \frac{F_H}{1 + GC_{id}} r(t) - \frac{1}{1 + GC_{id}} \times (F_G + F_G GC_{id} + F_H HC_{id}) e(t). \tag{29}$$

It follows from (25)–(29) that

$$E |\Delta C_N(e^{i\omega})|^2 \approx \frac{n}{N} \frac{|H|^2 \sigma^2 \phi_z}{\phi_w} \approx \frac{n}{N} |H|^2 \left\{ |F_H|^2 + \frac{\sigma^2}{\phi_r} |F_G + (F_G G + F_H H) C_{id}|^2 \right\}. \tag{30}$$

We observe from this last expression that the variance of the controller error is minimized at every frequency if the identification is performed in closed loop with an operating† controller:

$$C_{id}^{inf} = - \frac{F_G}{F_G G + F_H H}. \tag{31}$$

This optimal choice of course also minimizes the closed-loop degradation criterion J_V : see (22).

There is no guarantee that the controller defined by (31) is stabilizing or proper, and in fact it will not always be, as we shall illustrate later. In order to obtain the stabilizing controller that minimizes (30), we introduce the Youla parameterization of all stabilizing controllers: see e.g. Vidyasagar (1985).

Proposition 1. Let G and C_0 have fractional representations $G = N_G D_G^{-1}$ and $C_0 = N_C D_C^{-1}$, where N_G , D_G , N_C and D_C belong to \mathbf{S} , the ring of proper stable transfer functions. Assume that the Bézout identity $N_G N_C + D_G D_C = 1$ holds. For any $S \in \mathbf{S}$, define

$$N_S = N_C - D_G S, \quad D_S = D_C + N_G S. \tag{32}$$

1. Then $C(S) = N_S D_S^{-1}$ is a stabilizing controller for $G = N_G D_G^{-1}$.
2. Furthermore, any linear time-invariant (LTI)

† The superscript ‘inf’ stands for infimum.

controller that stabilizes G has a fractional representation (32) for some $S \in \mathbf{S}$.

Let $C_0 = N_C D_C^{-1}$ be any stabilizing controller of G , and let the stabilizing controller used for identification, C_{id} , correspond to $S_{id} \in \mathbf{S}$ such that $C_{id} = C(S_{id})$. Using the coprime factors of G and C_0 , we can rewrite (30) as

$$E |\Delta C_n|^2 \approx \frac{n}{N} |H|^2 |F_H|^2 \times \left(1 + \frac{\sigma^2 |H|^2}{\phi_r} \left| \frac{F_G}{F_H H D_G} + N_C - S_{id} D_G \right|^2 \right). \quad (33)$$

We observe that the variance of the controller error is minimized at every frequency if the identification is performed in closed loop with the controller (31) that corresponds to

$$S_{id}^{\text{inf}} = \frac{1}{D_G} \left(N_C + \frac{F_G}{F_H H D_G} \right). \quad (34)$$

Observe that S_{id}^{inf} need not be proper and stable, and hence the corresponding controller C_{id}^{inf} defined in (31) is not necessarily stabilizing and proper. The stabilizing optimal controller for identification, C_{id}^{opt} , is obtained by minimizing (33) over all $S_{id} \in \mathbf{S}$. Let S_{id}^{opt} be defined by

$$S_{id}^{\text{opt}} = \arg \min_{S_{id} \in \mathbf{S}} \{J_V(S_{id})\}. \quad (35)$$

Then the stabilizing optimal controller for identification is given by $C_{id}^{\text{opt}} = C(S_{id}^{\text{opt}})$.

The optimality of closed-loop identification follows from the fact that C_{id}^{opt} is nonzero. We observe that the only case where open-loop identification is optimal is when $F_G = 0$. This corresponds to a control design criterion in which the controller is independent of the input-output model—a rather unlikely situation. In such a case, we obtain

$$E |\Delta C_N(e^{i\omega})|^2 \approx \frac{n}{N} |H|^2 |F_H|^2 \left(1 + \frac{\sigma^2 |H|^2}{\phi_r} \left| \frac{N_C - S_{id} D_G}{D_C + S_{id} N_G} \right|^2 \right). \quad (36)$$

If the plant is stable, this is indeed minimized by taking $S_{id} = N_C D_G^{-1}$, i.e. $C_{id} = 0$. This is open-loop identification, leading to

$$E |\Delta C_N(e^{i\omega})|^2 \approx \frac{n}{N} |H|^2 |F_H|^2. \quad (37)$$

We now consider two special cases.

Case 1: the controller does not depend on the estimated noise model

This corresponds to $F_H = 0$. We then have

$$E |\Delta C_N(e^{i\omega})|^2 \approx \frac{n}{N} \frac{|H|^2 |F_G|^2 |1 + G C_{id}|^2 \sigma^2}{\phi_r} = \frac{n}{N} \frac{|H|^2 |F_G|^2 \sigma^2}{|D_G (D_C + S_{id} N_G)|^2 \phi_r}, \quad (38)$$

$$J_V \approx \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|G|^2 \phi_y}{|1 + G C|^2 N} \frac{n |H|^2 |F_G|^2 |1 + G C_{id}|^2 \sigma^2}{\phi_r} d\omega. \quad (39)$$

We observe that J_V is minimized by performing closed-loop identification with a reference spectrum ϕ_r that is as large as possible, or with a controller $C_{id} = C(S_{id})$ that makes $|1 + G C_{id}|$ as small as is compatible with the constraints on S_{id} and on the input spectrum ϕ_u ; see (26).

Case 2: open-loop identification

If G is stable, one can always take $N_G = G$, $D_G = 1$, $N_C = 0$ and $D_C = 1$. With $C_{id} = 0$, we get

$$E |\Delta C_N(e^{i\omega})|_{\text{ol}}^2 \approx \frac{n}{N} |H|^2 \left(|F_H|^2 + \frac{|F_G|^2 \sigma^2}{\phi_u} \right). \quad (40)$$

We observe that, when the input spectrum goes to infinity, this controller error variance goes to

$$E |\Delta C_N(e^{i\omega})|^2 \approx \frac{n}{N} |H|^2 |F_H|^2.$$

Comments.

1. As is typical of optimal experiment design results, the optimal design depends on the unknown system $[G, H]$ and is therefore not feasible. If also depends on the control design criterion through the sensitivity functions F_G and F_H . However, we shall exploit this result later to propose a feasible suboptimal design.
2. For MV control, it was shown in Gevers and Ljung (1986) that $C_{id}^{\text{opt}}(q) = C(q)$. One contribution of this paper will be to show in Section 6 that this also holds for MR control design. Thus, for these control design criteria (and possibly for others), the optimal experiment design is to perform closed-loop identification with the ideal controller $C(q)$ operating in the loop.
3. The sensitivity functions F_G and F_H are easily computable only in the cases where the controller depends explicitly on the plant $[G, H]$. In other cases, such as LQG or H_∞ control, it is not clear how to define these sensitivity functions.

4. A particular intriguing question† is for what other classes of controllers, besides MV and MR control, does the property hold that the optimal controller for identification coincides with the ideal controller, i.e. $C_{id}^{opt} = C$? This is an open question!
5. In general, the optimal controller for identification, C_{id}^{opt} , depends on the reference signal spectrum ϕ_r , because S_{id}^{opt} depends on ϕ_r . However, the optimal experiment design is indeed independent of the reference signal spectrum when $C_{id}^{opt} = C_{id}^{inf}$.

If identification can be performed under the ideal closed-loop condition with $C_{id}^{opt}(q) = C_{id}^{inf}(q)$, the control error variance is‡

$$E |\Delta C_N(e^{i\omega})|_{idel}^2 \approx \frac{n}{N} |H|^2 |F_H|^2, \quad (41)$$

and the error degradation measure J_v becomes

$$J_v^{opt} \approx \frac{n}{2\pi N} \int_{-\pi}^{\pi} \frac{|G|^2 |H|^2 |F_H|^2}{|1 + GC|^2} \phi_y d\omega. \quad (42)$$

By comparison, with open-loop identification, the control error variance has the additional term (see (40)):

$$\frac{n \sigma^2 |H|^2 |F_G|^2}{N \phi_u}$$

We have thus proved the following result. It is an extension to arbitrary control design criteria of a result obtained in Gevers and Ljung (1986) for MV control design.

Theorem 1. Let G have a fractional representation $G = N_G D_G^{-1}$. Consider some stabilizing controller $C_0 = N_C D_C^{-1}$ and assume that N_G, D_G, N_C and D_C belong to \mathbf{S} with the Bézout identity $N_G N_C + D_G D_C = 1$ holding. Consider the direct prediction error identification of a system $\mathcal{S} = [G, H]$ using a model set $\mathcal{M} = \{[G(\theta), H(\theta)], \theta \in D_\theta\}$ with $\mathcal{S} \in \mathcal{M}$, for the purpose of designing a controller C using a mapping $C(\theta) = c(G(\theta), H(\theta))$. Assume that the model set \mathcal{M} and the control design mechanism $c(\cdot)$ are respectively subject to Assumptions A_1 and A_2 . Let \hat{C}_N denote the controller obtained from $[G(\hat{\theta}_N), H(\hat{\theta}_N)]$ by this indirect scheme, where $\hat{\theta}_N$ is the parameter estimate obtained from N input-output data; let C denote the ideal controller, $C = c(G, H)$, and assume that the controller \hat{C}_N depends on both $G(\hat{\theta}_N)$ and $H(\hat{\theta}_N)$.

† At least one that has intrigued the authors of this paper for months.

‡ The subscript 'idel' stands for ideal closed loop.

Under these assumptions, whatever the control design criterion, the variance of $C - \hat{C}_N$ is minimized by performing the identification in closed loop with a feedback controller:

$$C_{id}^{opt} = \frac{N_C - D_G S_{id}^{opt}}{D_C + N_G S_{id}^{opt}} \quad (43)$$

where S_{id}^{opt} is defined in (35).

In the particular case where S_{id}^{inf} defined in (34) belongs to \mathbf{S} , we have

$$C_{id}^{opt} = C_{id}^{inf} = - \frac{F_G}{F_G G + F_H H}. \quad (44)$$

Under those ideal closed-loop experimental conditions, at every frequency $E |\Delta C_N(e^{i\omega})|^2$ is smaller than the corresponding variance under open-loop identification, *whatever the input or reference signal power*, provided identifiability is guaranteed.

We observe that the optimal control error variance (41) is indeed independent of ϕ_r , and that the open-loop variance error formula (40) approaches (41) for infinite input energy. A similar observation had already been made in Gevers and Ljung (1986) for the case of minimum-variance control design.

As already pointed out, the result of Theorem 1 is theoretically interesting but practically not feasible, because C_{id}^{opt} depends on the unknown system. In the next section, we build on this result to produce a feasible identification design.

4. ITERATIVE CLOSED-LOOP DESIGN OUTPERFORMS OPEN-LOOP IDENTIFICATION

In this section we shall assume that the controller $C_{id}^{inf}(q)$ defined in (31) is stabilizing, i.e. (44) is satisfied. We first elaborate on (30) to connect the variance of ΔC_N , the error on the designed controller, with the 'error' between the optimal controller for identification, $C_{id}^{opt}(q)$, and the controller that is actually applied during identification, $C_{id}(q)$. We denote this error by $\Delta C_{id}(q) \triangleq C_{id}(q) - C_{id}^{opt}(q)$. By replacing C_{id} by $C_{id}^{opt} + \Delta C_{id}$ in (30), we get immediately

$$E |\Delta C_N|^2 \approx \frac{n}{N} |H|^2 \times \left[|F_H|^2 + \frac{\sigma^2}{\phi_r} |(F_G G + F_H H) \Delta C_{id}|^2 \right]. \quad (45)$$

This expression holds for open-loop and closed-loop identification, regardless of the control design criterion§ and regardless of the size of ΔC_{id} . The open-loop formula (40) is

§ The control design criterion enters into the expression through F_G and F_H .

recovered by setting $\Delta C_{id} = C_{id}^{opt}$ and $\phi_r = \phi_u$. The following result follows immediately from (45).

Theorem 2. At some frequency ω , the designed controller error variance $E|\Delta C_N|^2$ obtained by closed-loop identification with an operating controller $C_{id}(q)$ and a reference signal spectrum $\phi_r(\omega)$ is smaller than that obtained by open-loop identification with input spectrum $\phi_u(\omega)$ if and only if

$$\left| \frac{C_{id}(e^{i\omega}) - C_{id}^{opt}(e^{i\omega})}{C_{id}^{opt}(e^{i\omega})} \right|^2 \leq \frac{\phi_r(\omega)}{\phi_u(\omega)} \quad (46)$$

This holds for any control design mechanism subject to the set of Assumptions A₂.

Comment. This theorem tells us how close (in relative terms) the controller C_{id} applied during identification must be to the optimal controller C_{id}^{opt} for closed-loop identification to outperform open-loop identification in terms of yielding a smaller controller error $E|\Delta C_N|^2$. This relative distance is a function of the ratio between the signal spectra, ϕ_r and ϕ_u , applied in closed-loop and open-loop identification respectively. An alternative way to read this result is that, for any given relative distance $\alpha(\omega)$ between C_{id} and C_{id}^{opt} at frequency ω , closed-loop identification will outperform open-loop identification if and only if $\phi_r(\omega) \geq \alpha(\omega)\phi_u(\omega)$.

We now turn to iterative identification. We assume again that (44) is satisfied. We shall show in Section 6 that this is indeed the case for MV and MR control design, in which case we even have $C_{id}^{opt}(q) = C(q)$. We then consider the idea of splitting up the total data collection interval of length N into two subintervals. In the first subinterval the identification is performed in open loop with input signal spectrum ϕ_u . Let $\hat{\theta}_1$, \hat{G}_1 and \hat{H}_1 denote the estimates of θ , G and H at the end of the first subinterval of length N_1 , and let $\hat{C}_{id,1}^{opt}$ be the corresponding optimal controller for identification:

$$\hat{C}_{id,1}^{opt} = - \frac{\hat{F}_G}{\hat{F}_G \hat{G}_1 + \hat{F}_H \hat{H}_1} \quad (47)$$

In the second subinterval the data are collected in closed loop, with this controller applied to the system and reference signal spectrum ϕ_r . The parameter vector $\hat{\theta}_N$ is computed at the end of the second subinterval with $\hat{\theta}_1$ as initial value.

We now compare two strategies—open-loop identification and iterative identification—with the objective being the best possible controller estimate at the end of the total interval of length N .

Consider first that a data set of length N is collected with the plant operating in open loop and that the identified model is used to compute the certainty equivalence controller $\hat{C}_{ol,N}$. Denote by $J_V^{(1)}$ the corresponding degradation measure. Then

$$J_V^{(1)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|G|^2 \phi_y}{|1 + GC|^2} E|\Delta C_{ol,N}|^2 d\omega \quad (48)$$

with $\Delta C_{ol,N} \triangleq \hat{C}_{ol,N} - C$ and, by (40),

$$E|\Delta C_{ol,N}|^2 = \frac{n}{N} |H|^2 \left(|F_H|^2 + \frac{|F_G|^2 \sigma^2}{\phi_u} \right) \quad (49)$$

Consider next that the first $(1-\alpha)N$ data ($0 < \alpha < 1$) are collected on the plant operating in open loop with input spectrum ϕ_u , and that the model that is estimated from these $(1-\alpha)N$ data is used to compute the optimal certainty equivalence controller for identification design, $\hat{C}_{id,(1-\alpha)N}^{opt}$, which is then applied to the plant for the remaining interval of length αN . A new model is estimated using the data from the subinterval of length αN . Let $\hat{C}_{it,N}$ be the certainty equivalence controller computed from this model and let $J_V^{(2)}$ denote the corresponding performance degradation measure. Denote $\Delta C_{id,(1-\alpha)N}^{opt} \triangleq \hat{C}_{id,(1-\alpha)N}^{opt} - C_{id}^{opt}$. Using the same arguments as in the previous section, it is possible to compute the variance of this error:

$$E|\Delta C_{id,(1-\alpha)N}^{opt}|^2 = \frac{n}{(1-\alpha)N} |H|^2 \left(|\bar{F}_H|^2 + \frac{|\bar{F}_G|^2 \sigma^2}{\phi_u} \right) \quad (50)$$

where \bar{F}_G and \bar{F}_H are defined as in (11) with C replaced by C_{id}^{opt} .

At the end of the second interval, we have

$$J_V^{(2)} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|G|^2 \phi_y}{|1 + GC|^2} E|\Delta C_{it,N}|^2 d\omega \quad (51)$$

with $\Delta C_{it,N} \triangleq \hat{C}_{it,N} - C$ and

$$E|\Delta C_{it,N}|^2 = \frac{n}{\alpha N} |H|^2 \times \left(|F_H|^2 + \frac{\sigma^2}{\phi_r} |F_G G + F_H H|^2 E|\Delta C_{id,(1-\alpha)N}^{opt}|^2 \right) \quad (52)$$

where E denotes the expectation over the whole interval of length N .

† The subscript 'it' stands for iterative.

Denote

$$f(\omega) \triangleq \frac{\sigma^2}{|F_H(e^{i\omega})|^2 \phi_r(\omega)} |F_G(e^{i\omega})G(e^{i\omega}) + F_H(e^{i\omega})H(e^{i\omega})|^2, \quad (53)$$

$$g(\omega) \triangleq n |H(e^{i\omega})F_H(e^{i\omega})|^2, \quad (54)$$

$$h(\omega) \triangleq \frac{1}{2\pi} \frac{|G|^2 \phi_y}{|1 + GC|^2}, \quad (55)$$

$$\gamma_1 \triangleq \int_{-\pi}^{\pi} hg \, d\omega, \quad (56)$$

$$\gamma_2 \triangleq \int_{-\pi}^{\pi} hg \frac{|F_G|^2 \sigma^2}{|F_H|^2 \phi_u} \, d\omega, \quad (57)$$

$$\gamma_3 \triangleq \int_{-\pi}^{\pi} h g f n |H|^2 |\bar{F}_H|^2 \left(1 + \frac{|\bar{F}_G|^2 \sigma^2}{|\bar{F}_H|^2 \phi_u}\right) \, d\omega. \quad (58)$$

Note that the quantities γ_1, γ_2 and γ_3 are positive. Then

$$J_V^{(1)} = \frac{1}{N}(\gamma_1 + \gamma_2), \quad J_V^{(2)} = \frac{1}{\alpha N} \left[\gamma_1 + \frac{\gamma_3}{(1 - \alpha)N} \right]. \quad (59)$$

Therefore $J_V^{(2)} < J_V^{(1)}$ if

$$\gamma_1 + \frac{\gamma_3}{(1 - \alpha)N} < \alpha(\gamma_1 + \gamma_2).$$

A necessary condition for this is

$$\frac{\gamma_1}{\gamma_1 + \gamma_2} < \alpha < 1. \quad (60)$$

Thus, choose any α such that (60) holds and denote

$$N_0(\alpha) \triangleq \frac{\gamma_3}{(1 - \alpha)[\alpha\gamma_2 - (1 - \alpha)\gamma_1]}. \quad (61)$$

Observe that the necessary condition (60) makes the denominator of (61) positive. Then, for any $N > N_0(\alpha)$, we have $J_V^{(2)} < J_V^{(1)}$.

This holds regardless of which reference signal spectrum is used in the second subinterval, as long as N can be chosen large enough; it is, for example true even if the power of the reference signal in the closed-loop experiment is smaller than the power of the input signal in the open-loop experiment. This remarkable fact is because the minimal criterion (42) does not depend on the reference signal used in the closed loop identification step.

The idea developed here with two subintervals can of course be repeated, leading to iterative identification and control design. We conclude that the iterative approach can always yield a

smaller degradation than open-loop identification, provided enough data can be collected.

We summarize these results on iterative identification in the following theorem.

Theorem 3. Consider that a model $[\hat{G}_N, \hat{H}_N]$ of a system $[G, H]$ is identified by a direct prediction error method using N noisy input-output data with the purpose of computing the certainty equivalence controller $\hat{C}_N = c(\hat{G}_N, \hat{H}_N)$. Assume that the system is in the model set, and that the controller design mapping $C = c(G, H)$ is such that (44) holds. Then iterative closed-loop identification leads to a smaller performance degradation measure J_V than open-loop identification, provided the number of data N is large enough. In particular, if the total interval of length N is split up in a fraction of $(1 - \alpha)N$ data collected in open loop using an input spectrum ϕ_u , followed by αN data collected in closed loop with an arbitrary reference signal spectrum, then the performance measure J_V achieved by such an iterative design is better than that achieved by open-loop identification using the same input spectrum ϕ_u as in the first step of the iterative procedure, provided α is larger than the bound given by (60) and N is larger than $N_0(\alpha)$ given by (61).

We make the final comment that the scheme presented here relies crucially on the relation (44). We shall prove in Section 6 that this result holds for MV and for MR control design with the additional property that $C_{id}^{opt} = C$. But first we present some simulations to illustrate our results and to give the reader a respite from heavy-duty formulae.

5. SIMULATIONS

In order to get a feeling for the performance improvements that can be achieved by using the iterative scheme suggested in Section 4, we have performed the following simulations.

We have taken a 'true system' that has the following ARX structure:

$$(1 - 1.5q^{-1} + 0.7q^{-2})y_t = q^{-1}(1 + 0.5q^{-1})u_t + e_t,$$

with e_t white noise of unit variance. The optimal MV controller (corresponding to the true system) is

$$u_t = -\frac{1.5 - 0.7q^{-1}}{1 + 0.5q^{-1}} y_t + r_t.$$

By the results of Section 3 (or by reading Gevers and Ljung, 1986) we know that the optimal, but infeasible, identification conditions (in order to compute the best certainty equivalence MV

controller) are to perform the identification in closed loop with this optimal controller in the loop. The objective of our simulations was to compare the performance degradation J_V obtained with this infeasible optimal design with that obtained by open-loop identification and by the feasible iterative scheme.

With $N=1000$ data points allowed to be collected and a MV control design criterion, we have compared the following three experiments.

1. The identification is performed in closed loop using 1000 data generated with the ideal MV controller operating in the loop, using an external white reference signal with spectrum $\phi_r = \beta$.
2. The identification is performed in open loop using 1000 data generated with a white input with spectrum $\phi_u = \beta$.
3. The identification is performed in open loop first using $(1-\alpha)1000$ data (with $0 < \alpha < 1$) generated with a white input with spectrum $\phi_u = \beta$; at the end of that first interval, the corresponding certainty equivalence MV controller $\hat{C}_{(1-\alpha)1000}$ is applied to the plant, and the remaining $\alpha 1000$ data are collected on the closed-loop system with an external white reference signal with spectrum $\phi_r = \beta$. The final parameter estimate $\hat{\theta}_{1000}$ is computed using all the 1000 data† thus collected. This has been done for all values of α between 0 and 1.

In each case the certainty equivalence MV controller \hat{C}_{1000} is computed from the model $[\hat{G}_{1000}, \hat{H}_{1000}]$ obtained at the end of the 1000 data, and the corresponding performance de-

† If a model other than ARX is used (such as OE or BJ) then the estimate $\hat{\theta}_{(1-\alpha)1000}$ obtained at the end of the open-loop phase is used as initial estimate for the closed-loop phase.

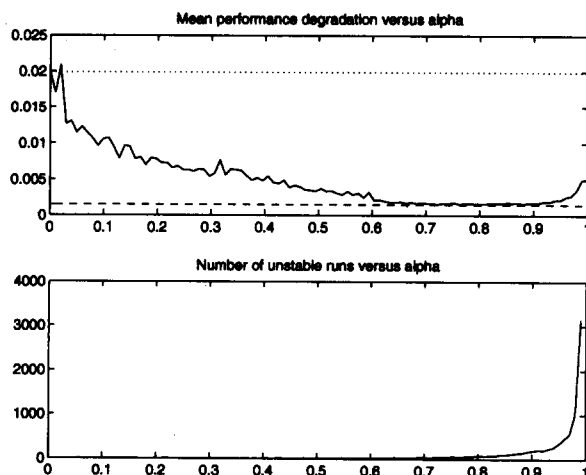


Fig. 3. Mean performance degradation and number of unstable closed loops versus α for $\beta = 1$.

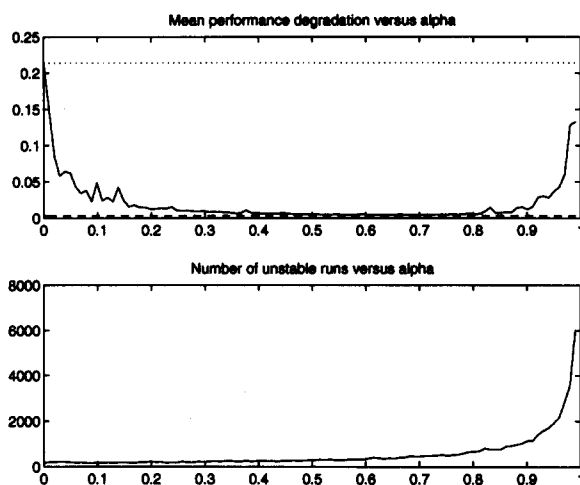


Fig. 4. Mean performance degradation and number of unstable closed loops versus α for $\beta = 0.1$.

gradation J_V *vis-à-vis* the optimal controller achieved on the 'true system' is computed experimentally. To do so, 1000 Monte Carlo simulations have been run for the ideal closed-loop experiment, for the open-loop experiment and for each of the iterative open/closed-loop experiments in order to compute the experimental estimates of J_V . The results are shown respectively in Figs 3(a) and 4(a) for $\beta = 1$ and $\beta = 0.1$.

The dashed lines in Figs 3(a) and 4(a) represent the performance degradation that results from identifying the system using 1000 data collected under the ideal experimental conditions, i.e. with the optimal controller in the loop. We get $J_V^{(1)} = 0.0014$ and $J_V^{(1)} = 0.0029$ respectively for $\beta = 1$ and $\beta = 0.1$. Observe that, even under these ideal experimental conditions, $J_V^{(1)} \neq 0$ since the ideal controller C is estimated on the basis of 1000 noisy data. The dotted lines represent the performance degradation that results from identifying the system using 1000 data collected in open loop. We get $J_V^{(2)} = 0.0199$ and $J_V^{(2)} = 0.214$ respectively for $\beta = 1$ and $\beta = 0.1$; i.e. open-loop identification performs respectively 14 and 74 times worse† than closed-loop identification with the ideal controller.

The full lines represent the performance degradation that results from using open-loop

† Now, $\phi_r = 1$ and $\phi_r = 0.1$ in closed-loop identification lead to an input variance that is approximately 9 and 54 times larger than that in open-loop identification. However, both for $\phi_r = 1$ and $\phi_r = 0.1$, using a reference signal for the ideal identification design that yields the same input variance as in the open loop case would still give more than 50% performance improvement over the open-loop design, but with a considerably smaller output variance. Notice also that it follows from Theorem 3 that, provided N is large enough, the iterative procedure outperforms the open-loop procedure for any reference signal spectrum.

followed by closed-loop identification, for different values of the fraction α of 'closed-loop identification time'. We observe that for most values of α , this feasible two-stage scheme performs almost as well as the infeasible ideal experiment design. Except for α close to 0 (mostly open-loop identification) or close to 1 (mostly closed-loop identification), the curve is fairly insensitive to the choice of α ; thus it appears that it would not make much sense to optimize the design parameter α .

A word of caution is in order, however. If α is taken too large, that is, if the open-loop fraction of the total experiment time is too short, then the model obtained at the end of open-loop identification may be so poor that the corresponding controller destabilizes the true system. This is indeed what happens. To give a honest representation of this phenomenon, we have represented in Figs 3(b) and 4(b) the total numbers of runs that produced unstable closed loops as a function of α , until 100 stable runs were produced. The averages in Figs 3(a) and 4(a) are computed on the stable runs only.

6. OPTIMAL IDENTIFICATION CONDITION FOR MV AND MR CONTROL DESIGN

In this section we show that, for minimum-variance (MV) and model reference (MR) control design, the optimal design (31) yields $C_{id}^{inf}(q) = C$, and hence $C_{id}^{inf}(q)$ is stabilizing. Thus, for these control design criteria, if the objective of the identification of $[G, H]$ is to design the controller C then the best experimental condition is to perform closed-loop identification with the ideal (MV or MR) controller operating. This is a lovely theoretical, but apparently useless result—except that it implies (44), which is essential for our feasible iterative design of Section 4.

We first consider MV control in the situation where the plant is minimum-phase with unit delay. The MV control law is then given by $C = (H - 1)/G$: see Ljung (1987). For this situation, it was shown in Gevers and Ljung (1986) that the optimal identification design is closed-loop identification with the optimal controller in the loop during identification. The proof in Gevers and Ljung (1986) did not use (31). From Theorem 1 and this formula, we get the desired result immediately, noting that $F_G = -(H - 1)/G^2$ and $F_H = 1/G$.

We now consider that the control design criterion is the model reference (MR) design criterion (9), which defines the two-degree-of-freedom controller $\hat{C} = [\hat{C}_1 \ \hat{C}_2]$ as a function of $[\hat{G}, \hat{H}]$ and of a prespecified model $[G_{yr}, G_{ye}]$.

If C is the MR controller corresponding to $[G, H]$, we write

$$\frac{GC_1}{1 + GC_2} = G_{yr}, \quad \frac{\hat{G}\hat{C}_1}{1 + \hat{G}\hat{C}_2} = \hat{G}_{yr}, \quad (62)$$

$$\frac{H}{1 + GC_2} = \frac{\hat{H}}{1 + \hat{G}\hat{C}_2} = G_{ye}. \quad (63)$$

We first compute the error $y(t) - y_N(t)$ between the outputs of the ideal and actual closed-loop systems. We assume again that the number of observed data is large enough that ΔG_N , ΔH_N , $\hat{G}_{yr} - G_{yr}$ and $\Delta C_N \triangleq \hat{C}_N - C$ at any given frequency ω are small.† Using the same arguments as in Section 3, we can write

$$\begin{aligned} y(t) - y_N(t) &= \left(\frac{GC_1}{1 + GC_2} - \frac{\hat{G}\hat{C}_1}{1 + \hat{G}\hat{C}_2} \right) r(t) \\ &\quad + \left(\frac{H}{1 + GC_2} - \frac{\hat{H}}{1 + \hat{G}\hat{C}_2} \right) e(t) \\ &\approx \frac{G}{(1 + GC_2)^2} \{ [-(1 + GC_2) \Delta C_1 \\ &\quad + GC_1 \Delta C_2] r(t) + H \Delta C_2 e(t) \} \\ &\approx \frac{G}{(1 + GC_2)^2} [\Delta C_1 \ \Delta C_2] M \begin{bmatrix} r(t) \\ e(t) \end{bmatrix}, \end{aligned} \quad (64)$$

where

$$M \triangleq \begin{bmatrix} -(1 + GC_2) & 0 \\ GC_1 & H \end{bmatrix}. \quad (65)$$

Compare this expression with (21) for the one-degree-of-freedom controller. The closed-loop degradation criterion J_V can then be written as

$$\begin{aligned} J_V &\approx \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|G|^2}{|1 + GC_2|^4} \\ &\quad \times E \left\{ [\Delta C_1 \ \Delta C_2] M \text{diag} \{ \phi_r, \sigma^2 \} M^* \begin{bmatrix} \Delta C_1^* \\ \Delta C_2^* \end{bmatrix} \right\} d\omega. \end{aligned} \quad (66)$$

In this expression $[\Delta C_1 \ \Delta C_2]$ is a random vector, a function of the random model error $[\Delta G_N, \Delta H_N]$. We now compute these sensitivities, i.e. the relationships between errors in $[G, H]$ and errors in $[C_1, C_2]$, assuming small errors in the plant model. From (62) and (63), we get, after some calculations in which we neglect all second-order error terms,

$$[\Delta C_1 \ \Delta C_2] = [\Delta G_N \ \Delta H_N] F, \quad (67)$$

† Recall that ΔC_N denotes $[\Delta C_{1,N} \ \Delta C_{2,N}]$ in the present case of a two-degree-of-freedom controller.

where the sensitivity matrix F is given by

$$F = \begin{bmatrix} -\frac{C_1}{G} & -\frac{C_2}{G} \\ \frac{C_1}{H} & \frac{1+GC_2}{GH} \end{bmatrix}. \quad (68)$$

Therefore

$$J_V \approx \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{|G|^2}{|1+GC_2|^4} \text{tr} \left(\text{diag} \{ \phi_r, \sigma^2 \} M^* F^* \right. \\ \left. \times E \left\{ \begin{bmatrix} \Delta G_N^* \\ \Delta H_N^* \end{bmatrix} \begin{bmatrix} \Delta G_N & \Delta H_N \end{bmatrix} \right\} FM \right) d\omega. \quad (69)$$

We now compute

$$FM = \frac{1}{G} \begin{bmatrix} -C_1 & -C_2 H \\ 0 & 1+GC_2 \end{bmatrix}, \quad (70)$$

and we substitute (70) and the expression (24) for the covariance matrix of the transfer function error $[\Delta G_N \ \Delta H_N]$ into (69). After rather lengthy manipulations, this leads to

$$J_V \approx \frac{n}{2\pi N} \int_{-\pi}^{\pi} \frac{\sigma^4 |H|^2}{|1+GC_2|^4 (\sigma^2 \phi_u - |\phi_{ue}|^2)} \\ \times \{ |C_1|^2 \phi_r + \sigma^2 |C_2|^2 |H|^2 \\ + 2 \text{Re} [\phi_{ue} C_2 H (1+G^* C_2^*)] \\ + \phi_u |1+GC_2|^2 \} d\omega. \quad (71)$$

We now need to minimize J_V with respect to the experimental conditions, i.e. with respect to ϕ_u and ϕ_{ue} . We write J_V as the sum of two terms, and show that they are both minimized by the same experimental conditions.

Consider first

$$J_V^{(1)} = \frac{n}{2\pi N} \int_{-\pi}^{\pi} \frac{\sigma^4 |H|^2 |C_1|^2 \phi_r}{|1+GC_2|^4 (\sigma^2 \phi_u - |\phi_{ue}|^2)} d\omega. \quad (72)$$

During closed-loop identification with a two-degree-of-freedom controller $[C_{id,1}(q) \ C_{id,2}(q)]$, we have

$$u(t) = \frac{C_{id,1}}{1+GC_{id,2}} r(t) - \frac{C_{id,2}H}{1+GC_{id,2}} e(t). \quad (73)$$

Therefore

$$\phi_{ue} = -\frac{C_{id,2}H}{1+GC_{id,2}} \sigma^2, \quad (74)$$

and

$$\sigma^2 \phi_u - |\phi_{ue}|^2 = \frac{\sigma^2 |C_{id,1}|^2}{|1+GC_{id,2}|^2} \phi_r, \quad (75)$$

Thus, minimizing $J_V^{(1)}$ with respect to (ϕ_u, ϕ_{ue}) is equivalent with

$$\min_{C_{id,1}, C_{id,2}} \frac{n}{2\pi N} \int_{-\pi}^{\pi} \frac{\sigma^2 |H|^2}{|1+GC_2|^2} \\ \times \left| \frac{C_1}{1+GC_2} \right|^2 \left| \frac{1+GC_{id,2}}{C_{id,1}} \right|^2 d\omega. \quad (76)$$

By the same argument as used in Ljung (1987, p. 226), it follows that the optimum is achieved by

$$C_{id,1}(q) = C_1(q), \quad C_{id,2}(q) = C_2(q). \quad (77)$$

Note that

$$J_V^{(1)\text{opt}} = \frac{n}{2\pi N} \int_{-\pi}^{\pi} \frac{\sigma^2 |H|^2}{|1+GC_2|^2} d\omega.$$

Now consider the remaining term:

$$J_V^{(2)} = \frac{n}{2\pi N} \int_{-\pi}^{\pi} \frac{\sigma^2 |H|^2}{|1+GC_2|^4} \\ \times \frac{1}{\sigma^2 \phi_u - |\phi_{ue}|^2} \{ \sigma^4 |G|^2 |H|^2 \\ + \sigma^2 \text{Re} [\phi_{ue} C_2 H (1+G^* C_2^*)] \\ + \sigma^2 \phi_u |1+GC_2|^2 \} d\omega.$$

After some manipulations, it can be rewritten as

$$J_V^{(2)} = \frac{n}{2\pi N} \int_{-\pi}^{\pi} \frac{\sigma^2 |H|^2}{|1+GC_2|^2} \\ \left(1 + \frac{|\sigma^2 C_2 H + \phi_{ue}|^2}{\sigma^2 \phi_u - |\phi_{ue}|^2} \right) d\omega. \quad (78)$$

Clearly, the minimum is achieved for

$$\phi_{ue} \triangleq -\frac{C_{id,2}H}{1+GC_{id,2}} \sigma^2 = \frac{C_2 H}{1+GC_2} \sigma^2, \quad (79)$$

and hence $C_{id,2} = C_2$. We note that for this optimal value of $C_{id,2}$, $J_V^{(2)\text{opt}}$ becomes independent of $C_{id,1}$:

$$J_V^{(2)\text{opt}} = \frac{n}{2\pi N} \int_{-\pi}^{\pi} \frac{\sigma^2 |H|^2}{|1+GC_2|^2} d\omega.$$

Theorem 4. Consider the identification of a system $\mathcal{S} = [G, H]$ using a model set $\mathcal{M} = \{[G(\theta), H(\theta)], \theta \in D_\theta\}$ with $\mathcal{S} \in \mathcal{M}$, for the purpose of designing a two-degree-of-freedom model reference controller $[C_1, C_2]$ specified by the mapping (62), (63). Then the closed-loop degradation measure J_V defined by (15) is minimized if the identification is performed in closed loop with the ideal controller $[C_1 \ C_2]$ operating on the plant. Under those optimal experimental conditions,

$$J_V^{\text{opt}} \approx \frac{2n}{2\pi N} \int_{-\pi}^{\pi} \frac{\sigma^2 |H|^2}{|1+GC_2|^2} d\omega \\ = \frac{2n}{N} \sigma_y^2, \quad (80)$$

where σ_y^2 is the variance of the noise contribution in the output of the ideal closed-loop system.

7. CONCLUSIONS

We have presented three distinct contributions. The first was about optimal experiment design. We showed that, when a model is identified with a view to computing a controller $C = c(G, H)$ that, at each frequency ω , is a smooth function of the model at that same frequency, and when the system is in the model set so that no bias errors occur, then the optimal experimental setup is to identify the system in closed loop with some optimal controller C_{id}^{opt} operating on the plant. The second was to show that, for MV and MR control criteria, this controller C_{id}^{opt} happens to be the ideal controller: $C_{id}^{opt} = C$. Here ideal means that it is the controller that would be designed if the true plant were known.

Our third contribution has been to exploit this apparently useless result (at least from a practical point of view) to establish that, provided the identification time is long enough, one can always obtain a more accurate controller estimate (and better closed-loop behaviour on the real plant) by performing at least part of the identification on closed loop data, with the current controller estimate operating on the plant. This gives theoretical support to the idea of iterative identification and control design—at least when the system is in the model set.

Several interesting open problems remain, the most intriguing being a characterization of all control design criteria for which the best controller to be applied during identification, C_{id}^{opt} , is the ideal controller C .

Acknowledgements—This paper presents research results of the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State Prime Minister's Office for Science, Technology and Culture. The scientific responsibility rests with its authors. Håkan Hjalmarsson was also with CESAME on postdoctoral leave from the Department of Electrical Engineering, Linköping University, Sweden when this work was completed.

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