ON MULTIVARIABLE POLE-ZERO CANCELLATIONS AND THE STABILITY OF FEEDBACK SYSTEMS*

by

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ABSTRACT

We study conditions for pole-zero cancellation including unstable pole-zero cancellation in the product of two transfer function matrices G and H. Pole-zero cancellation is defined using McMillan degree ideas, and conditions for cancellation are phrased in terms of the coprimeness of matrices associated with matrix fraction descriptions of G and H. Using the condition for unstable pole-zero cancellation, we obtain a new set of conditions for the stability of linear MIMO feedback systems. We show that such a feedback system is stable if and only if there is no unstable pole-zero cancellation in GH and if $(I+GH)^{-1}$ is stable. On the other hand, if there is no unstable pole-zero cancellation in GH and any or all of $(I+HG)^{-1}$, $G(I+HG)^{-1}$ and $H(I+GH)^{-1}$ are stable, the closed-loop may be unstable - but only if there is an unstable pole zero cancellation in HG.

1. INTRODUCTION

Consider the discrete-time feedback system illustrated in Figure 1. We shall assume G and H are proper rational transfer function matrices represented by coprime matrix fraction descriptions

$$G = A_\ell^{-1}B_\ell = B_rA_r^{-1} \qquad H = C_\ell^{-1}D_\ell = D_rC_r^{-1} \tag{1}$$

though extensions to more general transfer function matrices using the ideas of [1, 2] are undoubtedly possible.

There are two main thrusts of the note. The first is to state conditions on A_ℓ, B_ℓ, etc. for the existence of pole-zero cancellations in the product GH. This first of all requires us to define what is meant by pole-zero cancellation, and it proves convenient to also define the notions of pole-zero cancellation at a point z_0, and unstable pole-zero cancellation. These definitions involve the concept of McMillan degree.

The second thrust of the note is to develop conditions for the stability of the scheme of Figure 1. To understand these conditions in perspective, we recall the following results in [3, 4]:

<u>Theorem 1</u>: Suppose that $\det[I+HG] \not\equiv 0$. Then the transfer function matrix

$$W(z) = \begin{bmatrix} (I+HG)^{-1} & -H(I+GH)^{-1} \\ G(I+HG)^{-1} & (I+GH)^{-1} \end{bmatrix} \tag{2}$$

which links the z-transforms of $[u_1^T \;\; u_2^T]^T$ to $[e_1^T \;\; e_2^T]^T$ has all poles* in $|z| < 1$ if and only if any one of the

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* The poles of a matrix are taken to be the poles of the entries of the matrix.

the following conditions holds

$$\det\begin{bmatrix} A_r(z) & D_r(z) \\ -B_r(z) & C_r(z) \end{bmatrix} \neq 0 \qquad \forall\ |z| \geq 1 \tag{3a}$$

$$\det\begin{bmatrix} C_\ell(z) & D_\ell(z) \\ -B_\ell(z) & A_\ell(z) \end{bmatrix} \neq 0 \qquad \forall\ |z| \geq 1 \tag{3b}$$

$$\det[C_\ell(z)A_r(z) + D_\ell(z)B_r(z)] \neq 0 \qquad \forall\ |z| \geq 1 \tag{3c}$$

$$\det[A_\ell(z)C_r(z) + B_\ell(z)D_r(z)] \neq 0 \qquad \forall\ |z| \geq 1 \tag{3d}$$

Moreover, if the blocks $G(z)$ and $H(z)$ physically correspond to minimal state-variable realizations, the closed-loop is asymptotically stable in the sense of Lyapunov.

Reference [3] also demonstrates that there exist pairs G, H such that any 3 block entries of $W(z)$ in (2) have all poles in $|z| < 1$ while the fourth block entry has a pole in $|z| > 1$. Thus stability of the closed loop cannot be concluded by establishing, say, that $(I+HG)^{-1}$ is stable. For completeness however, we note that [4] establishes any of the following sets of conditions are sufficient to guarantee stability of the closed-loop.

G, H have all poles in $|z| < 1$ and $\det(I+HG) \neq 0$ in $|z| \geq 1$ (4a)

H and $G(I+HG)^{-1}$ have all poles in $|z| < 1$ (4b)

$H(I+GH)^{-1}$ and $G(I+HG)^{-1}$ have all poles in $|z| < 1$ and either G, H have no common pole in $|z| \geq 1$ or G and H are scalar (4c)

Lemma 2: $\delta[P_+] = \sum_{z_i} \delta_{z_i}[P]$, where the sum is taken over all poles of P in $|z| \geq 1$, including ∞ if P is not proper.

Definition: With G, H rational transfer functions for which the product GH exists, GH contains no pole-zero cancellation at z_0 if

$$\delta_{z_0}[GH] = \delta_{z_0}[G] + \delta_{z_0}[H] \tag{5a}$$

and no unstable pole-zero cancellation if

$$\delta[(GH)_+] = \delta[G_+] + \delta[H_+] \tag{5b}$$

and no pole-zero cancellation if

$$\delta[GH] = \delta[G] + \delta[H] \tag{5c}$$

The motivation for this definition should be clear. It is trivial to note GH may contain no unstable pole-zero cancellation while HG does contain a cancellation. Example:

$$G = \begin{bmatrix} \frac{1}{z-2} \\ \frac{1}{z-2} \end{bmatrix} \qquad H = [1 \quad -1] \tag{6}$$

We shall now obtain an alternative characterization of the above definition in terms of coprime matrix fraction descriptions.

Our main tool will be the following lemma. The second result at least of the lemma is undoubtedly widely known and can be established using the equivalence ideas of [6].

Lemma 3: Let A, B, C, D, E, F, be polynomial matrices such that

$$P = D^{-1}EF^{-1} = AB^{-1}C \tag{7}$$

for some rational matrix transfer function P and let z_0 be finite. Then

$$\delta_{z_0}[P] = \partial_{z_0}[|D|] + \partial_{z_0}[|F|] \tag{8}$$

if and only if

$$[D(z_0)\ \ E(z_0)] \text{ and } \begin{bmatrix} E(z_0) \\ \\ F(z_0) \end{bmatrix} \text{ have full rank} \tag{9}$$

and

$$\delta_{z_0}[P] = \partial_{z_0}[|B|] \tag{10}$$

if and only if

$$\begin{bmatrix} A(z_0) \\ \\ B(z_0) \end{bmatrix} \text{ and } [B(z_0)\ \ C(z_0)] \text{ have full rank} \tag{11}$$

Proof: We shall prove only the first result, the second being very similar. We prove the only if statement first.

Suppose $\text{rank}[D(z_0)\ E(z_0)]$ is not full. Then there exists a square polynomial matrix $Q(z)$ with $|Q(z_0)| = 0$ and

$$[D(z)\ \ E(z)] = Q(z)[D_1(z)\ \ E_1(z)]$$

with $[D_1\ \ E_1]$ left coprime. It follows that

$$P = D_1^{-1}E_1F^{-1} = D_1^{-1}K^{-1}L$$

for some coprime K, L. Clearly,

$$\delta_{z_0}[P] \le \partial_{z_0}[|KD_1|] \text{ (with equality guaranteed iff } [KD_1, L] \text{ is coprime)}$$

$$= \partial_{z_0}[|D_1|] + \partial_{z_0}[|K|]$$

$$\le \partial_{z_0}[|D_1|] + \partial_{z_0}[|F|] \quad \text{(since } K^{-1}L = E_1F^{-1} \text{ and } [K,\ L] \text{ is coprime)}$$

$$= \partial_{z_0}[|D|] + \partial_{z_0}[|F|] - \partial_{z_0}[|Q|]$$

$$< \partial_{z_0}[|D|] + \partial_{z_0}[|F|]$$

Likewise, if $\begin{bmatrix} E(z_0) \\ F(z_0) \end{bmatrix}$ fails to have full rank, this inequality results. So failure of (9) implies failure of (8), or (8) holds only if (9) holds.

Now suppose (9) holds. Let $EF^{-1} = K^{-1}L$, with $[K\ L]$ left coprime. Then $P = (KD)^{-1}L$. We claim that

$$[K(z_0)D(z_0) \quad L(z_0)] \text{ has full rank} \tag{12}$$

We argue by contradiction. If (12) fails, there exists a row vector $\alpha \neq 0$ such that

$$\alpha[K(z_0)D(z_0) \quad L(z_0)] = 0 \tag{13}$$

implying

$$\alpha[K(z_0)D(z_0) \quad L(z_0)F(z_0)] = 0$$

and then, because $KE = LF$,

$$\alpha K(z_0)[D(z_0) \quad E(z_0)] = 0 \tag{14}$$

Using (9), we conclude from (14) that $\alpha K(z_0) = 0$, while also $\alpha L(z_0) = 0$ by (13), contradicting the coprimeness of K, L. This establishes (12).

Now using (12), we have

$$\delta_{z_0}[P] = \delta_{z_0}[(KD)^{-1}L]$$
$$= \partial_{z_0}[|KD|]$$
$$= \partial_{z_0}[|D|] + \partial_{z_0}[|K|]$$
$$= \partial_{z_0}[|D|] + \delta_{z_0}[K^{-1}L] \tag{15}$$

(Recall that $[K, L]$ are left coprime). Now use the fact that $EF^{-1} = K^{-1}L$, with the second assumption in (9). Thus (15) yields

$$\delta_{z_0}[P] = \partial_{z_0}[|D|] + \delta_{z_0}[EF^{-1}]$$
$$= \partial_{z_0}[|D|] + \partial_{z_0}[|F|]$$

as required.

<u>Theorem 2</u>: With G, H given by the coprime matrix fraction descriptions (1), with both matrices proper, and with the product GH existing,

$$\delta_{z_0}[GH] = \delta_{z_0}[G] + \delta_{z_0}[H] \tag{16}$$

if and only if one of the following three equivalent conditions holds:

(i) $\begin{bmatrix} B_r(z_0) \\ C_\ell(z_0)A_r(z_0) \end{bmatrix}$ and $[C_\ell(z_0)A_r(z_0) \; \vdots \; D_\ell(z_0)]$ (17)

have full rank.

(ii) $\begin{bmatrix} C_r(z_0) \\ B_\ell(z_0)D_r(z_0) \end{bmatrix}$ and $[A_\ell(z_0) \; \vdots \; B_\ell(z_0)D_r(z_0)]$ (18)

have full rank.

ensure complete observability of product GH

ensure complete controllability of product GH

(iii) $\begin{bmatrix} B_\ell(z_0) \\ \\ C_\ell(z_0) \end{bmatrix}$ and $[A_r(z_0) \,\vdots\, D_r(z_0)]$ (19)

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have full rank. c.c.

c.o.

Proof: There are no poles of G, H, or GH at $z_0 = \infty$. So it is enough to establish the equivalence at every finite z_0. By (1):

$$GH = B_r(C_\ell A_r)^{-1}D_\ell = A_\ell^{-1}(B_\ell D_r)C_r^{-1} \tag{20}$$

$$= A_\ell^{-1}B_\ell C_\ell^{-1}D_\ell = B_r A_r^{-1}D_r C_r^{-1} \tag{21}$$

Also:

$$\delta_{z_0}[G] = \partial_{z_0}[|A_\ell|] = \partial_{z_0}[|A_r|] \tag{22}$$

$$\delta_{z_0}[H] = \partial_{z_0}[|C_\ell|] = \partial_{z_0}[|C_r|] \tag{23}$$

The equivalence between (16), (17) and (18) follows immediately from (20), (22), (23) and Lemma 3. The equivalence between (16) and (19) is proved as follows.

1) Only if: Suppose, e.g. that $[A_r(z_0) \,\vdots\, D_r(z_0)]$ does not have full rank. Then $A_r^{-1}D_r = A_{r1}^{-1}D_{r1}$ and $\partial_{z_0}[|A_{r1}|] < \partial_{z_0}[|A_r|]$ (see proof of Lemma 3). Therefore by (21)

$$\delta_{z_0}[GH] \le \partial_{z_0}[|A_{r1}|] + \partial_{z_0}[|C_r|]$$

$$< \partial_{z_0}[|A_r|] + \partial_{z_0}[|C_r|] = \delta_{z_0}[G] + \delta_{z_0}[H].$$

This is a contradiction.

2) If: Assume that $[A_r(z_0)\ D_r(z_0)]$ has full rank. We show

that then $[A_\ell(z_0) \quad B_\ell(z_0)D_r(z_0)]$ has full rank. We argue by contradiction.

Suppose there exists $\alpha \neq 0$ such that

$$\alpha[A_\ell(z_0) \quad B_\ell(z_0)D_r(z_0)] = 0 \tag{24}$$

Then

$$\alpha[A_\ell(z_0)B_r(z_0) \quad B_\ell(z_0)D_r(z_0)] = 0$$

Then by (1)

$$\alpha[B_\ell(z_0)A_r(z_0) \quad B_\ell(z_0)D_r(z_0)] = 0$$

and, by assumption, $\alpha B_\ell(z_0) = 0$. But then, by (24), $\alpha[A_\ell(z_0) \quad B_\ell(z_0)] = 0$. This is a contradiction, because A_ℓ, B_ℓ are left coprime.

The same method of proof shows also that the first condition of (19) implies the first condition of (18). So (19) implies (18), and thus (16).

Corollary 1: With G, H given by the coprime matrix fraction descriptions (1), with both matrices proper, and with the product GH existing, GH has no unstable pole-zero cancellations if and only if one of the three equivalent conditions (17)-(19) holds for all z_0 in $|z| \geq 1$.

Proof: The proof follows immediately from Lemma 2 and Theorem 2.

Corollary 2: Let G and H be proper rational transfer function matrices given by the coprime matrix fraction descriptions (1) and assume that the product GH exists. Then GH contains no pole-zero cancellation if and only if one of the following three equivalence conditions holds:

i) $(B_r, C_\ell A_r)$ are right coprime, and $(C_\ell A_r, D_\ell)$ are left coprime. (25)

(ii) $(C_r, B_\ell D_r)$ are right coprime, and $(A_\ell, B_\ell D_r)$ are left coprime (26)

(iii) (B_ℓ, C_ℓ) are right coprime, and (A_r, D_r) are left coprime. (27)

A theorem of [5] can now be restated very simply.

Theorem 3 [5]: Consider the system obtained by the series connection of a minimal state-variable realization of H followed by a minimal state-variable realization of G. Then this cascade system with the natural state variable is minimal if and only if there is no pole-zero cancellation in GH.

Proof: In [5] it is shown that the cascade realization is minimal if and only if any one of the conditions (25) to (27) holds.

The following result is also clear, as pointed out by a reviewer:

Theorem 4: Consider the system obtained by the series connection of a minimal state-variable realization of H followed by a minimal state-variable realization of G. Then this cascade system with the natural state variable is stabilizable and detectable if and only if there is no pole-zero cancellation in GH.

Comment: We believe that the definition of pole-zero cancellation that we have proposed is the most natural one to use, because it is a natural extension of the scalar definition. In the process of proving certain results on the stability of feedback systems, we proved its equivalence with (25)-(26). It was only later that we became aware of

[5], where Theorem 3 is proved. This made us aware of the fact that Theorem 2 and Corollary 2 could be extended to also include the conditions (19) and (27). Notice however that each of the conditions (25) or (26) requires only one matrix factorization for G and H, while use of (27) requires the computation of both right coprime and left coprime factorizations for each of G and H.

3. CONDITION FOR CLOSED-LOOP SYSTEM STABILITY

Our main result is as follows.

Theorem 5: Let G, H be proper rational transfer function matrices and suppose that $\det[I + G(\infty)H(\infty)] \neq 0$. Then the transfer function matrix $W(z)$ in (2) has all poles in $|z| < 1$ if and only if

$$GH \text{ has no unstable pole-zero cancellation} \tag{28}$$

and

$$(I + GH)^{-1} \text{ has all its poles in } |z| < 1, \tag{29}$$

Proof: If: Assume $(I + GH)^{-1}$ has all its poles in $|z| < 1$, and adopt the matrix fraction descriptions of (1). Then

$$\begin{aligned}(I + GH)^{-1} &= (I + A_\ell^{-1} B_\ell D_r C_r^{-1})^{-1} \\ &= C_r (A_\ell C_r + B_\ell D_r)^{-1} A_\ell \end{aligned} \tag{30}$$

Because GH has no unstable pole-zero cancellations, the matrices

$$[A_\ell \quad B_\ell D_r] \quad \text{and} \quad \begin{bmatrix} B_\ell D_r \\ C_r \end{bmatrix}$$

have full rank in $|z_0| \geq 1$, by Theorem 2, see (18). Consequently, the same is true of the matrices

$$[A_\ell \quad A_\ell C_r + B_\ell D_r] \quad \text{and} \quad \begin{bmatrix} A_\ell C_r + B_\ell D_r \\ C_r \end{bmatrix}$$

Now $(I+GH)^{-1}$ is proper, since $\det[I+G(\infty)H(\infty)] \neq 0$ and G, H are proper. Therefore,

$$\delta\left\{[(I+GH)^{-1}]_+\right\} = \sum_{|z_0|\geq 1} \delta_{z_0}[(I+GH)^{-1}]$$

$$= \sum_{|z_0|\geq 1} \partial_{z_0}[|A_\ell C_r + B_\ell D_r|]$$

on using Lemma 3. Because $(I+GH)^{-1}$ has all poles in $|z| < 1$, for any $|z_0| \geq 1$,

$$\partial_{z_0}[|A_\ell C_r + B_\ell D_r|] = 0$$

and the claim of the theorem follows by Theorem 1.

<u>Only if</u>: Obviously it is necessary for $(I+GH)^{-1}$ to have all poles in $|z| < 1$. Suppose there is an unstable pole-zero cancellation in GH. Then by Theorem 2 there exists z_0 with $|z_0| \geq 1$ such that either

$$[A_\ell(z_0) \quad B_\ell(z_0)D_r(z_0)] \quad \text{or} \quad \begin{bmatrix} B_\ell(z_0)D_r(z_0) \\ C_r(z_0) \end{bmatrix}$$

does not have full rank, and in either case

$$A_\ell(z_0)C_r(z_0) + B_\ell(z_0)D_r(z_0)$$

is singular. By Theorem 1, $W(z)$ does not have all its poles in $|z| < 1$.

It might be conjectured that if there is no pole zero cancellation in GH, and any of $(I+HG)^{-1}$, $G(I+HG)^{-1}$ and $H(I+GH)^{-1}$ is stable, then $(I+GH)^{-1}$ and so $W(z)$ would be stable. That this is not so can be checked with the following example.

Example:

$$G = \begin{bmatrix} \frac{z+1}{z} & \frac{z+1}{z-1} \\ 0 & \frac{z+1}{z} \end{bmatrix} \qquad H = \begin{bmatrix} \frac{z-1}{z} & \frac{z+1}{z-1} \\ 0 & \frac{z+1}{z-1} \end{bmatrix}$$

for which $\delta[(GH)_+] = 2$, $\delta[G_+] = 1$, $\delta[H_+] = 1$. $(I+HG)^{-1}$, $H(I+GH)^{-1}$ and $G(I+HG)^{-1}$ are stable, but $(I+GH)^{-1}$ is unstable. The reason is an unstable pole-zero cancellation in HG at $z = 1$.

From the complete symmetry of Figure 1 it follows that Theorem 5 can also be stated with G and H interchanged. Therefore Theorem 5 also implies the following result.

Corollary 3: Let G and H be proper rational transfer function matrices and suppose $\det[I+G(\infty)H(\infty)] \neq 0$. Then the transfer function matrix $W(z)$ in (2) has a pole in $|z| \geq 1$ if either GH or HG has an unstable pole-zero cancellation.

4. CONCLUSIONS

We have shown the equivalence between two alternative definitions of multivariable pole-zero cancellations. These definitions have also been specialized to unstable pole-zero cancellations. This has enabled us to give a new and very simple set of necessary and sufficient conditions for the stability of linear multivariable feedback systems, which requires the computation of only one of the four submatrices of the feedback system transfer function matrix. Of course, when G and H are scalar, the results are very well-known. The matrix results are, naturally, quite evidently generalizations of the scalar results. But as the work of [1-5] shows, especially those results concerned with stability, the exact form of the generalization is not always intuitively clear, and occasionally, results run counter to intuition. We believe our condition gives new insight into the stability properties of feedback systems in the multivariable case. We also believe that the proposed equivalent definitions of multivariable pole-zero cancellations will prove helpful in a variety of other multivariable problems.

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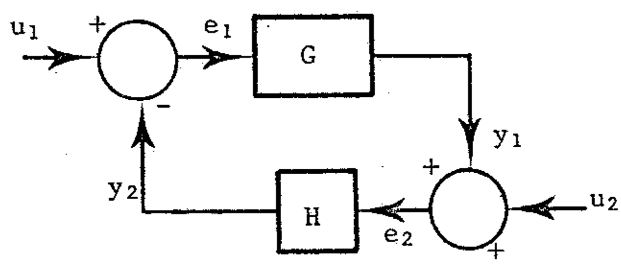

Figure 1: Closed-loop system S