Identifiability of Dynamic Networks: The Essential Rôle of Dources and Dinks

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***Abstract*—The article (Bazanella et al., 2019) presented the first results on generic identifiability of dynamic networks with partial excitation and partial measurements. All previous papers assumed that either all nodes are excited or all nodes are measured. One key contribution of that paper was to establish a set of necessary conditions on the excitation and measurement pattern (EMP) that guarantee generic identifiability: all sources must be excited and all sinks measured, and all other nodes must be either excited or measured. In this article, we show that two other types of nodes, which are defined by the local topology of the network, play an essential rôle in the search for a valid EMP, i.e., one that guarantees generic identifiability. We have called these nodes dources and dinks. We show that a network is generically identifiable only if, in addition to the abovementioned conditions, all dources are excited and all dinks are measured. We also show that sources and dources are the only nodes in a network that always need to be excited, and that sinks and dinks are the only nodes that need to be measured for an EMP to be valid.**

***Index Terms*—Dynamic networks, network analysis and control, network identification.**

I. Introduction

This work deals with identifiability of dynamic networks. The network framework used in this article was introduced in [2], where signals are represented as nodes of the network, which are related to other nodes through transfer functions. To such dynamic network, one can associate a directed graph, where the transfer functions, also called modules, are the edges of the graph and the node signals are its vertices.

In [2], it was assumed that all nodes are excited and measured. As a result, an input–output matrix of the network, denoted $T(z)$, can be defined, which can always be identified from these excitation and measurement data. The network identifiability question is then whether the network matrix, denoted $G(z)$, whose elements are the transfer functions relating the nodes, can be recovered from $T(z)$. In subsequent works, a range of new objectives were defined, from the identification of the whole network to identification of some specific part of the network [3], [4], [5], [6], [7], [8], [9], [10], [11]. As for the assumptions on the signals, up to 2019, all contributions assumed that either all nodes are excited, or all nodes are measured.

The first identifiability results for networks with partial excitation and measurement were presented in [1]. Thus, in that paper, the assumption that either all nodes are measured, or all nodes are excited, was removed. The paper first provided a necessary condition for identifiability of any network: each node must be either excited or measured, at least one node must be excited and at least one node measured, all sources must be excited, and all sinks must be measured.[1]

The results of [1] inspired a new way of looking at the network identifiability problem. The question now becomes: what is a combination of excited nodes and measured nodes, defined as an excitation and measurement pattern (EMP), that allows the identifiability of the network? The concept of EMP was introduced in [12], where an EMP was called *valid* if it guarantees the identifiability of the whole network; it was called *minimal* if it guarantees the identifiability of the network using the smallest possible combination of excited and measured nodes. This number is the *cardinality of the EMP.* Achieving identifiability of a network with a minimal EMP is of both theoretical and practical interest, since the excitation of a node or its measurement has a cost. On the other hand, having some flexibility in the choice of a valid EMP is also of practical interest, since these costs may be significantly different for different nodes. One must of course remember that each node must be either excited or measured or both, as shown in [1]. As a result, the cardinality of a valid EMP is always at least equal to n, the number of nodes.

The search for valid, and possibly minimal, EMPs began by looking at special structures. In [1], a necessary and sufficient condition was given for the identifiability of a tree; it showed that a tree can possibly be identified with an EMP of cardinality n. In [13], necessary and sufficient conditions were derived for the identifiability of some classes of parallel networks. In [14], necessary and sufficient conditions were given for the identifiability of loops. It was shown that any loop with more than three nodes can be identified with a minimal EMP of cardinality n, and that constructing EMPs for loops—even minimal EMPs—is very easy. A novel approach to the generic identifiability of a network with partial excitation and measurement was developed in [15], where a local identifiability analysis allows the authors to determine which transfer functions are identifiable with probability one.

In [16], we generalized the results of [1] for the identification of trees to a much wider class of networks, namely, those that have the structure of a directed acyclic graph (DAG), i.e., a directed graph that has no cycles. In the process of deriving valid EMPs for the identifiability of a DAG, we came up with a completely unexpected result. Whereas it has been known for some time that identifiability of any network requires that all sources must be excited and all sinks must be measured, we showed that, besides sources and sinks, two other types of nodes play a

[1] See Section II for the definitions.

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particular rôle in the construction of a valid EMP for DAGs. We called them *dources* and *dinks*,[2] and we showed that identifiability of a DAG requires that all its dources be excited and all its dinks be measured.

To our surprise, we have since realized that dources must be excited and dinks must be measured for the identifiability of any dynamic network, not just DAGs. This is the main message of this article, which is organized as follows. In Section II, we define the dynamic networks we deal with, we recall the definition of generic identifiability, and the existing conditions on excitation and measurement for networks, and we introduce the new concept of dources and dinks. In Section III, we show that, in addition to the previously known conditions for identifiability of a network, a network is generically identifiable only if all its dources are excited and all its dinks are measured. The results of Section IV show that sources and dources are the only nodes in a network that always require excitation, and that sinks and dinks are the only nodes that need to be measured for an EMP to be valid. The essential rôle of dources and dinks for generic identifiability of a dynamic network may appear bizarre at first. In Section V, we illustrate with an example the intuition that underpins these two new concepts. We also show that, unless a node is a source or a dource, one can always identify the network without exciting that node. Finally, Section VI concludes this article.

II. Definitions, Notations, and Preliminaries

We consider dynamic networks composed of n nodes (or vertices), which represent internal scalar signals $\{w_k(t)\}$ for $k \in \{1, 2, \ldots, n\}$. These nodes are interconnected by discrete time transfer functions, represented by edges, which are entries of a *network matrix* $G(z)$. The dynamics of the network is given by the following:

$$w(t) = G(z)w(t) + Br(t) \tag{1a}$$

$$y(t) = Cw(t) \tag{1b}$$

where $w(t) \in \mathbb{R}^n$ is the node vector, $r(t) \in \mathbb{R}^m$ is the input vector, and $y(t) \in \mathbb{R}^p$ is the set of measured nodes, considered as the output vector of the network. The matrix $B \in \mathbb{Z}_2^{n\times m}$, where $\mathbb{Z}_2 \triangleq \{0, 1\}$, is a binary selection matrix with a single 1 and $n-1$ zeros in each column; it selects the inputs affecting the nodes of the network. Similarly, $C \in \mathbb{Z}_2^{p\times n}$ is a matrix with a single 1 and $n-1$ zeros in each row that selects which nodes are measured.

To each network matrix $G(z)$, we can associate a directed graph $\mathcal{G}$ defined by the tuple $(\mathcal{V}, \mathcal{E})$, where $\mathcal{V}$ is the set of vertices and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. It defines the topology of the network. A particular transfer function $G_{ij}(z)$ of $G(z)$ is called an incoming edge of node i and outgoing edge of node j. For this transfer function, node i is an out-neighbor of node j, and node j is an in-neighbor of node i. A node j is connected to node i if there exists a directed edge from node j to node i. For the graph $\mathcal{G}$ associated with $G(z)$, we introduce the following notations.

1) $\mathcal{W}$—the set of all n nodes.
2) $\mathcal{B}$—the set of excited nodes, defined by B in (1a).
3) $\mathcal{C}$—the set of measured nodes, defined by C in (1b).
4) $\mathcal{F}$—the set of sources: nodes with no incoming edges.
5) $\mathcal{S}$—the set of sinks: nodes with no outgoing edges.
6) $\mathcal{N}_j^-$—the set of in-neighbors of node j.
7) $\mathcal{N}_j^+$—the set of out-neighbors of node j.

In addition, we introduce the following two types of nodes discussed in Section I.

Definition 1: A node j is called a *dource* if it has at least one out-neighbor to which all its in-neighbors have a directed edge.

Definition 2: A node j is called a *dink* if it has at least one in-neighbor that has a directed edge to all its out-neighbors.

[2] See Section II for their definition.

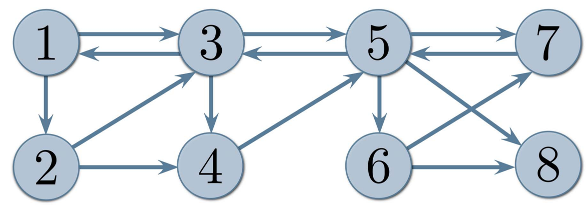

Fig. 1. Example of a network where nodes 2 and 6 are dources, and nodes 4 and 6 are dinks.

Observe that a node can be both a dource and a dink. Fig. 1 illustrates an 8-node network, in which node 2 is a dource, node 4 is a dink, and node 6 is both a dource and a dink.

Assumptions on the network matrix $G(z)$

We make the following assumptions on the network matrix.

1) The diagonal elements are zero and all other elements are proper.
2) $(I - G(z))^{-1}$ is proper and all its elements are stable.

One can represent the dynamic network in (1a) and (1b) as an input–output model as follows:

$$y(t) = M(z)r(t), \text{ with } M(z) \triangleq CT(z)B. \tag{2}$$

where

$$T(z) \triangleq (I - G(z))^{-1}. \tag{3}$$

Observe that $T(z)$ is generically nonsingular by construction.

In analyzing the identifiability of the network matrix, it is assumed that the input–output model $M(z)$ is known; the identification of $M(z)$ from input–output data $\{y(t), r(t)\}$ is a standard identification problem, provided the input signal $r(t)$ is sufficiently rich. The question of identifiability of the network is then whether the network matrix $G(z)$ can be fully recovered from the transfer matrix $M(z)$. We now give a formal definition of generic identifiability of the network matrix from the data $\{y(t), r(t)\}$ and from the graph structure.

Definition 3 [6]: The network matrix $G(z)$ is generically identifiable from excitation signals applied to $\mathcal{B}$ and measurements made at $\mathcal{C}$ if, for any rational transfer matrix parameterization $G(P, z)$ consistent with the directed graph associated with $G(z)$, there holds

$$C[I - G(P, z)]^{-1}B = C[I - \tilde{G}(z)]^{-1}B \Rightarrow G(P, z) = \tilde{G}(z)$$

for all parameters P except possibly those lying on a zero measure set in $\mathbb{R}^N$, where $\tilde{G}(z)$ is any network matrix consistent with the graph.

In this article, we discuss generic identifiability in terms of which nodes must be excited and/or measured in order to guarantee identifiability of the network. Thus, we do not assume that either all nodes are excited or all nodes are measured, which was the common assumption until the publication of [1]. Our results expand on the following proposition, which gives a necessary condition for generic identifiability of a network; it combines Theorem III.1 and Corollary III.1 of [1].

Proposition II.1: The network matrix $G(z)$ is generically identifiable only if the following holds.

1) At least one node is excited and one node is measured.
2) All sources are excited.
3) All sinks are measured.
4) All other nodes are either excited or measured.

Finding conditions that guarantee generic identifiability of a given network is equivalent to constructing an EMP that guarantees identifiability; such EMP is then called a valid EMP. The concept of EMP and of valid EMP, which led to the concept of minimal EMP, was introduced in [12]. They are defined in the following.

Definition 4: A pair of selection matrices B and C, with its corresponding pair of node sets $\mathcal{B}$ and $\mathcal{C}$, is called an EMP. An EMP is said to be *valid* if it is such that the network (1a) and (1b) is generically identifiable. Let $\nu = |\mathcal{B}| + |\mathcal{C}|$[3] be the cardinality of an EMP. A given EMP is said to be *minimal* if it is valid and there is no other valid EMP with smaller cardinality.

The following result establishes a lower and an upper bound for the cardinality of a valid EMP for any network.

Lemma II.1: The cardinality of a minimal EMP for the identification of a dynamic network with n nodes is at least equal to n and at most equal to $2n - f - s$, where f is the number of sources and s the number of sinks.

Proof: The lower bound results from Proposition II.1; it can actually be achieved for trees and loops, as shown in [1] and [14]. As for the upper bound, we know by Proposition II.1 that all sources must be excited and all sinks measured, while the remaining $n - f - s$ nodes must be excited or measured. Assuming that these are all excited and measured, then the cardinality of the EMP is $f + s + 2(n - f - s) = 2n - f - s$. ■

Proposition II.1 showed that, inter alia, all sources must be excited and all sinks must be measured for an EMP to be valid. The main objective of this article is to show that, in addition, all dources must be excited and all dinks must be measured for an EMP to be valid. Beyond this extension of the necessary conditions of Proposition II.1, we will also present necessary and sufficient conditions on dources and dinks that any valid EMP must satisfy.

From now on, we drop the arguments z and t used in (1a) and (1b) whenever there is no risk of confusion. We first present a preliminary technical lemma.

Lemma II.2: Consider a dynamic network with network matrix G and transfer matrix $T = (I - G)^{-1}$. The following relationships hold:

$$T_{ii} = 1 + \sum_{j=1}^{n} T_{ij} G_{ji} = 1 + \sum_{j=1}^{n} G_{ij} T_{ji} \tag{4}$$

$$T_{ik} = \sum_{j=1}^{n} T_{ij} G_{jk} = \sum_{j=1}^{n} G_{ij} T_{jk}, \text{ for } k \neq i. \tag{5}$$

Proof: The proof follows from:

$$T(I - G) = I \iff T = I + TG. \tag{6}$$

The ith row of T, denoted T_i, can be written as $T_i = I_i + T_i G$, from which the first equalities of (4) and (5) follow. As for the last equalities of (4) and (5), they follow from the fact that $(I - G)T = I = T(I - G) \iff GT = TG$. ■

III. Necessary Condition on Dources and Dinks

Our main result in this section is the following theorem. It presents a new set of necessary conditions for the identifiability of a network, which is a sharpening of those of Proposition II.1.

Theorem III.1: The network matrix $G(z)$ is generically identifiable only if the following holds.

1) At least one node is excited and one node is measured.
2) All sources and dources are excited.
3) All sinks and dinks are measured.
4) All other nodes are either excited or measured.

Proof: Observe that the necessary conditions of Theorem III.1 are the same as those of Proposition II.1, with the addition that all dources must be excited and all dinks must be measured. All conditions except the excitation of dources and the measurement of dinks have been proved in Proposition II.1. We thus prove that dources must be excited; the proof for dinks follows by duality.

Let D be a dource (see Definition 1), and let O be an out-neighbor of D to which all its k in-neighbors I are connected. Both D and O have dimension 1. Let the remaining $n - 2 - k$ nodes of the network be labeled as S. The network matrix G can then be partitioned as follows:

$$G = \begin{bmatrix} 0 & G_{DO} & G_{DI} & G_{DS} \\ G_{OD} & 0 & G_{OI} & G_{OS} \\ G_{ID} & G_{IO} & G_{II} & G_{IS} \\ G_{SD} & G_{SO} & G_{SI} & G_{SS} \end{bmatrix}. \tag{7}$$

Notice that from the definition of dource we have $G_{DO} = 0$[4] and $G_{DS} = 0$ since all in-neighbors of D are collected in I. We now assume that the dource D is not excited and we show that we can then not uniquely recover G_{OD} and G_{OI}. From the relationship $(I - G)T = I$, we write down all equations in which G_{OD} and G_{OI} appear. This yields the following:

$$\begin{bmatrix} -G_{OD} \\ 1 \\ -G_{OI} \\ -G_{OS} \end{bmatrix}^T \left[\begin{array}{cc|cc} T_{DD} & T_{DO} & T_{DI} & T_{DS} \\ T_{OD} & T_{OO} & T_{OI} & T_{OS} \\ \hline \text{ine}T_{ID} & T_{IO} & T_{II} & T_{IS} \\ T_{SD} & T_{SO} & T_{SI} & T_{SS} \end{array}\right] = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^T. \tag{8}$$

Since D is a dource, it follows that $G_{OD} \neq 0$ and that all elements of G_{OI} are nonzero, whereas the elements of G_{OS} can be zero or nonzero. Since it is assumed that D is not excited, the first column of the T matrix is unknown. Disregarding this unknown first column of T, we get the following equation:

$$\begin{bmatrix} G_{OD} & G_{OI} \end{bmatrix} \begin{bmatrix} T_{DO} & T_{DI} & T_{DS} \\ T_{IO} & T_{II} & T_{IS} \end{bmatrix}$$
$$= \begin{bmatrix} T_{OO} - 1 - G_{OS} T_{SO} & T_{OI} - G_{OS} T_{SI} & T_{OS} - G_{OS} T_{SS} \end{bmatrix}. \tag{9}$$

The question is whether or not G_{OD} and G_{OI} can be uniquely identified from (9), even in the situation where all nodes other than D are both excited and measured, i.e., even if all T_{XY} elements in (9) are known. Equation (9) has a unique solution for $[G_{OD}\ G_{OI}]$ only if the matrix

$$\begin{bmatrix} T_{DO} & T_{DI} & T_{DS} \\ T_{IO} & T_{II} & T_{IS} \end{bmatrix} \tag{10}$$

has full generic row rank. We show that this matrix does not have full generic row rank. From (5) in Lemma 2.2, we have

$$T_{ik} = T_{i,:} G_{:k} = G_{i,:} T_{:,k} \tag{11}$$

where $T_{i,:}$ and $T_{:,k}$ denote the ith row and kth column of T, respectively. Therefore, we can write

$$T_{DO} = G_{D,:} T_{:,O} = \begin{bmatrix} 0 & G_{DO} & G_{DI} & G_{DS} \end{bmatrix} \begin{bmatrix} T_{DO} \\ T_{OO} \\ T_{IO} \\ T_{SO} \end{bmatrix}$$
$$= \begin{bmatrix} 0 & 0 & G_{DI} & 0 \end{bmatrix} T_{:,O} = G_{DI} T_{IO}.$$

Similarly, we have that $T_{DI} = G_{D,:} T_{:,I} = G_{DI} T_{II}$, and $T_{DS} = G_{D,:} T_{:,S} = G_{DI} T_{IS}$. Thus, the generic row rank of the matrix in (10) is not full, since the first row is a linear combination of the elements of the second block row. As a result, G_{OD} and G_{OI} cannot be uniquely

[3] $|\cdot|$—Denotes the cardinality of a set.

[4] Otherwise O would also be an in-neighbor of D, and since all in-neighbors of D must be connected to O, this would create a self-loop, which is not allowed.

identified when the dource D is not excited, which proves condition 2) of the Theorem. The result for dinks, i.e., condition 3) is obtained by duality, and the proof is therefore omitted. ■

These new necessary conditions for identifiability of a dynamic network are both surprising and practically important. The definitions of dources and dinks are based on the local topology around each node of the network. In the search for a valid EMP, the first task is therefore to locate the dources and dinks in the network and to enforce, in the EMP, an excitation at each dource and a measurement at each dink; in addition, of course, to the excitation of all sources and the measurement of all sinks. An algorithm that detects all dources and dinks in the network from its matrix G has been developed by Mapurunga [17]. It is based on the fact that if the submatrix $G_{\mathcal{N}_i^+,\mathcal{N}_i^-}$ has at least one complete nonzero row (column), then node i is a dource (dink).

IV. Only Sources and Dources Need to Be Excited

In this section, we move from the necessary condition on the excitation of sources and dources of Theorem III.1 to a necessary and sufficient condition. We first show that sources and dources are the only type of nodes that need to be excited for the generic identifiability of a dynamic network, regardless of the EMP on all other nodes of the network.

Theorem IV.1: Consider a dynamic network with network matrix G and let D be a node of interest. There exists at least one valid EMP, in which node D is measured but not excited, if and only if node D is neither a source nor a dource.

Proof: The necessity of excitation follows directly from Theorem III.1. We thus need to prove that, if node D is neither a source nor a dource, then there exists an EMP, in which D is measured but not excited and for which G is generically identifiable. We define an EMP, in which D is measured but not excited, all sources are excited and all sinks are measured, and all other nodes are both excited and measured. We show that if D is not a source or a dource, then, even if $T_{:,D}$ is not available (because node D is not excited), we can identify all transfer functions from the network.

Consider first all nodes of the network that are not out-neighbors of D. These nodes are excited and measured, and their in-neighbors are excited; thus their incoming edges are all identifiable by the dual of [6, Corollary V.2]. Thus, we only need to consider the out-neighbors of D, and prove that, for each of them, all its incoming edges are identifiable. Let i be an out-neighbor of D, and consider its in-neighbors $\mathcal{N}_i^-$; node D is one of them. Since D is not a dource, its in-neighbors, which are all excited, are not all connected to this out-neighbor i. Thus, the in-neighbors of node i are not all in-neighbors of node D. Therefore, there exist $|\mathcal{N}_i^-|$ vertex disjoint paths from excited nodes to node i; recall that all nodes of the network are excited except D. In addition, by the assumption on the EMP, node D is measured, as well as all its in-neighbors. The result follows from [1, Corollary IV.3]. ■

The dual version of Theorem IV.1 for sinks and dinks is expressed as follows. The proof is dual to that of Theorem IV.1 and is therefore omitted.

Theorem IV.2: Consider a dynamic network with network matrix G, and let D be a node of interest. There exists at least one valid EMP, in which node D is excited but not measured, if and only if node D is neither a sink nor a dink.

The main message of Theorems IV.1 and IV.2 is the following. Provided some node of interest in a network is neither a source nor a dource, one can design a valid EMP for which this node is not excited. Dually, provided some node is neither a sink nor a dink, one can design a valid EMP for which this node is not measured. This has important practical implications for the design of EMPs, for example when a node is difficult to excite or difficult to measure.

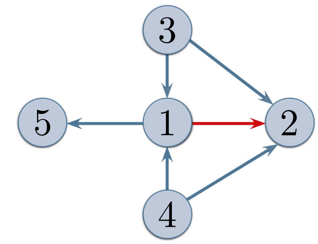

Fig. 2. Why does a dource require excitation?

V. Understanding Dources and Dinks

In this section, we first explain the intuition behind the requirement for the excitation of a dource, using a very simple example. We then present another example to illustrate the result of Theorem IV.1.

A. Example A

Consider the network of Fig. 2.

Its network matrix G is

$$G = \begin{bmatrix} 0 & 0 & G_{13} & G_{14} & 0 \\ G_{21} & 0 & G_{23} & G_{24} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ G_{51} & 0 & 0 & 0 & 0 \end{bmatrix}. \quad (12)$$

We observe that node 1 is a dource. Just as in the proof of Theorem III.1, we identify D as the dource, I as its in-neighbors, O as the out-neighbor of D to which all its in-neighbors I are connected, and S as the remaining nodes of the network. Both D and O have dimension 1. It follows that, for Example A, we have $D = \{1\}, O = \{2\}, I = \{3, 4\}$, and $S = \{5\}$. The matrix in (10) is given by

$$\begin{bmatrix} T_{DO} & T_{DI} & T_{DS} \\ T_{IO} & T_{II} & T_{IS} \end{bmatrix} = \begin{bmatrix} T_{12} & T_{1(3,4)} & T_{15} \\ T_{(3,4)2} & T_{(3,4)(3,4)} & T_{(3,4)5} \end{bmatrix} = \begin{bmatrix} 0 & G_{13} & G_{14} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad (13)$$

and it is clearly seen that there are only two nonzero equations to identify the three unknowns $G_{OD} = G_{21}$ and $G_{OI} = [G_{23}\ G_{24}]$. Hence, any EMP in which node 1 is not excited is not valid; in other words, in all valid EMPs node 1 is excited.

The intuition behind this result can be observed from the graph of Fig. 2. Nodes 3 and 4 are sources, and they must therefore be excited, while node 2 must be measured because it is a sink. There are two parallel paths from the in-neighbors of the dource (i.e., nodes 3 and 4) to its out-neighbor node 2. This makes the identification of the red transfer function G_{21} impossible unless node 1 is excited. The crucial condition for node 1 to be a dource is that *all* its neighbors must be connected to the out-neighbor 2. If, on the other hand, $G_{24} = 0$, then node 1 is no longer a dource and G_{21} becomes identifiable using the excitation of node 4, without exciting node 1.

B. Example B

To illustrate the result of Theorem IV.1, we use the example of Fig. 3.

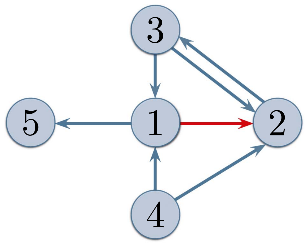

Fig. 3. Illustration of Theorem IV.1.

Its network matrix is given by

$$G = \begin{bmatrix} 0 & 0 & G_{13} & G_{14} & 0 \\ G_{21} & 0 & G_{23} & G_{24} & 0 \\ 0 & G_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ G_{51} & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{14}$$

We observe that node 1 is a dource. With the same definitions as in Example A, we now have $D = \{1\}$, $O = \{2\}$, $I = \{3, 4\}$, and $S = \{5\}$. The input–output matrix T is given in (19) as shown at the bottom of this page, with $\Delta = 1 - G_{23}G_{32} - G_{21}G_{13}G_{32}$.

We first establish, for this Example, that node 1 must be excited. If it is not excited, then the first column of the T matrix in (8) is unknown, and the question is whether or not G_{OD} and G_{OI} can be uniquely identified from (9), even if all T_{XY} elements in (9) are known. The relevant matrix multiplying $[G_{OD}\ G_{OI}]$ in (9) is therefore (10), which is given by

$$\begin{bmatrix} T_{DO} & T_{DI} & T_{DS} \\ T_{IO} & T_{II} & T_{IS} \end{bmatrix} = \begin{bmatrix} T_{12} & T_{1(3,4)} & T_{15} \\ T_{(3,4)2} & T_{(3,4)(3,4)} & T_{(3,4)5} \end{bmatrix}$$

$$= \frac{1}{\Delta} \begin{bmatrix} G_{13}G_{32} & G_{13} & G_{13}G_{32}G_{24} - G_{14}G_{23}G_{32} + G_{14} & 0 \\ G_{32} & 1 & G_{32}G_{14}G_{21} + G_{32}G_{24} & 0 \\ 0 & 0 & \Delta & 0 \end{bmatrix}$$

with $\Delta = 1 - G_{23}G_{32} - G_{21}G_{13}G_{32}$. It is not difficult to check that the first row of this matrix is a linear combination of the other two: $l_1 = G_{13}l_2 + G_{14}l_3$. As a result, there are only two linearly independent equations to identify the three unknowns $G_{OD} = G_{21}$ and $G_{OI} = [G_{23}\ G_{24}]$. Hence, any valid EMP requires excitation of node 1, which confirms Theorem III.1.

We now illustrate Theorem IV.1 for this example. Since node 1 is the only dource, and node 4 is the only source, the theorem states that the excitation of any other node is dispensable, in the sense that there is at least one valid EMP for which this node need not be excited. We illustrate the result with node 3. Suppose that we do not excite it, so that the third column of T in (19) shown at the bottom of this page, is unknown. Identification of the network matrix must then rely on solving the set of equations

$$G\tilde{T} = \tilde{T} - \tilde{I} \tag{15}$$

where $\tilde{T}$ is the (5×4) matrix obtained by removing the third column of T, and $\tilde{I}$ is the (5×4) matrix obtained by removing the third column of the identity matrix I_5. This set of equations can be organized as four different systems of linear equations, one for each row of G

$$\begin{bmatrix} G_{13} \\ G_{14} \end{bmatrix}^T \begin{bmatrix} T_{31} & T_{32} & T_{34} \\ 0 & 0 & T_{44} \end{bmatrix} = \begin{bmatrix} T_{11} - 1 \\ T_{12} \\ T_{14} \end{bmatrix}^T \tag{16}$$

$$\begin{bmatrix} G_{21} \\ G_{23} \\ G_{24} \end{bmatrix}^T \begin{bmatrix} T_{11} & T_{12} & T_{14} \\ T_{31} & T_{32} & T_{34} \\ 0 & 0 & T_{44} \end{bmatrix} = \begin{bmatrix} T_{21} \\ T_{22} - 1 \\ T_{24} \end{bmatrix}^T \tag{17}$$

$$G_{32}T_{21} = T_{31}, \quad G_{51}T_{11} = T_{51} \tag{18}$$

where in all cases we have omitted the last column, which is zero. In (16), it is clear from the matrix structure that there are two linearly independent equations for the two unknowns. In (17), it can be verified that the rank of the matrix is generically 3 by checking that $\det\left(\begin{bmatrix} T_{11} & T_{12} \\ T_{31} & T_{32} \end{bmatrix}\right) = \frac{1}{\Delta}[G_{32}(1 - G_{23}G_{32}) - G_{21}G_{13}G_{32}^2)] = G_{32} \neq 0$ and that $T_{44} = 1$. The equations in (18) are scalar, with $T_{21},\ T_{11} \neq 0$.

Hence, each one of the four equations (17)–(19) has a unique solution and thus all the elements of the network matrix can be identified from knowledge of the first, second, and fourth columns, confirming that there is no need to excite node 3. One valid EMP would thus be $\mathcal{B} = \{1, 2, 4\}$ and $\mathcal{C} = \{1, 2, 3, 5\}$.

On the other hand, it is easy to see that node 4, which is a source, must be excited: if the fourth column of T is unknown, then there are no coefficients multiplying G_{14} or G_{24} in any equations, so these transfer functions cannot be identified.

VI. Conclusion

This article has put the spotlight on the essential rôle of two types of nodes, dources and dinks, whose existence in a network is easy to detect and depends on the local topology only. Their rôle is essential in that they enforce constraints on the design of any valid EMP. Designing a valid EMP is the key to the identification of a dynamic network. With the results of this article, we now know that without exciting all sources and all dources, and without measuring all sinks and all dinks, it is impossible to identify a dynamic network. The first task of any experiment designer is to detect all dources and dinks in the network, which is easily achieved using the algorithm [17], that is freely available.

But in addition to this constraint, in the form of a necessary condition, this article has shown that if a specific node is not a source or a dource, the network is always identifiable without exciting it. Dually, if a specific node is not a sink or a dink, the network is always identifiable without measuring it.

$$T = \frac{1}{\Delta} \begin{bmatrix} 1 - G_{23}G_{32} & G_{13}G_{32} & G_{13} & G_{13}G_{32}G_{24} - G_{14}G_{23}G_{32} + G_{14} & 0 \\ G_{21} & 1 & G_{23} + G_{21}G_{13} & G_{21}G_{14} + G_{24} & 0 \\ G_{32}G_{21} & G_{32} & 1 & G_{32}(G_{21}G_{14} + G_{24}) & 0 \\ 0 & 0 & 0 & \Delta & 0 \\ G_{51}(1 - G_{23}G_{32}) & G_{51}G_{13}G_{32} & G_{51}G_{13} & G_{51}(G_{13}G_{32}G_{24} - G_{14}G_{23}G_{32} + G_{14}) & \Delta \end{bmatrix}. \tag{19}$$

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