Systems & Control Letters 5 (1985) 309–315
North-Holland

April 1985

Monotonicity and stabilizability properties of solutions of the Riccati difference equation: Propositions, lemmas, theorems, fallacious conjectures and counterexamples *

Robert R. BITMEAD, Michel R. GEVERS, Ian R. PETERSEN

Department of Systems Engineering, Research School of Physical Sciences, Australian National University, GPO Box 4, Canberra, ACT 2601, Australia

R. John KAYE

School of Electrical Engineering, University of New South Wales, Kensington, N.S.W. 2033, Australia

Received 19 July 1984
Revised 6 December 1984

The problem considered is that of selecting an initial covariance matrix for the Kalman filter to ensure that the closed-loop filter at every subsequent time instant is exponentially asymptotically stable as a time-invariant filter. Sufficient conditions are derived based on monotonicity properties of the solution of the Riccati difference equation. The results have application in observer design, and the cases of filtering for nonstabilizable systems and systems with singular system matrices are included.

Keywords: Riccati difference equation, Kalman filtering, Stabilizability.

1. Introduction

The results of this paper are concerned with the stabilizing properties of solutions $\{P_i\}$ of the Riccati difference equation (RDE) of optimal filtering. In particular, we ask the question: For what initial condition values, P_0, is the sequence of closed-loop filtering matrices $\{F-K_iH\}$ a sequence of exponentially asymptotically stable matrices? The motivation for studying this problem comes from considering the use of a Kalman filter to derive optimal estimates of the state of a constant parameter model. It is well known that there are certain conditions under which the Kalman filter is asymptotically stable as a time-varying filter [1] and converges, but the limiting steady-state filter need not be asymptotically stable. The new results presented here allow the use of the Kalman filter to generate optimal state estimates as a time-varying filter from an initial start-up time until a switching-off or freezing time, from which the Kalman gain update is stopped and the filter is run as a time-invariant observer. The mechanism utilizes the learning aspects of the Kalman filter's transient response but can avoid the stability and numerical problems of the steady state object. This particular problem arises naturally in short-time Fourier analysis [2].

In the process of deriving a sufficient condition on P_0 that guarantees exponential stability all along the trajectory of the Kalman filter, we have been led to disprove a number of conjectures about the Riccati equation, which corresponded to popular beliefs, at least among the group of authors of this paper. We believe it serves a useful purpose to share the insights we gained by concocting some of the couterexamples, because they show that the Riccati equation is a difficult beast whose behaviour can often be counterintuitive.

The Riccati equation is one of the most studied objects of optimal control and filtering theory and it is rather surprising that new results can still be derived. In this short paper, we cannot cite all the people who have contributed to the vast mountain of results about the behaviour of the RDE but, for our purpose, the most relevant previous contributions are those of Willems [3], Kucera [4],

* Work supported by the Radio Research Board of Australia.

0167-6911/85/$3.30

Martensson [5], Caines and Mayne [6], Payne and Silverman [7] and, most recently, Chan, Goodwin and Sin [8].

To keep this paper short, our results will be presented in discrete time, and for the filtering problem. The results and conclusions apply equally to the dual control problem and to the continuous-time Riccati equation, although in some instances the importance of the problem interpretation is not duality invariant.

2. The discrete-time Riccati equation

We consider the standard Kalman filter

$$\hat{x}_{i+1} = F\hat{x}_i + K_i(y_i - C\hat{x}_i), \tag{2.1}$$

$$K_i = FP_iC^{\mathrm{T}}(CP_iC^{\mathrm{T}} + R)^{-1}, \tag{2.2}$$

and the associated Riccati difference equation (RDE):

$$P_{i+1} = FP_iF^{\mathrm{T}} - FP_iC^{\mathrm{T}}(CP_iC^{\mathrm{T}} + R)^{-1}CP_iF^{\mathrm{T}} + Q \tag{2.3}$$

with initial conditions P_0. This is the Kalman filter for the one-step ahead prediction of the state of the following model:

$$x_{i+1} = Fx_i + w_i, \tag{2.4a}$$

$$y_i = Cx_i + v_i, \tag{2.4b}$$

where x and y have dimension n and p respectively, and where

$$E\left\{\begin{bmatrix} w_i \\ v_i \end{bmatrix}\begin{bmatrix} w_j^{\mathrm{T}} & v_j^{\mathrm{T}} \end{bmatrix}\right\} = \begin{bmatrix} Q & 0 \\ 0 & R \end{bmatrix}\delta_{ij} \tag{2.5}$$

with Q nonnegative definite ($Q \geqslant 0$) and R positive definite ($R > 0$). By factoring R and Q has

$$R = (R^{1/2})(R^{1/2})^{\mathrm{T}} \quad \text{and} \quad Q = LL^{\mathrm{T}}, \tag{2.6}$$

and by defining $H = R^{-1/2}C$, we can rewrite the Riccati equation in a normalized form:

$$P_{i+1} = FP_iF^{\mathrm{T}} - FP_iH^{\mathrm{T}}(HP_iH^{\mathrm{T}} + I)^{-1}HP_iF^{\mathrm{T}} + LL^{\mathrm{T}}. \tag{2.7}$$

The closed-loop state transition matrix of the Kalman filter is

$$\bar{F}_i = F - FP_iH^{\mathrm{T}}(HP_iH^{\mathrm{T}} + I)^{-1}H = F - K_iH. \tag{2.8}$$

We shall assume throughout that C (and hence H) has full rank. Let us also note that, if $P_0 \geqslant 0$, then the solution P_i of (2.7) is nonnegative definite for all i. This follows easily by rewriting (2.7) as

$$P_{i+1} = (F - K_iH)P_i(F - K_iH)^{\mathrm{T}} + K_iK_i^{\mathrm{T}} + LL^{\mathrm{T}}.$$

This paper deals with some properties of the RDE (2.7) and its associated algebraic Riccati equation (ARE)

$$P = FPF^{\mathrm{T}} - FPH^{\mathrm{T}}(HPH^{\mathrm{T}} + I)^{-1}HPF^{\mathrm{T}} + LL^{\mathrm{T}}. \tag{2.9}$$

The main result is the derivation of sufficient conditions on P_0 that guarantee the stability of the closed loop at any time, i.e. $|\lambda_k(\bar{F}_i)| < 1$ for all $i \geqslant 0$ and for $k = 1, \ldots, n$. This main result will be established in Section 3. We first recall some basic results on the RDE and the ARE that will be needed in the sequel.

Definition 1 [8]. A real symmetric nonnegative definite solution P^+ of the ARE (2.9) is called a *strong solution* if the corresponding filter state-transition matrix

$$\bar{F} \triangleq F - FP^+H^{\mathrm{T}}(HP^+H^{\mathrm{T}} + I)^{-1}H$$

has all its eigenvalues inside or on the unit circle. It is called stabilizing if $\bar{F}$ has all its eigenvalues inside the unit circle.

Proposition 1 [8,12]. *If $[H, F]$ is detectable, then*

(1) *the strong solution of the ARE exists and is unique*;

(2) *if $[F, L]$ is stabilizable, the strong solution is the only non-negative definite solution of the ARE*;

(3) *if $[F, L]$ has no unreachable modes on the unit circle, the strong solution is stabilizing*;

(4) *if $[F, L]$ has an unreachable mode on the unit circle, there is no stabilizing solution*;

(5) *if $[F, L]$ has an unreachable mode on the unit circle, the strong solution is not positive definite.*

Remark. The results of [8] have all been established for nonsingular F, but these have been extended to Riccati equations with singular F in [12]. Some of these results, with no restriction on F, can also be found in [7].

Proposition 2 [8,12]. *If either*

(1) $[F, L]$ *has no unreachable modes on the unit circle*; $[H, F]$ *is detectable*; $P_0 > 0$;

or (2) $[F, L]$ *is stabilizable*; $[H, F]$ *is detectable*; $P_0 \geqslant 0$;

then $\lim_{i \to \infty} P_i = P^+$, *where* $\{P_i\}$ *is the solution of the RDE with initial condition* P_0, *and* P^+ *is the unique stabilizing solution of the ARE.*

We should stress at this juncture some of the similarities and differences between the RDE and the ARE, since shortly we shall be making a somewhat unorthodox description of the solutions to the RDE as solutions of distinct ARE's. The RDE is a recurrence relation for the evolution of a time-varying sequence of matrices $\{P_i\}$. Should this equation possess a fixed point P then clearly the P satisfies the ARE (2.3). By the same token any $n \times n$ matrix $\bar{P}$ defines an associated matrix $\bar{Q}$ according to

$$\bar{Q} = \bar{P} - F\bar{P}F^{\mathrm{T}} + F\bar{P}H^{\mathrm{T}}(H\bar{P}H^{\mathrm{T}} + I)^{-1}H\bar{P}F^{\mathrm{T}} \tag{2.10}$$

which, should $\bar{Q}$ be nonnegative definite, expresses $\bar{P}$ as the solution of an ARE. This connection will be exploited more fully to develop the stability conditions.

3. A stability result

We now turn to the stability of the filter transition matrix $\bar{F}_i$ of (2.8). Our result will require a monotonic behaviour of the Riccati equation matrices $\{P_i\}$. It is well known (see e.g. [6]) that if $P_0 = 0$, then the sequence P_i increases monotonically, in the sense that $P_{i+1} \geqslant P_i$ ($P_{i+1} \geqslant P_i$ meaning that $P_{i+1} - P_i$ is nonnegative definite). However, this monotonic behaviour breaks down if P_0 is an arbitrary nonnegative definite matrix: this will be illustrated by an example later.

We recall first a device due to Nishimura [9]. Consider two RDE's (2.7) with the same F and H matrices but possibly different $Q = LL^{\mathrm{T}}$ matrices, $\hat{Q}$ and $\bar{Q}$, and possibly different initial conditions. Let the solutions to these RDE's be written

$$\hat{P}_{i+1} = f(\hat{P}_i, \hat{Q}), \qquad \bar{P}_{i+1} = f(\bar{P}_i, \bar{Q}),$$

where the functional form of f is identical and is given by (2.7) with LL^{T} replaced by Q. Then

$$\begin{aligned}\hat{P}_{i+1} - \bar{P}_{i+1} = F&\left[I - \bar{P}_i H^{\mathrm{T}}(H\bar{P}_i H^{\mathrm{T}} + I)^{-1}\right] \\ &\cdot\left\{(\hat{P}_i - \bar{P}_i) - (\hat{P}_i - \bar{P}_i)H^{\mathrm{T}}\right. \\ &\left.\cdot(H\hat{P}_i H^{\mathrm{T}} + I)^{-1}H(\hat{P}_i - \bar{P}_i)\right\} \\ &\cdot\left[I - \bar{P}_i H^{\mathrm{T}}(H\bar{P}_i H^{\mathrm{T}} + I)^{-1}\right]^{\mathrm{T}} F^{\mathrm{T}} \\ &+ \hat{Q} - Q.\end{aligned} \tag{3.1}$$

Now define the terms between the brackets { } in (3.1) to be $g(\hat{P}_i - \bar{P}_i)$, and assume first that $\hat{P}_i - \bar{P}_i$ is positive definite. Then it can be shown (see [9]) that

$$\begin{aligned}g(\hat{P}_i - \bar{P}_i) = (\hat{P}_i - \bar{P}_i)H^{\mathrm{T}}&\left[H(\hat{P}_i - \bar{P}_i)H^{\mathrm{T}}\right. \\ &+ H(\hat{P}_i - \bar{P}_i)H^{\mathrm{T}}(H\bar{P}_i H^{\mathrm{T}} + I)^{-1} \\ &\left.\cdot H(\hat{P}_i - \bar{P}_i)H^{\mathrm{T}}\right]^{-1} H(\hat{P}_i - P_i) \\ &+ \{I - (\hat{P}_i - \bar{P}_i)H^{\mathrm{T}}[H(\hat{P}_i - \bar{P}_i)H^{\mathrm{T}}\} \\ &\cdot(\hat{P}_i - \bar{P}_i)\{I - (\hat{P}_i - \bar{P}_i)H^{\mathrm{T}} \\ &\cdot[H(\hat{P}_i - \bar{P}_i)H^{\mathrm{T}}]^{-1}H]^{\mathrm{T}}.\end{aligned} \tag{3.2}$$

The inverses exist by our standing full rank assumption on H and the positive definiteness of $\hat{P}_i - \bar{P}_i$. Therefore, if $\hat{P}_i > \bar{P}_i$ and $\hat{Q} \geqslant Q$, then $\hat{P}_{i+1} \geqslant \bar{P}_{i+1}$, since by (3.1) and (3.2) $\hat{P}_{i+1} - \bar{P}_{i+1}$ is the sum of 3 nonnegative definite matrices. Suppose now that $\hat{P}_i - \bar{P}_i$ is nonnegative definite and consider $g(\hat{P}_i - \bar{P}_i + \varepsilon I)$ with $\varepsilon > 0$. Using (3.2) again, with $\hat{P}_i - \bar{P}_i$ replaced by $\hat{P}_i - \bar{P}_i + \varepsilon I$, it follows that $g(\hat{P}_i - \bar{P}_i + \varepsilon I)$ is nonnegative definite for all $\varepsilon > 0$. And since $g(\cdot)$ is a continuous function of ε, this remains true when $\varepsilon \to 0$. Therefore $\hat{P}_{i+1} \geqslant \bar{P}_{i+1}$ if $\hat{P}_i \geqslant \bar{P}_i$ and $\hat{Q} \geqslant Q$. We have from this:

Lemma 1. *Let* P^+ *be any nonnegative definite solution of the ARE* (2.9) *and suppose that* $P_0 \geqslant P^+$. *Then* $\{P_i\colon i = 1, 2, \dots\}$ *generated by the RDE* (2.7) *satisfies* $P_i \geqslant P^+$.

Proof. Consider $P^+ = f(P^+, Q)$ and $P_{i+1} = f(P_i, Q)$ in (3.1). A simple induction argument yields the result. □

Lemma 2. *Consider the RDE* (2.7) *and suppose that for some* i, $P_i \geqslant P_{i+1} \geqslant 0$. *Then* $P_{i+k} \leqslant P_{i+k-1}$ *for all* $k \geqslant 1$.

Proof. Again this follows from (3.1) by considering $\hat{P}_i = P_i$, $\bar{P}_i = P_{i+1}$ and $\hat{Q} = \bar{Q} = LL^T$, and using induction. □

As remarked earlier, the proofs of our major theorems depend on the monotonicity properties of the solution sequences of the RDE and on the stabilizing properties of solutions of certain ARE's. The key device is to use the RDE (2.7) to define a symmetric matrix Q_i and an ARE associated with each P_i by

$$Q_i = P_i - FP_iF^T + FP_iH^T(HP_iH^T + I)^{-1}HP_iF^T. \tag{3.3}$$

Comparing the RDE (2.7) to the ARE (3.3) we have

$$Q_i = LL^T + P_i - P_{i+1}. \tag{3.4}$$

This yields:

Lemma 3. *If $\{P_i\}$ is a monotonic nonincreasing sequence of nonnegative definite matrices satisfying the RDE* (2.7) *then $\{Q_i\}$ is a sequence, not necessarily monotonic, of nonnegative definite matrices defined by* (3.3) *satisfying*

$$Q_i \geqslant LL^T. \tag{3.5}$$

This lemma shows that monotonicity of $\{P_i\}$ satisfying the RDE implies that each P_i is a nonnegative definite solution of an ARE (3.3) which is well formed, i.e. $Q_i \geqslant 0$. The stabilizing properties will follow from this.

Before proceeding to the main results we shall require the following technical result which is easily proved.

Lemma 4. *Suppose $[F, L]$ is a stabilizable pair and that matrix S satisfies $SS^T \geqslant LL^T$. Then $[F, S]$ is a stabilizable pair.*

We are now in a position to derive the following theorem.

Theorem 1. *Consider the RDE* (2.7) *with initial condition P_0 and solution sequence $\{P_i\}$. Define the sequence of matrices $\{Q_i\}$ by* (3.3). *If*

(1) *$[H, F]$ is detectable,*
(2) *$[F, L]$ is stabilizable,*
(3) *$P_0 \geqslant 0$ is such that $Q_0 \geqslant LL^T$,*

then the solution sequence $\{P_i\}$ of the RDE is stabilizing for all $i \geqslant 0$, i.e. $|\lambda_k(\bar{F}_i)| < 1$ for all $i \geqslant 0$ and for $k = 1, 2, \ldots, n$ with $\bar{F}_i$ defined by (2.8) *and $\lambda_k(\cdot)$ denoting the individual eigenvalues.*

Proof. First note that the set of P_0 satisfying (3) is not empty: take any $Q_0 \geqslant LL^T$ and let P_0 be the corresponding strong solution of the ARE (2.9), with LL^T replaced by Q_0. By Proposition 1, P_0 exists and is unique.

From Proposition 2 we know that P_i converges to P^+, the strong (and in this case stabilizing) solution of the ARE (2.9). Further (2.9) shows that $P_1 = P_0 + LL^T - Q_0$ and hence $P_1 \leqslant P_0$. Therefore, from Lemma 2, $P_{i+1} \leqslant P_i$ for all $i \geqslant 0$. Lemma 3 establishes that each P_i is a nonnegative definite solution of an ARE (3.3) with $Q_i \geqslant LL^T$. Lemma 4 and Proposition 1 show that, by condition (2) of the theorem statement, P_i is stabilizing. □

An examination of the conditions of Theorem 1 is in order before extending the results. The first requirement of detectability is crucial to the well-posedness of any state estimation problem — the admission of unstable, unobservable modes negates all possible worth of state estimates. The third condition is the necessary requirement for monotonicity of $\{P_i\}$. It is well known from Willems [3] and others that $Q_0 \geqslant LL^T$ implies $P_0 \geqslant P^+$. This is easily extracted from the proof of Theorem 1 since $\{P_i\}$ is monotonic nonincreasing and convergent to P^+, implying $P_i \geqslant P^+$ for all $i \geqslant 0$. It is in regard to the second condition that improvement is desirable. Condition (2) states that the underlying problem generates a stabilizing limiting value P^+. As we argued earlier, we envisage these stability results on P_i to be of use in instances where numerical problems arise due to the $[F, L]$ pair not necessarily being stabilize. Indeed, the motivation for our particular analysis is from Fourier analysis where ostensibly $L = 0$ and F has eigenvalues equally spaced around the unit circle. In generalizing Theorem 1 our aim is to replace the stabilizability condition on $[F, L]$ by a condition on $[F, Q_0^{1/2}]$.

Our approach to proving Theorem 2 will be to present the theorem, then to derive some intermediate results before knitting the threads of the proof together.

Theorem 2. *Consider the RDE* (2.7). *Define Q_0 as* *in* (3.3) *and assume that*

(1) $[H, F]$ *is detectable*,

(2) $[F, Q_0^{1/2}]$ *is stabilizable*,

(3) $P_0 \geqslant 0$ *and* $Q_0 \geqslant LL^T$.

Then the solution sequence $\{P_i\}$ *with initial condition* P_0 *is stabilizing for each* i.

In the light of the previous proofs, our broad aim will be to show that $[F, Q_i^{1/2}]$ is a stabilizable pair for each i. We remark here that the nonstrict inequalities of condition (3) of Theorem 2 and the possible singularity of F obstruct a simple proof.

To study the stabilizability of $[F, Q_i^{1/2}]$ one needs to focus attention on the eigenvectors of F^T. For simplicity, we consider only simple eigenvectors and omit the extension to generalized eigenvectors.

We have two ancillary results:

Lemma 5. *Let* x *be an eigenvector of* F^T *with eigenvalue* λ. *Consider the RDE* (2.7) *and suppose* $\{P_i\}$ *is nonnegative definite. Then*

(1) $x^*P_{i+1}x = 0$, *and*

(2) $\lambda \neq 0$

imply $x^*P_ix = 0$, $x^*LL^Tx = 0$ *and* $x^*Q_ix = 0$ (*where* x^* *denotes the conjugate transpose of* x).

Proof. Write $P_i = V_i^TV_i$ for some matrix V_i. The RDE becomes

$$\begin{aligned}P_{i+1} &= F\Big[V_i^TV_i - V_i^T(V_iH^T)\big[(V_iH^T)^T\\ &\qquad \cdot(V_iH^T)+I\big]^{-1}(V_iH^T)^TV_i\Big]F^T + LL^T\\ &= F\Big[V_i^TV_i - V_i^T\big\{I-(I\\ &\qquad + V_iH^THV_i^T)^{-1}\big\}V_i\Big]F^T + LL^T\end{aligned}$$

or

$$P_{i+1} = FV_i^T\left[I + V_iH^THV_i^T\right]^{-1}V_iF^T + LL^T. \quad (3.6)$$

Multiply on the left by x^* and on the right by x:

$$\begin{aligned}x^*P_{i+1}x = |\lambda|^2 x^*V_i^T\left[I + V_iH^THV_i^T\right]^{-1}V_ix\\ + x^*LL^Tx.\end{aligned}$$

The right-hand side above is zero for $\lambda \neq 0$ if and only if $V_ix = 0$ and $L^Tx = 0$ since both terms on this side are nonnegative. That $x^*Q_ix = 0$ follows from (3.4). □

This lemma effectively states that a drop in rank from P_i to P_{i+1} can occur only if $[F, L]$ has an unreachable eigenvalue at zero. It is unlikely that this would affect stability as we shall show later.

Lemma 6. *Suppose* $\{P_i\}$ *is nonnegative definite and monotonic nonincreasing and suppose that* $HP_ix = 0$ *for a given vector* x. *Then* $HP_{i+k}x = 0$ *for all* $k \geqslant 0$.

Proof. Since P_i is nonnegative definite write

$$P_i = V_{i,1}^TV_{i,1} + V_{i,2}^TV_{i,1}$$

for $V_{i,1}^T$ in the range space of H^T, (H^T), and $V_{i,2}^T$ in the null space of H, (H). Similarly, write $x = x_1 + x_2$. Then $HP_ix = 0$ implies $V_{i,1}x_1 = 0$. The property that $P_{i+1} \leqslant P_i$ implies that $V_{i+1,1}^TV_{i+1,1} \leqslant V_{i,1}^TV_{i,1}$ and the result follows. □

Proof of Theorem 2. The existence of P_0 satisfying (3) has been established in the proof of Theorem 1. Now, the ARE (3.3) may be written as a Lyapunov equation

$$(F - K_iH)P_i(F - K_iH)^T - P_i = -Q_i - K_iK_i^T \quad (3.7)$$

where

$$K_i = FP_iH^T\left(HP_iH^T + I\right)^{-1}.$$

Let x be an eigenvector of $F^T - H^TK_i^T$ with eigenvalue λ. Then (3.7) yields

$$|\lambda|^2 x^*P_ix - x^*P_ix = -x^*Q_ix - x^*K_iK_i^Tx. \quad (3.8)$$

We now consider two cases.

Case 1. $X^*P_ix \neq 0$. In this case, equation (3.8) implies $|\lambda| \leqslant 1$, since $Q_i \geqslant 0$ and $K_iK_i^T \geqslant 0$. We show by contradiction that $|\lambda| \neq 1$. For suppose $|\lambda| = 1$; then

$$x^*Q_ix = 0 = x^*K_iK_i^Tx$$

and hence x is an eigenvector of F^T, also with eigenvalue λ. However, the fact that $K_i^Tx = 0$ implies $HP_ix = 0$. By Lemma 6, this implies that $HP_{i+k}x = 0$ for all $k \geqslant 0$. From assumption (3) of the theorem it follows that $P_1 \leqslant P_0$. Hence, using Lemma 2, we conclude that $P_i \geqslant P_{i+1}$. This, together with the fact that $x^*Q_ix = 0$ implies that $x^*LL^Tx = 0$. (See (3.4).) We now use equation

(2.7) to conclude that $x^*P_ix = x^*P_{i+k}x$ for all $k \geqslant 0$. From this and from $HP_{i+k}x = 0$ for all $x \geqslant 0$, it follows that $x^*P_ix = x^*P^+x$ and $HP^+x = 0$ since P_i converges to P^+ by Proposition 2. Now consider the ARE (2.9) with P replaced by P^+. Multiplying this equation to the right hand by x and denoting $P^+x \triangleq z$ yields $Fz = \lambda^{-1}z$ with $|\lambda| = 1$. But we also have $Hz = 0$. This contradicts the detectability of $[H, F]$.

Case 2. $x^*P_ix = 0$, $\lambda \neq 0$. In this case, it again follows from (3.8) that

$$x^*Q_ix = 0 = x^*K_iK_i^{\mathrm{T}}x.$$

Hence, x is an eigenvector of F^{T}. Furthermore, we may apply Lemma 5 to show that $x^*P_{i-1}x = 0, \ldots, x^*P_0x = 0$. Hence from (3.7), $x^*Q_0x = 0$. Thus, by stabilizability of $[F, Q_0^{1/2}]$, we have $|\lambda| < 1$. □

4. Counterexamples and fallacious conjectures

In deriving the results of this paper, the authors explored many false trails while attempting to establish desirable results. The most informative of these blind alleys and their corresponding counterexamples are presented here since they serve best to indicate inappropriate procedures for establishing stability. They also perform a role in defining a regime of necessity for our sufficient conditions and emphasize the value of our results.

Fallacious Conjecture 1. If P_0 and P^+ are stabilizing and $P_0 > 0$ then P_i is stabilizing.

Counterexample. Let

$$H^{\mathrm{T}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad F = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix},$$

$$Q = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}, \qquad P_0 = \begin{bmatrix} 0.3 & 0.4 \\ 0.1 & 1.2 \end{bmatrix} > 0.$$

Then

$$F - K_0H = \begin{bmatrix} 1 & -0.182 \\ 1 & 0.273 \end{bmatrix}$$

is exponentially asymptotically stable,

$$P_1 = \begin{bmatrix} 2.2273 & -0.5909 \\ -0.5909 & 2.1364 \end{bmatrix}$$

and

$$F - K_1H = \begin{bmatrix} 1 & 0.1884 \\ 1 & 0.5072 \end{bmatrix}$$

is unstable. Further

$$\lim_{i \to \infty} P_i = P^+ = \begin{bmatrix} 6.6347 & 4.1201 \\ 4.1201 & 7.4877 \end{bmatrix}$$

with

$$F - K^+H = \begin{bmatrix} 1 & -0.4854 \\ 1 & -0.3676 \end{bmatrix},$$

which is exponentially asymptotically stable again.

This counterexample clearly does not generate monotonic nonincreasing $\{P_i\}$. Indeed, $P_0 < P^+$ and we next consider possible conditions to ensure monotonicity.

Fallacious Conjecture 2. If $P_0 > P^+$ then $\{P_i\}$ is monotonic nonincreasing.

Counterexample. Let

$$H^{\mathrm{T}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then

$$P^+ \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 + \sqrt{3} \end{bmatrix}.$$

Taking

$$P_0 = \begin{bmatrix} 100 & 0 \\ 0 & 10 \end{bmatrix} > P^+,$$

it is easy to check that

$$P_1 = \begin{bmatrix} 1.9 & 0 \\ 0 & 101 \end{bmatrix} \not< P_0.$$

Statements about failure of (nondecreasing) monotonicity have also been given by Caines and Mayne [6], who state also that $\{P_i\}$ need not even be cyclomonotonic, i.e. $P_{i+N} \leqslant P_i$ for all i and some fixed N. Examples of this nonmonotonicity can be constructed by considering

$$F = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix},$$

for a θ which is an irrational multiple of π, and a variety of P_0's. It turns out that the crucial condition for monotonicity is $Q_0 \geqslant LL^{\mathrm{T}}$ which implies but is not implied by $P_0 \geqslant P^+$. This can be seen from:

Fallacious Conjecture 3. Let $P_0 > P^+$. Then $Q_0 > Q$.

Counterexample. Let

$$H^{\mathrm{T}} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad F = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix}, \quad Q = 0.$$

Then

$$\bar{P} = \begin{bmatrix} 15 & 0 \\ 0 & \frac{15}{4} \end{bmatrix}.$$

Take

$$P_0 = \begin{bmatrix} 16 & 0 \\ 0 & 16 \end{bmatrix}.$$

Then

$$Q_0 = \begin{bmatrix} -48 & 0 \\ 0 & 12.2 \end{bmatrix}$$

which is not greater than Q.

This collection of conjectures and counterexamples serves to demonstrate some of the difficulties of dealing with the RDE just as the paper's valid results illustrate the utility of some methods concerning the solutions. We have not as yet found a counterexample to the following.

Possibly Fallacious Conjecture. Suppose $P_0 > P^+$ and P_0 and P^+ are both stabilizing. Then $\{P_i\}$ is stabilizing.

5. Conclusion

Despite the enormous amount of published literature on the Riccati equation, it still maintains a few mysteries and its behaviour is sometimes counterintuitive. In this paper, we have unveiled part of this remaining mystery. Our result provides a fairly easy procedure to initialize a Riccati equation in such a way that the RDE can be stopped at any time, while guaranteeing the exponential stability of the corresponding Kalman filter. While our conditions are only sufficient, it has proved remarkably difficult to narrow them further or to obtain necessary and sufficient conditions.

We have concentrated on the discrete-time filtering formulation for our study because, firstly, our motivation and rationale come from this area and, secondly, the results are extensible to continuous-time and optimal control most easily. Further extensions to these results (at least initially when F and P^+ are invertible) may be possible by generalizing the Lyapunov stability work of Deyst and Price [10]. This has been the approach of Parker [11] from which the authors first became aware of the connection between monotonicity and stability.

Acknowledgement

The authors are grateful to Philip J. Parker for access to his unpublished research notes on the problem, to Graham C. Goodwin and his coauthors for early access to [8] and [12], and to Marie-Antoinette Poubelle for her technical assistance.

References

[1] B.D.O. Anderson and J.B. Moore, *Optimal Filtering* (Prentice-Hall, Englewood Cliffs, NJ, 1979).
[2] R.R. Bitmead, A.-C. Tsoi and P.J. Parker, A Kalman-filtering approach to short-time Fourier analysis, submitted for publication.
[3] J.C. Willems, Least squares stationary optimal control and the algebraic Riccati equation, *IEEE Trans. Automat. Control* **16** (1971) 621–634.
[4] V. Kucera, The discrete Riccati equation of optimal control, *Kybernetika* **8** (1972) 430–447.
[5] K. Martensson, On the matrix Riccati equation, *Inform. Sci.* **3** (1971) 17–49.
[6] P.E. Caines and D.Q. Mayne, On the discrete-time matrix Riccati equation of optimal control, *Internat. J. Control* **12** (1970) 785–794.
[7] H.J. Payne and L.M. Silverman, On the discrete-time algebraic Riccati equation, *IEEE Trans. Automat. Control* **18** (1973) 226–234.
[8] S.W. Chan, G.C. Goodwin and K.S. Sin, Convergence properties of the Riccati difference equation in optimal filtering of nonstabilizable systems, *IEEE Trans. Automat. Control* **29** (1984) 110–118.
[9] T. Nishimura, On the a priori information in sequential estimation problems, *IEEE Trans. Automat. Control* **11** (1966) 147–204. See also **12** (1976) 123.
[10] J.J. Deyst Jr. and C.F. Price, Conditions for asymptotic stability of the discrete minimum-variance linear estimator, *IEEE Trans. Automat. Control* **13** (1968) 178–181. Correction **18** (1973) 562–563.
[11] P.J. Parker, Private communication (1983).
[12] C.E. de Souza and G.C. Goodwin, Riccati equation in optimal filtering of nonstabilizable systems having singular state transition matrices, Dept. Electrical Engineering, Universidade Federal de Uberlandia, 38400 Uberlandia, MG, Brazil, Tech. Report 84/02 (February 1984).