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Simultaneous Stabilizability of Three Linear Systems Is Rationally Undecidable*

V. Blondel† and M. Gevers‡

Abstract. We show that the simultaneous stabilizability of three linear systems, that is the question of knowing whether three linear systems are simultaneously stabilizable, is rationally undecidable. By this we mean that it is not possible to find necessary and sufficient conditions for simultaneous stabilization of the three systems in terms of expressions involving the coefficients of the three systems and combinations of arithmetical operations (additions, subtractions, multiplications, and divisions), logical operations ("and" and "or"), and sign test operations (equal to, greater than, greater than or equal to, ...).

Key words. Simultaneous stabilization, Decidability, Rationally decidable question.

1. Introduction

When is it possible to find a single rational controller that simultaneously stabilizes three, or more, linear systems? At present there is no comprehensive answer to this question, and this paper is devoted to it.[1]

We restrict our attention to single-input single-output linear, time-invariant systems that are rational but not necessarily causal, so $p_i(s) \in \mathbb{R}(s)$, $i = 1, \ldots, k$, and we allow ourselves the use of a linear, time-invariant rational controller. Our goal is to achieve closed-loop internal stability with the controller. That is, we require that the four closed-loop transfer functions, $p_i(s)c(s)(1 + p_i(s)c(s))^{-1}$, $p_i(s)(1 + p_i(s)c(s))^{-1}$, $c(s)(1 + p_i(s)c(s))^{-1}$, and $(1 + p_i(s)c(s))^{-1}$, associated with each of the k systems have no poles in the extended right half-plane. A controller that satisfies this condition is said to be a *simultaneously stabilizing controller* for $\{p_i(s)\}$.

* Date received: June 1, 1992. Date revised: November 20, 1992.

† Now at Optimization and Systems Theory, Royal Institute of Technology, S-10044 Stockholm, Sweden.

‡ Université Catholique de Louvain, Cesame, Bâtiment Euler, B-1348 Louvain-La-Neuve, Belgium.

[1] Consider for instance the three linear systems

$$p_1(s) = 0, \qquad p_2(s) = \frac{2}{17}\frac{s-1}{s+1}, \quad \text{and} \quad p_3(s) = \frac{(s-1)^2}{(9s-8)(s+1)}.$$

At present, it is not known whether these three systems are simultaneously stabilizable or not. A bottle of good French champagne is offered by the authors to the first person who either gives the expression of a stabilizing controller or proves that no such controller exists.

The search for conditions on the $p_i(s)$ for the existence of a stabilizing controller was first addressed in [SM] and [VV].

The simultaneous stabilization question for two systems can be reformulated into one of strong stabilization—stabilization by a stable controller—for a single system [V]. The strong stabilization question was solved in 1974 by Youla *et al.* [YBL] and has an elegant solution: a system is stabilizable by a stable controller if and only if it has an even number of real unstable zeros between each pair of real unstable poles. Anderson [A] proved that this condition on real poles and zeros can be checked by performing only elementary arithmetic operations (additions, subtractions, multiplications, and divisions) on the coefficients of the system: the real poles and zeros do not have to be computed explicitly.

The situation is different for three systems. The simultaneous stabilization question for three (or more) systems is recognized as one of the difficult open problems in linear system theory, and it has attracted much attention during the last decade. The presently available results are in the form of necessary conditions [G1], [W], sufficient conditions [WB], [BCG], [K], or untractable necessary and sufficient conditions [G2], [BGMR]. Despite all these efforts, *at present no tractable necessary and sufficient conditions exist for testing simultaneous stabilizability for three (or more) systems.*

In the central result of this paper, Theorem 5, we show that unlike the case of two systems, the simultaneous stabilization question for three linear systems is rationally undecidable. In other words, it is not possible to find a general criterion involving only the coefficients of the three linear systems, arithmetical operations (additions, subtractions, multiplications, and divisions), logical operations ("and" and "or"), and sign test operations (equal to, greater than, greater than or equal to, ...), that is both necessary and sufficient for simultaneous stabilizability of the three systems.

We close this section with some notation used in what follows. Our notion of a rationally decidable question is presented in Section 2. Section 3 introduces a result on the range of analytic functions that is used in our main result. Finally, in Section 4, we show in Theorem 5 that the simultaneous stabilization question for three systems is not rationally decidable.

This paper contains some of the material in the first author's Ph.D. thesis [B3].

Notations

$\mathbb{R}$ is the set of real numbers and $\mathbb{Q}$ is the set of rational numbers. $\mathbb{R}[s]$ is the set of real polynomials in the variable s. $\mathbb{R}(s)$ is the set of real rational functions. $\mathbb{Q}(\beta)$ is the set of rational functions in the variable β and with coefficients in $\mathbb{Q}$. $\mathbb{C}_\infty$ is the extended complex plane $\mathbb{C} \cup \{\infty\}$ topologized with the Riemann sphere topology and $\mathbb{R}_\infty$ is the extended real line, $\mathbb{R} \cup \{\infty\}$. D is the open unit disk $\{s \in \mathbb{C}: |s| < 1\}$, $\bar{D}$ is the closed unit disk $\{s \in \mathbb{C}: |s| \leq 1\}$, and $\mathbb{C}_{+\infty} = \{s \in \mathbb{C}: \Re(s) \geq 0\} \cup \{\infty\}$ is the extended closed right half-plane. Assume that Ω is a subset of $\mathbb{C}_\infty$. A real rational function $f(s) \in \mathbb{R}(s)$ is Ω-*stable* if it has no poles in Ω.[2] $S(\Omega)$ is the set of all Ω-stable

[2] We note that this is simply a convention; other authors define Ω-stability in the opposite way.

rational functions. We use $U(\Omega)$ to denote the set of functions in $S(\Omega)$ whose inverse are in $S(\Omega)$ and we call such rational functions *Ω-bistable* rational functions. Finally, to shorten the notations, we denote $U = U(\mathbb{C}_{+\infty})$ and $S = S(\mathbb{C}_{+\infty})$.

2. Rational Decidability and Algebraic Numbers

This section is in three parts. We first give our definition of rational decidability, then that of algebraic and transcendental numbers, and finally we prove a result that links rational decidable questions and algebraic numbers.

2.1. Rational Decidability

The test for polynomial stability is a good example of what we mean by a rationally decidable question. A polynomial is called stable if and only if all its zeros have negative real part. For third-degree polynomials,

$$p(s) = a_1 + a_2 s + a_3 s^2 + a_4 s^3$$

(where $a_4 \neq 0$), the Routh–Hurwitz test states that $p(s)$ is stable if and only if the logical sentence

$$(((a_1 > 0) \wedge (a_2 > 0) \wedge (a_3 > 0) \wedge (a_4 > 0) \wedge (a_2 a_3 - a_1 a_4 > 0))$$
$$\vee ((a_1 < 0) \wedge (a_2 < 0) \wedge (a_3 < 0) \wedge (a_4 < 0) \wedge (a_2 a_3 - a_1 a_4 < 0)))$$

is true. Here the symbols $\wedge$ and $\vee$ stand for the logical operations "and" and "or," respectively.

For every third-degree polynomial specified by its four coefficients, the question of whether or not the polynomial is stable is a typical example of what we mean by a *rationally decidable* question: it can be answered by using a finite number of elementary operations where elementary operations are defined as follows.

Definition 1. An elementary operation is any one of:

1. The four arithmetical operations: addition, subtraction, multiplication, and division. These are commonly referred to as rational operations.
2. The two logical operations: "and" and "or."
3. The five test operations: $=$, $>$, $<$, $\geq$, and $\leq$.

We say that the polynomial stability question is rationally decidable because, for every n, stability of degree n polynomials is rationally decidable.

The general notion of rational decidability is a formalization of this idea.

Definition 2. A true/false question Q associated with an n-tuple $(a_1, \ldots, a_n) \in \mathbb{R}^n$ is rationally decidable if and only if a logical sentence L of finite length exists that involves only elementary operations on the entries a_i of the n-tuple and such that L is true if and only if Q is.

With the obvious abuse of terminology, a finite or countable family of true/false questions is called a rationally decidable question just in case all members of the family are rationally dedicable questions as defined above. Thus, the following questions are rationally decidable: the stability of a polynomial, the positive definiteness of a matrix, the coprimeness of two polynomials, and the simultaneous stabilizability of two linear systems. On the other hand, we show in Section 4 that simultaneous stabilizability of three linear systems is *not* rationally decidable.

2.2. Algebraic Numbers

Algebraic numbers are numbers that are roots of polynomials whose coefficients are integers; see, for example, [B1].

Definition 3. A real number is *algebraic* if and only it is the root of a polynomial that has integer (or rational) coefficients. A real number that is not algebraic is called *transcendental.*

For example, $-1, \sqrt{2}, i = \sqrt{-1}$, and $\sqrt{\sqrt{7} + 31/(\sqrt{13} - 5)}$ are algebraic numbers whereas π, e, and $\Gamma(\frac{1}{4})$ are transcendental. It is in general not true that the ratio of two transcendental numbers is a transcendental number.

For our analysis of simultaneous stabilization we need the following nontrivial result. The proof of this theorem is independent of the rest of the paper.

Theorem 1. *The real number* $4\pi^2/\Gamma^4(\frac{1}{4})$ *is transcendental.*

Proof. Our proof is based on a result found on p. 158 of [B1]. This result states: "The transcendence degree of the field L generated by $\omega_1 = \Gamma^2(\frac{1}{4})/\sqrt{8\pi}$, $\omega_2 = i\omega_1$, $\eta_1 = \pi/\omega_1$, and $\eta_2 = -i\eta_1$ over the rationals $\mathbb{Q}$ is at least 2." Since $\omega_1^2 + \omega_2^2 = 0$ and $\eta_1^2 + \eta_2^2 = 0$ this means that ω_1 and η_1 are transcendental and algebraically independent. However, then $\eta_1/2\omega_1^3 = 4\pi^2/\Gamma^4(\frac{1}{4})$ is transcendental and so the theorem is proved. ∎

2.3. Rational Decidability and Algebraic Numbers

In this section we establish a link between rational decidability and algebraic numbers. We first illustrate our point with the example of polynomial stability and then generalize the concept in an abstract setting.

Assume that $\beta \in \mathbb{R}$. By the previous section we know that the polynomial

$$p(s) = 1 + \beta s + \beta s^2 + 2s^3$$

is stable if and only if the logical sentence

$$\begin{aligned}&(((1 > 0) \wedge (\beta > 0) \wedge (\beta > 0) \wedge (2 > 0) \wedge (\beta^2 - 2 > 0))\\ &\vee ((1 < 0) \wedge (\beta < 0) \wedge (\beta < 0) \wedge (2 < 0) \wedge (\beta^2 - 2 < 0)))\end{aligned}$$

is true. Some trivial simplifications show that this logical sentence is true if and only if

$$\beta \in (\sqrt{2}, \infty).$$

In this formulation the stability condition is expressed by means of an open interval $(\sqrt{2}, \infty)$ whose endpoints are the point at infinity and the algebraic number $\sqrt{2}$. A similar feature remains true in the abstract general case. Recall that $\mathbb{Q}(\beta)$ denotes the set of rational functions of β with coefficients in $\mathbb{Q}$.

Theorem 2. *If $Q(a_1, \ldots, a_n)$ is a rationally decidable binary question associated with an n-tuple $(a_1, \ldots, a_n)$ and if all the entries a_i of the n-tuple are in $\mathbb{Q}(\beta)$ $(a_i(\beta) \in \mathbb{Q}(\beta))$, then values $\bar{\sigma}_{k,j}$ and $\underline{\sigma}_{k,j}$ $(k = 1, 2$ and $j = 1, \ldots, m_k)$ exist that are either equal to $\pm\infty$ or to algebraic numbers, such that*

$$Q(a_1(\beta), \ldots, a_n(\beta)) \text{ is true} \quad \Leftrightarrow \quad \beta \in \left(\bigcup_{j=1}^{m_1} (\underline{\sigma}_{1,j}, \bar{\sigma}_{1,j}]\right) \cup \left(\bigcup_{j=1}^{m_2} [\underline{\sigma}_{2,j}, \bar{\sigma}_{2,j})\right).$$

Proof. Since the question $Q(a_1, \ldots, a_n)$ is rationally decidable, a logical sentence $L(a_1, \ldots, a_n)$ of finite length exists that involves only elementary operations on the entries a_i of the n-tuple and such that $L(a_1, \ldots, a_n)$ is true if and only if $Q(a_1, \ldots, a_n)$ is. Thus,

$$\forall \beta \in \mathbb{R}, \qquad Q(a_1(\beta), \ldots, a_n(\beta)) \text{ is true} \quad \Leftrightarrow \quad L(a_1(\beta), \ldots, a_n(\beta)) \text{ is true.}$$

It remains to show that values $\bar{\sigma}_{k,j}$ and $\bar{\sigma}_{k,j}$ $(k = 1, 2$ and $j = 1, \ldots, m_k)$ exist that are either equal to $\pm\infty$ or to algebraic numbers, such that,

$$\forall \beta \in \mathbb{R}, \qquad L(a_1(\beta), \ldots, a_n(\beta)) \text{ is true} \quad \Leftrightarrow$$
$$\beta \in \left(\bigcup_{j=1}^{m_1} (\underline{\sigma}_{1,j}, \bar{\sigma}_{1,j}]\right) \cup \left(\bigcup_{j=1}^{m_2} [\underline{\sigma}_{2,j}, \bar{\sigma}_{2,j})\right).$$

To prove this we proceed by induction on the size of the logical sentence $L(a_1, \ldots, a_n)$.

The logical sentence $L(a_1, \ldots, a_n)$ is either made up of two smaller logical sentences $L_1(a_1, \ldots, a_n)$ and $L_2(a_1, \ldots, a_n)$ linked by an "and" or an "or" logical operation ($L(a_1, \ldots, a_n) = L_1(a_1, \ldots, a_n) \wedge L_2(a_1, \ldots, a_n)$ or $L(a_1, \ldots, a_n) = L_1(a_1, \ldots, a_n) \vee L_2(a_1, \ldots, a_n)$) or is a nucleus expression of the form $L(a_1, \ldots, a_n) = R_1(a_1, \ldots, a_n) \diamondsuit R_2(a_1, \ldots, a_n)$, where $R_1(a_1, \ldots, a_n)$ and $R_2(a_1, \ldots, a_n)$ are rational expressions of the coefficients $a_1, \ldots, a_n$ ($R_i(a_1, \ldots, a_n) \in \mathbb{Q}(a_1, \ldots, a_n)$ for $i = 1, 2$) and $\diamondsuit$ is any one of the five sign test operations $<, \leq, >, \geq, =$.

We analyze these two cases successively.

First, if $L(a_1, \ldots, a_n)$ is a nucleus expression, then $L(a_1(\beta), \ldots, a_n(\beta))$ is true if and only if

$$R_1(a_1(\beta), \ldots, a_n(\beta)) \diamondsuit R_2(a_1(\beta), \ldots, a_n(\beta))$$

for some $\diamondsuit \in \{<, \leq, >, \geq, =\}$. By hypothesis $a_i(\beta)$ are rational expressions of β ($a_i(\beta) \in \mathbb{Q}(\beta)$ for $i = 1, 2, \ldots, n$) and $R_j(a_1, \ldots, a_n)$ are rational expressions of $a_1, \ldots, a_n$ ($R_j(a_1, \ldots, a_n) \in \mathbb{Q}(a_1, a_2, \ldots, a_n)$ for $j = 1, 2$). Hence, $R'_j(\beta) \triangleq R_j(a_1(\beta), \ldots, a_n(\beta))$ are also rational expressions of β. The condition $R_1(a_1(\beta), \ldots, a_n(\beta)) \diamondsuit R_2(a_1(\beta), \ldots, a_n(\beta))$ is satisfied if and only if $R'_1(\beta) \diamondsuit R'_2(\beta)$

is, and this last condition is equivalent to

$$\beta \in \left(\bigcup_{j=1}^{m_1} (\underline{\sigma}_{1,j}, \overline{\sigma}_{1,j}]\right) \cup \left(\bigcup_{j=1}^{m_2} [\underline{\sigma}_{2,j}, \overline{\sigma}_{2,j})\right)$$

for some $\overline{\sigma}_{k,j}$ and $\underline{\sigma}_{k,j}$ ($k = 1, 2$ and $j = 1, \ldots, m_k$) that are equal to $\pm\infty$ or to algebraic numbers. Thus the theorem is proved in the case of a nucleus expression.

Secondly, suppose that $L(a_1, \ldots, a_n)$ is made up of two logical sentences $L_1(a_1, \ldots, a_n)$ and $L_2(a_1, \ldots, a_n)$ linked by an "and" or an "or" logical operation. By induction hypothesis assume that the value $\overline{\sigma}^1_{k,j}$ and $\underline{\sigma}^1_{k,j}$ ($k = 1, 2$ and $j = 1, \ldots, m^1_k$) and $\overline{\sigma}^2_{k,j}$ and $\underline{\sigma}^2_{k,j}$ ($k = 1, 2$ and $j = 1, \ldots, m^2_k$) are equal to $\pm\infty$ or to algebraic numbers and are such that

$$L_1(a_1(\beta), \ldots, a_n(\beta)) \text{ is true} \quad \Leftrightarrow \quad \beta \in \left(\bigcup_{j=1}^{m_1} (\underline{\sigma}^1_{1,j}, \overline{\sigma}^1_{1,j}]\right) \cup \left(\bigcup_{j=1}^{m_1} [\underline{\sigma}^1_{2,j}, \overline{\sigma}^1_{2,j})\right)$$

and

$$L_2(a_1(\beta), \ldots, a_n(\beta)) \text{ is true} \quad \Leftrightarrow \quad \beta \in \left(\bigcup_{j=1}^{m_1} (\underline{\sigma}^2_{1,j}, \overline{\sigma}^2_{1,j}]\right) \cup \left(\bigcup_{j=1}^{m_2} [\underline{\sigma}^2_{2,j}, \overline{\sigma}^2_{2,j})\right).$$

Then, if $L(a_1, \ldots, a_n) = L_1(a_1, \ldots, a_n) \wedge L_2(a_1, \ldots, a_n)$ we have

$$\begin{aligned} L(a_1(\beta), \ldots, a_n(\beta)) \text{ is true} \quad \Leftrightarrow \quad \beta \in &\left(\left(\bigcup_{j=1}^{m_1} (\underline{\sigma}^1_{1,j}, \overline{\sigma}^1_{1,j}]\right) \cup \left(\bigcup_{j=1}^{m_2} [\underline{\sigma}^1_{2,j}, \overline{\sigma}^1_{2,j})\right)\right) \\ &\cap \left(\left(\bigcup_{j=1}^{m_1} (\underline{\sigma}^2_{1,j}, \overline{\sigma}^2_{1,j}]\right) \cup \left(\bigcup_{j=1}^{m_2} [\underline{\sigma}^2_{2,j}, \overline{\sigma}^2_{2,j})\right)\right), \end{aligned}$$

whereas, if $L(a_1, \ldots, a_n) = L_1(a_1, \ldots, a_n) \vee L_2(a_1, \ldots, a_n)$ we have

$$\begin{aligned} L(a_1(\beta), \ldots, a_n(\beta)) \text{ is true} \quad \Leftrightarrow \quad \beta \in &\left(\left(\bigcup_{j=1}^{m_1} (\underline{\sigma}^1_{1,j}, \overline{\sigma}^1_{1,j}]\right) \cup \left(\bigcup_{j=1}^{m_2} [\underline{\sigma}^1_{2,j}, \overline{\sigma}^1_{2,j})\right)\right) \\ &\cup \left(\left(\bigcup_{j=1}^{m_1} (\underline{\sigma}^2_{1,j}, \overline{\sigma}^2_{1,j}]\right) \cup \left(\bigcup_{j=1}^{m_2} [\underline{\sigma}^2_{2,j}, \overline{\sigma}^2_{2,j})\right)\right). \end{aligned}$$

It is trivial to see that in both cases we can rewrite the unions and intersections involved under the form

$$\left(\bigcup_{j=1}^{m_1} (\underline{\sigma}_{1,j}, \overline{\sigma}_{1,j}]\right) \cup \left(\bigcup_{j=1}^{m_2} [\underline{\sigma}_{2,j}, \overline{\sigma}_{2,j})\right)$$

for some $\overline{\sigma}_{k,j}$ and $\underline{\sigma}_{k,j}$ ($k = 1, 2$ and $j = 1, \ldots, m_k$) equal to $\pm\infty$ or to algebraic numbers. Thus, by induction on the size of L, the theorem is proved. ■

3. Analytic Functions

The results that we need are contained in two books on analytic functions, [N] and [G3]. We pick out a result from each of these sources and then merge them into a single formulation that is more suitable for our subsequent treatment. In all what follows we define $A \triangleq 4\pi^2/\Gamma^4(\frac{1}{4}) = 0.228\ldots$

Theorem 3 [G3, p. 89]. *Suppose that the function $F(z) = z^q + a_{q+1}z^{q+1} + a_{q+2}z^{q+2} + \cdots$, for $q \geq 1$, is regular (=analytic) in the disk $|z| < 1$. Then the image of that disk under the mapping $\xi = F(z)$ completely covers some segment of arbitrary predetermined slope that contains the point $\xi = 0$ and is of length no less than $2A$. The number A cannot be increased without additional restrictions on $F(z)$.*

The proof of this theorem is not contained in the book itself but in a Russian journal [B2] referenced in [G3].

That the bound A is the best achievable can be seen from a result contained in [N]. The function $f_e(z)$, denoted by $f(z)$ and introduced at the bottom of p. 330 in [N], is connected to the so-called elliptic modular function and is defined by the convergent infinite product

$$f_e(z) \triangleq \frac{4\pi^2}{\Gamma^4(\frac{1}{4})}\left(32 \exp\left(-\pi\frac{1+z}{1-z}\right)\prod_{n=1}^{\infty}\left(\frac{1+\exp(-2n\pi((1+z)/(1-z)))}{1-\exp(-(2n-1)\pi((1+z)/(1-z)))}\right)-1\right).$$

It is shown in [N] that $f_e(z)$ is an analytic function on $D = \{z \in \mathbb{C}: |z| < 1\}$ such that $f_e(\bar{z}) = \overline{f_e(z)}$, $f_e(0) = 0$, $f_e'(0) = 1$, and that $f_e(z)$ does not take the values $\pm A$ on D.

For further purposes we transform the formulation of Theorem 3 slightly by making use of the properties of the function $f_e(z)$.

Theorem 4. *Assume that $\beta \in \mathbb{R}$. An analytic function on D exists such that $f(\bar{z}) = \overline{f(z)}$, $f(0) = 0$, $f'(0) = 1$, and that omits the values $\pm\beta$ if and only if $|\beta| \geq A$.*

Proof. We first prove sufficiency. Let $f_e(z)$ be the function defined above, assume that $\beta \geq A$ and define

$$f(z) \triangleq \frac{\beta}{A} f_e\left(\frac{A}{\beta} z\right).$$

Due to the properties of $f_e(z)$ it is easy to check that $f(z)$ satisfies the conditions of the theorem. For necessity, assume by contradiction that $f(z)$ satisfies the conditions of the theorem and that $0 < \beta < A$. By assumption, the image of the disk D under the mapping $\xi = f(z)$ contains neither the value β nor the value $-\beta$. Thus, the image does not cover any segment of the real line that contains the origin and is of length $2A$. This contradicts Theorem 3, hence the result. ∎

This theorem is the crucial result that is needed for proving Theorem 5 below.

4. Simultaneous Stabilization of Three Systems: A Rationally Undecidable Question

From our definition of stabilization it is easy to see that a controller stabilizes the null plant if and only if it is stable. Therefore the three rational systems 0, p_1, and p_2 are simultaneously stabilizable if and only if the two systems p_1 and p_2 are simultaneously stabilizable by a stable controller. In the next theorem we prove that simultaneous stabilizability for two systems by a stable controller is rationally undecidable. By the short discussion above, this proves the simultaneous stabilizability of three systems to be a rationally undecidable question.

Theorem 5. *The simultaneous stabilizability of two systems by a stable controller is rationally undecidable.*

Proof. Discrete- and continuous-time stability regions are mapped into one another by the usual bilinear transformation. We present out proof in a discrete-time set up, in which a plant is called stable if it has no poles in the closed unit disk $\bar{D}$.

Assume that $\beta \in \mathbb{R}$ and consider the two systems $p_{1,\beta}(z) = z^2/(z^2 - \beta)$ and $p_{2,\beta}(z) = z^2/(z^2 + \beta)$. We proceed in two steps.

First, we show that when $\beta = 0$ or $|\beta| > A$ the two systems are simultaneously stabilizable by a stable controller, whereas when $0 < |\beta| < A$ they are *not* simultaneously stabilizable by a stable controller. Note that we omit the analysis of the case $\beta = A$.

Second, we show that the first step contradicts the fact that the simultaneous stabilizability question of two systems by a stable controller is a rationally decidable question.

Step 1. If $\beta = 0$, then $p_{1,\beta}(z) = p_{2,\beta}(z) = z$ and a stable stabilizing controller is given, for example, by $c(z) = 2$. If $\beta \neq 0$, then the two systems are simultaneously stabilizable by a stable controller if and only if a rational function $c(z)$ that has no poles in $\bar{D}$ exists such that

$$z^2 c(z) + z - \beta$$

and

$$z^2 c(z) + z + \beta$$

have no zeros in $\bar{D}$.

It remains to show that when $\beta < A$ such a function $c(z)$ does not exist whereas it does exist when $\beta > A$. We prove these two points in parts (a) and (b), respectively.

(a) Assume, by contradiction, that $\beta < A$, that $c(z)$ has no poles in $\bar{D}$ and that

$$z^2 c(z) + z - \beta$$

and

$$z^2 c(z) + z + \beta$$

have no zeros in $\bar{D}$. Then the function defined by

$$f(z) \triangleq z^2 c(z) + z$$

satisfies all the conditions of Theorem 4 and omits the values $\pm\beta$ with $\beta < A$. A contradiction is achieved and this part is proved.

(b) Assume that $\beta > A$. We construct a rational function $c(z)$ that satisfies all the requested conditions.

By Theorem 4, an analytic function $f(z)$ exists on D such that $f(0) = 0$, $f'(0) = 1$ and that does not take the values $\pm A$ on D. We define the function $g(z)$ by

$$g(z) \triangleq \frac{\beta}{A} f\left(\frac{A}{\beta} z\right).$$

Due to the properties of $f(z)$, the function $g(z)$ is such that

(1) $g(\bar{z}) = \overline{g(z)}$,
(2) $g(z)$ is analytic on $|z| < \beta/A$ (and $1 < \beta/A$),
(3) $g(0) = 0$ and $g'(0) = 1$, and
(4) $g(z)$ omits the values $\pm A$ on $|z| < \beta/A$.

With the help of this function $g(z)$, we construct a real *polynomial* $p(z) \in \mathbb{R}[z]$ such that $p(0) = 0$, $p'(0) = 1$, and the real number μ defined by

$$\mu \triangleq \min\left\{\inf_{z \in \bar{D}} |g(z) - A|,\ \inf_{z \in \bar{D}} |g(z) + A|\right\}$$

is strictly positive. Because of the first three points, the function $h(z)$ defined by

$$h(z) \triangleq \frac{g(z) - z}{z^2}$$

is real and analytic in $\{z : |z| < \beta/A\}$. By Runge's theorem (see [R]), a real polynomial $q(z)$ exists such that

$$|h(z) - q(z)| < \mu\left(\frac{A}{\beta}\right)^2, \qquad z \in \bar{D}.$$

This polynomial is then also such that

$$|g(z) - z - z^2 q(z)| < \mu, \qquad z \in \bar{D}.$$

Defining the polynomial $p(z) \triangleq z + z^2 q(z) \in \mathbb{R}[z]$, we have $p(0) = 0$ and $p'(0) = 1$, However, because

$$|g(z) - p(z)| < \mu, \qquad z \in \bar{D},$$

and

$$\mu \le \min\left\{\inf_{z \in \bar{D}} |g(z) - A|,\ \inf_{z \in \bar{D}} |g(z) + A|\right\},$$

it also follows that

$$|g(z) - p(z)| < |g(z) \pm A|, \qquad z \in \bar{D}.$$

Hence,

$$p(z) \neq \pm A, \qquad z \in \bar{D},$$

as requested.

A polynomial is a rational function with no poles of modulus less than or equal to one and, thus, point (b) is proved.

Step 2. Assume, for the purpose of obtaining a contradiction, that the simultaneous stabilizability of two systems by a stable controller is a rationally decidable question. Then so is the simultaneous stabilizability of the two systems $p_{1,\beta}(z) = z^2/(z^2 - \beta)$ and $p_{2,\beta}(z) = z^2/(z^2 + \beta)$ by a stable controller. However, then, using Theorem 2, values $\bar{\sigma}_{k,j}$ and $\underline{\sigma}_{k,j}$ ($k = 1, 2$ and $j = 1, \ldots, m_k$) exist that are either equal to $\pm\infty$ or that are algebraic numbers, such that our three systems are simultaneously stabilizable if and only if

$$\beta \in \left(\bigcup_{j=1}^{m_1} (\underline{\sigma}_{1,j}, \bar{\sigma}_{1,j}]\right) \cup \left(\bigcup_{j=1}^{m_2} [\underline{\sigma}_{2,j}, \bar{\sigma}_{2,j})\right).$$

This contradicts our first step since we know from there that the two systems are simultaneously stabilizable by a stable controller if and only if

$$\beta \in \left(-\infty, -\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}\right) \cup [0, 0] \cup \left(\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}, +\infty\right)$$

or if and only if

$$\beta \in \left(-\infty, -\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}\right] \cup [0, 0] \cup \left[\frac{4\pi^2}{\Gamma^4(\frac{1}{4})}, +\infty\right).$$

By Theorem 1, $4\pi^2/\Gamma^4(\frac{1}{4})$ is a transcendental number, a contradiction is achieved and the theorem is proved. ■

5. Conclusion

M. Vidyasagar, upon reading our paper, summarized its contribution in the following simple and elegant way.

> Assume $\beta > 0$. The two discrete-time plants $z^2/(z - \beta)$ and $z^2/(z + \beta)$ can be simultaneously stabilized by a stable controller if $\beta > A$ and cannot be so stabilized if $\beta < A$, where $A = 4\pi^2/\Gamma^4(\frac{1}{4})$.
>
> Now A is a transcendental number. If the upper limit for β could be decided using any combination of rational operations, then the limit so derived will be an algebraic number. Therefore there is no combination of rational operations that can determine the limit value of β. Consequently, given three plants, there is no set of rational operations that can be used to determine whether or not they are simultaneously stabilizable.

We believe that our result, so aptly summarized, closes much of the research on the simultaneous stabilization problem. Indeed, we have shown that no criterion exists for simultaneous stabilizability that involves only elementary operations on the coefficients. In particular, it is not possible to find a criterion that involves only, say, solving systems of linear equations, solving a Nevanlinna-type interpolation problem, or evaluating a Cauchy index, because all these operations are conducted by performing elementary operations only.

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Finally, the reviewers all made very interesting comments on a first version of this paper.

The first version of this paper was dedicated to all 1992 twins. Since then a number of 1992 twins were born and among them are Lolita and Merlin, and Adrien and Nicolas.

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Mathematics of Control, Signals, and Systems

Linear Stabilization of Nonlinear Cascade Systems*

A. Bacciotti,† P. Boieri,† and L. Mazzi†

Abstract. In this paper we consider the cascade connection of a nonlinear system and a system of integrators. Under suitable conditions we prove that if the nonlinear subsystem is stabilizable by means of a linear feedback, then a linear stabilizer exists for the overall system as well. In particular, we point out the role of the classical notion of k-asymptotic stability.

Key words. Nonlinear control, Cascade systems, Linear feedback, Stabilization, k-Asymptotic stability.

1. Introduction

This work is a contribution to the study of the stabilizability problem for finite-dimensional autonomous nonlinear control processes, briefly referred to as the *nonlinear stabilization problem* throughout this paper. This is a natural extension of the analogous problem for linear control processes, which played a crucial role in the development of linear system theory. At the same time, it exhibits deep connections with the classical literature on Liapunov stability for critical equilibria of ordinary differential equations. Finally, it is of interest for applications. For these and other reasons, the nonlinear stabilization problem has recently been considered by many authors. For a survey on the subject see [B1] and the references therein.

In order to make the paper self-contained we briefly recall certain general features of the problem. Let the plant to be stabilized be modeled by a system of ordinary differential equations

$$\dot{x} = f(x, u), \tag{1.1}$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, and $f\colon \mathbb{R}^{n+m} \to \mathbb{R}^n$ is a C^∞ map such that $f(0, 0) = 0$. Let $\mathscr{F}$ be a given class of functions $u\colon \mathbb{R}^n \to \mathbb{R}^m$ defined in a neighborhood of the origin and such that $u(0) = 0$. We say that $u \in \mathscr{F}$ is a *stabilizing feedback* for (1.1) if the *closed-loop system*

$$\dot{x} = g(x) = f(x, u(x)) \tag{1.2}$$

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† Dipartimento di Matematica, Politecnico di Torino, Corso Duca degli Abruzzi 24, 10129 Torino, Italy.