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INT. J. CONTROL, 1976, VOL. 24, NO. 3, 345–360

Combined identification of the input–output and noise dynamics of a closed-loop controlled linear system

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The open-loop input–output dynamics and the noise dynamics of a feedback controlled linear system perturbed by coloured noise admitting a Markov representation are identified in state variable form using a two-stage algorithm. This system is equivalent to an augmented system driven by white noise.

First the input–output dynamics are identified through a stochastic approximation algorithm using superimposed white noise. Subtracting the model output from the system output yields correlated residuals which are then used to identify the noise dynamics using stochastic realization theory. An innovations representation is obtained that is equivalent to the above defined augmented system. The two stages are combined by a judicious coordinate transformation. The method can be applied on an operating feedback controlled process, regardless of the structure of the unknown suboptimal regulator.

1. Introduction and outline of results

It is desired to identify and optimally regulate a linear discrete-time single-input single-output system of unknown parameters, with the following assumptions :

(1) the system is stationary and a finite state representation is assumed to exist ;

(2) it is perturbed by a coloured Gauss–Markov process noise of unknown dynamics ;

(3) during the identification phase the system is kept under closed-loop suboptimal control (e.g. the system could be a manually controlled industrial process).

We shall discuss the motivation for these assumptions and show that they have some nice implications in terms of practical applications. We shall also point out how these assumptions lead to some theoretical problems for which no solution was available in the existing identification literature. In order to clarify this discussion, let us state the equations first.

Following the various assumptions the system can be represented by the following model :

$$x(i+1) = Fx(i) + gu(i) + Dp(i) \qquad (1\,a)$$

$$p(i+1) = \psi p(i) + \gamma w(i) \qquad (1\,b)$$

$$y(i) = hx(i) + v(i) \qquad (1\,c)$$

where x is an n-vector, p and w are m-vectors, u, y and v are scalars. w and v are zero-mean white Gaussian processes with covariances 1 and r respectively ; for simplicity w and v will be assumed uncorrelated although the correlated

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Received 29 October 1975.

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case can be treated just as well. By the closed-loop assumption, the known (i.e. measurable) input u is an unknown function, possibly time-varying and non-linear, of past and present outputs, i.e. $u(i)=f[y(i), y(i-1), \ldots]$. The parameters of the matrices F, g, D, ψ, γ and h, as well as the covariance r, are unknown, but it is assumed that the system is minimal and stable, i.e. all eigenvalues of F and ψ lie inside the unit circle.

The assumption of a coloured process noise is not only quite realistic from a practical point of view (e.g. the external temperature acting on a glass-furnace process is clearly not white), but, from the regulator point of view, it has some nice conceptual interest as well. Indeed, it is intuitively clear that if the noise $p(\cdot)$ acting on the system is correlated and if the dynamics of this correlation (i.e. the ψ matrix) can be identified, then the noise process $p(\cdot)$ and is effect on the state can be predicted, which in turn should improve the performance of the regulator. In a sense the regulator should act as a predictor with respect to the noise process. This is exactly what happens, as is made apparent by the fact that in our framework the optimal regulator will have the form

$$u(i) = -C^{\mathrm{T}}\hat{X}(i|i) \tag{2}$$

where $\hat{X}(i|i)$ is the filtered estimated of the augmented state

$$X = \begin{bmatrix} x \\ \cdots \\ p \end{bmatrix} \tag{3}$$

and the gain C is a function of the estimated parameters of the system and noise process.

As is well know the computation of the filtered estimates $\hat{x}(i|i)$ and $\hat{p}(i|i)$, and of the optimal regulator gain C, requires the identification of both the open-loop system and noise dynamics. The assumption of coloured process noise complicates this identification problem in that the transfer function of the input–output dynamics is of a lower dimension than the transfer function of the dynamics of the noise process (i.e. that part of $y(\cdot)$ that is due to $w(\cdot)$ and $v(\cdot)$).

The third assumption, namely that the system is kept under closed-loop suboptimal control is even more appealing from a practical point of view: in many industrial applications one is not allowed to ' open the loop ' during the identification phase. However, this assumption results in a quite difficult identification problem for which, to our knowledge, no satisfactory solution has yet been offered. The main difficulties resulting from this closed-loop assumption can be summarized as follows:

(1) the input–output dynamics (i.e. F, g, h) cannot be correctly identified from input–output data since the feedback loop, whose dynamics are unknown, alters the dynamics of the input–output relation;

(2) on the other hand, the noise dynamics cannot be identified by output-correlation techniques (Mehra 1971, 1972) because the feedback control is correlated with the process noise.

One of the first authors to recognize the difficulties associated with the identification of feedback systems was Akaike (1967). He proposed a method

based on a ' causal chain model '. The idea is as follows: a new vector process $x(i)$ is defined, made up of the subprocesses $u(i)$ and $y(i)$:

$$x(i) = \begin{bmatrix} u(i) \\ y(i) \end{bmatrix}$$

For the components of $x(i)$ instantaneous causality is assumed to exist from $x^k(i)$ to $x^j(i)$ only if $k>j$. A transfer-function model $H(z)$ is then identified for $x(i)$ considered as a stochastic process driven by orthogonal internal noise sources, such that

$$S_x(z) = H(z)S_e H'(z^{-1})$$

where $S_x(z)$ is the spectral density function of z, and S_e is the variance of an orthogonal white noise $e(i)$. To obtain a unique model despite the presence of feedback Akaike imposed certain structural constraints on $H(z)$ (i.e. the causal chain structure) and on S_e (i.e. the orthogonality).

Expanding upon Akaike's idea, several authors (Akaike 1967, Caines and Wall 1972, Wellstead 1974, Phadke 1973, Chan 1974) have recently derived various identification methods under a variety of assumptions leading to different ' canonical ' representations. Caines and Wall (1972) assume that no instantaneous causality exists, but relax the condition of orthogonality on the internal white noise process $e(i)$. Wellstead (1974) shows that the estimates of the forward path transfer function are unique and consistent when orthogonal internal noise sources are present in both the feedforward and feedback loop, and when there is at least one lag in the loop.

Phadke (1973) has studied various canonical forms, and recommends a non-causal triangular two-sided moving average (TTSMA) model for $x(\cdot)$, from which the open-loop model can be derived. Chan (1974) shows that feedback systems with ' instantaneous causality ' lead to non-unique models; he obtains a ' unique ' model by imposing $H(0)=I$, while relaxing the condition that $e(i)$ is orthogonal.

As the preceding discussion shows, different sets of constraints on $H(z)$ and S_e have been proposed that lead to various ' unique ' or ' canonical ' models. Unfortunately, as Chan's study clearly shows, different models for $x(\cdot)$ will lead to different open-loop transfer functions, unless the structure of the model is known (e.g. it is known that the feedback transfer function is of order 2 with a unit delay). This is a severe drawback for all indirect identification methods.

Gustavsson *et al.* (1974) and Söderström *et al.* (1974), on the other hand, have derived various assumptions under which direct prediction error methods (such as the maximum likelihood method) can be used, treating the input u and the output y just as if the system were operating in open loop. They have shown that the system is thereby identifiable without knowledge of the true system structure provided the chosen model structure is able to reproduce the exact system transfer function and noise dynamics, and one of the following conditions hold:

(*a*) a measurable perturbation s is added to the input process u, that is both independent of w and v and persistently exciting of high order;

(*b*) a persistently exciting noise source is present in the feedback loop that is independent of w and v;

(*c*) the regulator is non-linear and non-degenerate;

(*d*) the regulator is persistently time-varying†.

In all these cases direct open-loop prediction error methods can be used. However, prediction error methods, such as maximum likelihood, usually require *a priori* parametrization of the model and a good initial guess for the parameter estimate. Therefore, even for open-loop systems, covariance factorization methods or impulse response models are often used as a first step.

One way of making the choice of the structure less critical is to add a measurable white perturbation input, whereby the open-loop input–output dynamics can be identified through a stochastic approximation method using an impulse response model, which is not as structure dependent. Saridis and Lobbia (1972) have proposed such a method; however, they did not identify the noise dynamics that are necessary for the state-estimation problem and, *a fortiori*, did not attempt to tackle the problem of a coloured process noise.

In this paper we proposed a two-stage method that separately identifies the input–output dynamics and the noise dynamics of the system. The matrices F, g, h of the input–output relation are identified first through a stochastic approximation algorithm using an external white noise input (Saridis and Lobbia 1972). Using the estimated $\hat{F}$, $\hat{g}$ and $\hat{h}$, the effect of the known input is subtracted from the system output and, using stochastic realization theory, a model is obtained for the resulting process. It is shown that this model converges to the model of the noise dynamics. A coordinate transformation is necessary to combine the two stages of the identification procedure.

Section 2 describes the main ideas of the identification method without any proofs. The technical details and the algorithmic forms of the three stages of the identification method are fully described in §§ 3, 4 and 5, respectively. Section 3 describes the stochastic approximation algorithm used to identify the input–output dynamics, § 4 deals with the identification of the noise dynamics, and § 5 describes the coordinate transformation necessary to combine these first two parts. The method has been tested on a simulated system; the results of this simulation are briefly described in § 6.

2. The method

By state-augmentation the system equations (1) can easily be put into the form

$$X(i+1)=\phi X(i)+Gu(i)+\Gamma w(i) \tag{4 a}$$

$$y(i)=HX(i)+v(i) \tag{4 b}$$

† The reader is referred to Gustavsson *et al.* (1974) for precise definitions of non-degenerate and persistently time-varying.

where

$$X=\begin{bmatrix} x \\ \hdashline p \end{bmatrix},\quad \phi=\begin{bmatrix} F & D \\ 0 & \psi \end{bmatrix},\quad G=\begin{bmatrix} g \\ \hdashline 0 \end{bmatrix},\quad \Gamma=\begin{bmatrix} 0 \\ \hdashline \gamma \end{bmatrix} \tag{5 a}$$

$$H=[h\,\vdots\,0] \tag{5 b}$$

X is an l-vector, with $l=n+m$, and ϕ, G, Γ, H have the appropriate dimensions. It is assumed that H, ϕ is an observable pair.

The output $y(\cdot)$ in (4 *b*) is the sum of two effects, a contribution $y_1(\cdot)$ due to the deterministic input $u(\cdot)$ and a noise process $y_2(\cdot)$ due to the noise sources $w(\cdot)$ and $v(\cdot)$:

$$y(i)=y_1(i)+y_2(i) \tag{6}$$

The state $X(\cdot)$ can be split similarly:

$$X(i)=X_1(i)+X_2(i) \tag{7}$$

This allows one to write two separate state-representations for $y_1(\cdot)$ and $y_2(\cdot)$. The equations of the input–output model are

$$X_1(i+1)=\phi X_1(i)+Gu(i) \tag{8 a}$$

$$y_1(i)=HX_1(i) \tag{8 b}$$

or, equivalently,

$$x_1(i+1)=Fx_1(i)+gu(i) \tag{9 a}$$

$$y_1(i)=hx_1(i) \tag{9 b}$$

with x_1 an n-vector, since the last m terms of X_1, are zero, i.e.

$$X_1 \triangleq \begin{bmatrix} x_1 \\ \hdashline 0 \end{bmatrix} \tag{10}$$

The equations of the noise process $y_2(\cdot)$ are

$$X_2(i+1)=\phi X_2(i)+\Gamma w(i) \tag{11 a}$$

$$y_2(i)=HX_2(i)+v(i) \tag{11 b}$$

By the innovations theory (Kailath 1968, 1970), the state model (11) with two white noise sources can be replaced by an equivalent innovations representation with a single noise source:

$$\theta(i+1)=\phi\theta(i)+K\epsilon(i) \tag{12 a}$$

$$y_2(i)=H\theta(i)+\epsilon(i) \tag{12 b}$$

where

$$\epsilon(i)=y_2(i)-\hat{y}_2(i|i-1) \tag{13 a}$$

the innovations.

We call σ_ϵ^2 the variance of the innovations:

$$\sigma_\epsilon^2=E\{\epsilon^2(i)\} \tag{13 b}$$

The method we propose separately identifies the model (9) for the input–output dynamics and the model (12) for the noise dynamics. These two parts are then combined to form the following global innovations model :

$$X(i+1)=\phi X(i)+Gu(i)+K\epsilon(i) \tag{14 a}$$

$$y(i)=HX(i)+\epsilon(i) \tag{14 b}$$

that is equivalent to (4).

Let us recall in passing the often-made remark that one of the main advantages of the innovations representation is that its predicted state estimate is the state itself, i.e.

$$\hat{X}(i|i-1)=X(i) \tag{15}$$

The filtered estimate is equally simple

$$\hat{X}(i|i)=X(i)+K\epsilon(i)=[I-KH]X(i)+Ky(i) \tag{16}$$

Hence, if the system is identified in the form (14), the state estimate that is required for the optimal regulator is immediately available, without an additional Kalman filter.

The two-stage method for the identification of the global system in the form (14) goes as follows :

(1) A first algorithm identifies the first $2n$ terms of the impulse response $h(1), \dots, h(2n)$ of the input–output system (9) through a stochastic approximation scheme proposed by Saridis and Lobbia (1972). In order to avoid identifying the regulator as part of the input–output relation, this algorithm requires the addition of a white noise perturbation $s(\cdot)$ to the control input $u(\cdot)$. The matrices h, F, g in observability canonical form (i.e. the form in which the observability matrix is the identity) are algebraically related to $h(1), \dots, h(2n)$. A recursive form of these relations is used to compute $\hat{h}$, $\hat{F}$, $\hat{g}$ on line. Asymptotic convergence of $\hat{h}$, $\hat{F}$, $\hat{g}$ to the parameters h, F, g of the open-loop system has been proved.

(2) A second algorithm identifies the matrices H, ϕ and K of the noise model (12), but in observability canonical form : H^*, ϕ^*, K^*. First an estimate y_M of the effect of the control input $s+u$ on the output is subtracted from the output y. This estimate y_M is obtained by passing the input $u+s$ through a model with parameters $\hat{h}$, $\hat{F}$, $\hat{g}$ as obtained from the first algorithm (see Fig. 1). The residuals are called z :

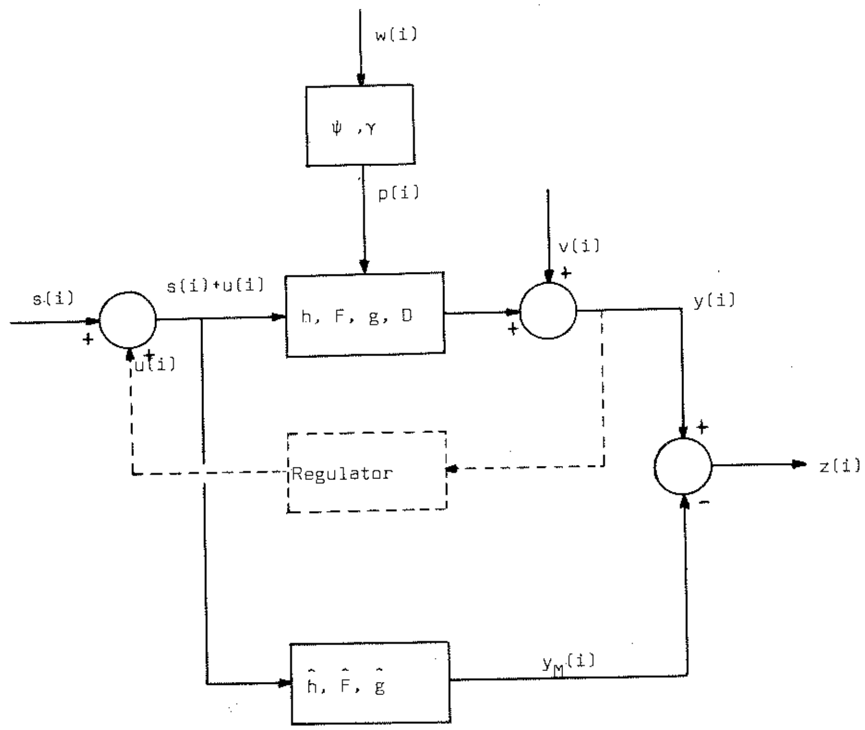
$$z(i)\triangleq y(i)-y_M(i) \tag{17}$$

These residuals contain a sum of three effects due respectively to the process noise $w(\cdot)$, the measurement noise $v(\cdot)$, and a modelling error as long as the first identification is not completed. The autocorrelations $c(k)\triangleq E\{z(i)\, z(i-k)\}$ are estimated and, using stochastic realization theory, Gevers and Kailath (1973), an innovations representation is obtained for $z(i)$, in observability canonical form :

$$\theta(i+1)=\hat{\phi}^*\theta(i)+\hat{K}^*\epsilon(i) \tag{18 a}$$

$$z(i)=H^*\theta(i)+\epsilon(i) \tag{18 b}$$

The unknown parameters in $\hat{\phi}^*$ and $\hat{K}^*$ can be estimated recursively.



When the first identification converges, the modelling error component in $z(i)$ vanishes, and it is easy to see that $z(i)$ converges to $y_2(i)$. It can be shown, therefore, that the matrices $\hat{\phi}^*$ and $\hat{K}^*$ converge asymptotically to the matrices ϕ^* and K^* (in observability canonical form) of the noise dynamics representation (12).

(3) In order to complete the identification it is necessary to transform the matrices H^*, ϕ^* and K^* into the matrices H, ϕ and K which appear in the forms (14) and (5), i.e. in a coordinate space that is compatible with the canonical form in which h, F and g were identified. This can be achieved through a judiciously chosen coordinate transformation that exploits the relationship that exists between the matrices F and ϕ^*.

3. First stage : Identification of the input–output dynamics

The procedure for the identification of the open-loop input–output model (9) has been described in § 2.1. The identification proceeds in two steps.

3.1. *Estimation of the truncated impulse response*

For reasons that have been stated above, a white gaussian noise perturbation s is added to the input u of the system. The following $2n$-vectors are defined :

$$V^{\mathrm{T}}\triangleq[h(1)\ \dots\ h(2n)] \tag{19}$$

$$S^{\mathrm{T}}(i)\triangleq[s(i-1)\ \dots\ s(i-2n)] \tag{20}$$

$$U^{\mathrm{T}}(i)\triangleq[u(i-1)\ \dots\ u(i-2n)] \tag{21}$$

where $h(1), \ldots, h(2n)$ are the first $2n$ terms of the input impulse response of the nth-order system (1).

The vector V is estimated recursively by the following stochastic approximation algorithm :

$$\hat{V}_{k+1} = \hat{V}_k + \mu_k[y(i) - \hat{V}_k{}^{\mathrm{T}}(U(i)+S(i))]S(i) \tag{22}$$

with $\hat{V}_0$ arbitrary but finite.

Proposition

If a $2n$-vector $\hat{V}_k$ is updated through (22) every $2m$ units of time with $m \geqslant n$ and if

$$\mu_k > 0, \quad \sum_{k=0}^{\infty} \mu_k = \infty, \quad \sum_{k=0}^{\infty} \mu_k{}^2 < \infty \tag{23}$$

then

$$\lim_{k\to\infty} E\{\| \hat{V}_k - V \|^2\} = 0 \tag{24}$$

i.e. the vector $\hat{V}_k$ converges asymptotically, in the mean square sense, to the vector V of the first $2n$ components of the impulse response of the system (one possible choice for μ_k is $\mu_k = a^2/(b^2+k)$, and a and b arbitrary).

Comments

It is worth noting that the convergence of this algorithm does not require any particular structure for the unknown regulator. The control input u can be any linear or non-linear, constant or time-varying function of past outputs.

3.2. *State representation*

From the estimates of $h(1), \ldots, h(2n)$ a state representation for the open-loop input–output dynamics can be obtained in observability canonical form :

$$x(i+1) = \left[\begin{array}{c|c} \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & I_{n-1} \\ \hline \multicolumn{2}{c}{-f^{\mathrm{T}}} \end{array}\right] x(i) + gu(i) \tag{25 a}$$

$$y(i) = [1 \; 0 \ldots 0]x(i) = hx(i) \tag{25 b}$$

where

$$f^{\mathrm{T}} = [f_n \; f_{n-1} \cdots f_1] \tag{26}$$

By the Cayley–Hamilton theorem and by virtue of the canonical form chosen for h and F, it is easy to establish the following algebraic relationships :

$$g^{\mathrm{T}} = [h(1) \ldots h(n)] \tag{27}$$

$$f = -\mathscr{H}^{-1}[h(n+1) \ldots h(2n)]^{\mathrm{T}} \tag{28}$$

where

$$\mathscr{H} = \begin{bmatrix} h(1) & \ldots & h(n) \\ & & \\ h(n) & \ldots & h(2n-1) \end{bmatrix} \tag{29}$$

$\mathscr{H}$ will be non-singular if the order of the system is at least n. The estimates of g and f can be computed recursively from the estimates of V. $\hat{g}_k$ is given by the first n terms of $\hat{V}_k$:

$$\hat{g}_k = [\hat{h}_k(1) \ldots \hat{h}_k(n)] \tag{30}$$

To obtain a recursive expression for f we define $\Delta\mathscr{H}_k$ as follows :

$$\mathscr{H}_{k+1} = \mathscr{H}_k + \Delta\mathscr{H}_k \tag{31}$$

where $\mathscr{H}_k$ is defined by replacing the elements of $\mathscr{H}$ in (29) by their kth estimates. For large k, $\Delta\mathscr{H}_k$ will be small compared to $\mathscr{H}_k$; therefore, by a matrix inversion lemma,

$$(\mathscr{H}_{k+1})^{-1} \cong (\mathscr{H}_k)^{-1} - (\mathscr{H}_k)^{-1}\Delta\mathscr{H}_k(\mathscr{H}_k)^{-1} \tag{32}$$

Using the approximation (32), the following recursive relation can be derived for $\hat{f}_k$:

$$\hat{f}_{k+1} = \hat{f}_k - \mu_k(\mathscr{H}_{k+1})^{-1}[y(i) - \hat{V}_k{}^{\mathrm{T}}(U(i)+S(i))] \begin{bmatrix} s(i-n-1) \\ s(i-n-2) \\ \vdots \\ s(i-2n) \end{bmatrix} \tag{33}$$

Equations (22) and (30)–(33) constitute the first stage of the identification. They provide a recursive algorithm for estimating F and g. Notice that no matrix inversion is necessary because $(\mathscr{H}_{k+1})^{-1}$ is computed recursively by (32) and $\Delta\mathscr{H}_k$ is computed directly from the data.

4. Second stage : Identification of the noise dynamics

The procedure for the identification of the noise dynamics (12) has been described in § 2.2.

4.1. *Autocorrelation of the residuals*

The residuals $z(i) \triangleq y(i) - y_M(i)$ are formed, where $y_M(i)$ is the output of the model (9) with F and g replaced by $\hat{F}$ and $\hat{g}$ as identified in the first part.

By the stationarity and ergodicity of the process $z(\cdot)$, an asymptotically unbiased, normal and consistent estimate of $c(k)$ (Parzen 1961) is

$$\hat{c}(k) = \frac{1}{N} \sum_{i=k}^{N} z(i)z(i-k) \tag{34}$$

These estimates can be computed recursively by

$$\hat{c}_{i+1}(k) = \hat{c}_i(k) + \frac{1}{i+1}[z(i+1)z(i+1-k) - \hat{c}_i(k)] \tag{35}$$

4.2. *Transition to a state-representation*

It is well known that a state-representation for $z(\cdot)$ of the form (18) can be obtained from the autocorrelations $c(k)$ using stochastic realization theory (Gevers and Kailath 1973). We briefly recall without proof how to obtain H^*, ϕ^*, K^* and σ_ε^2 from the $c(k)$'s for the particular case of observability canonical form. The reader is referred to Gevers and Kailath (1973) for the derivation of those results.

Assume that the representation (11) for $z(i)$ (recall that $z(i) \equiv y_2(i)$ upon convergence of the first identification) has been transformed to observability canonical form, i.e.

$$H^* = [1\ 0 \ldots 0] \tag{36 a}$$

$$\Phi^* = \left[\begin{array}{c|c} \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} & I_{l-1} \\ \hline \multicolumn{2}{c}{-\varphi^{\mathrm{T}}} \end{array}\right], \quad \varphi^{\mathrm{T}} = [\varphi_l\ \varphi_{l-1} \cdots \varphi_1] \tag{36 b}$$

Then the autocorrelations $c(k)$ can be expressed in terms of the model parameters as follows :

$$c(0) = H^* N + r \tag{37 a}$$

$$c(k) = H^* \Phi^{*k} N, \quad k > 0 \tag{37 b}$$

where

$$N = \pi^* H^{*\mathrm{T}}, \quad \text{an } l\text{-vector} \tag{38}$$

and

$$\pi^* = E\{X_2 X_2^{\mathrm{T}}\} \tag{39}$$

The unknown quantities φ, N, r, K^* and σ_ε^2 are successively derived from $c(0), \ldots, c(2l)$ as follows.

(*a*) From (37 *b*) and the Cayley–Hamilton theorem it follows that

$$c(l+k) = -\sum_{j=1}^{l} \varphi_j c(l+k-j), \quad k > 0 \tag{40}$$

where $l = n + m$ is the order of the augmented system.

Writing (40) for $k = 1, 2, \ldots, l$, leads to a system of Yule–Walker equations that are linear in the coefficient $\varphi_1, \ldots, \varphi_l$. The solution is

$$\varphi = -\mathscr{C}^{-1}[c(l+1)\ c(l+2) \ldots c(2l)]^{\mathrm{T}} \tag{41}$$

where

$$\mathscr{C} = \begin{bmatrix} c(1) & \ldots & c(l) \\ \vdots & & \vdots \\ c(l) & \ldots & c(2l-1) \end{bmatrix} \tag{42}$$

An estimate of φ is obtained by replacing the $c(k)$'s in (41) and (42) by their estimates obtained from (35). Mehra (1971) has shown how φ can be computed recursively :

$$\hat{\varphi}^{i+1} = \hat{\varphi}^i - \frac{(\mathscr{C}^{i+1})^{-1}}{i+1} \begin{bmatrix} z(i+1)z(i-l) - \hat{c}_i(l+1) \\ \vdots \\ z(i+1)z(i-2l+1) - \hat{c}_i(2l) \end{bmatrix} \tag{43}$$

$(\mathscr{C}^{i+1})^{-1}$ is obtained from $(\mathscr{C}^i)^{-1}$ by a matrix inversion lemma (cf. (32)) :

$$(\mathscr{C}^{i+1})^{-1} \cong (\mathscr{C}^i)^{-1} - (\mathscr{C}^i)^{-1} \Delta\mathscr{C}^i (\mathscr{C}^i)^{-1} \tag{44}$$

with the obvious definition for $\Delta\mathscr{C}^i$.

(*b*) From (37) and the Cayley–Hamilton theorem, it follows that

$$c(l) = -\sum_{j=1}^{l} \varphi_j c(l-j) + \varphi_l r$$

The covariance r of $v(\cdot)$ is therefore derived :

$$r = \frac{1}{\varphi_l} \sum_{j=0}^{l} \varphi_j c(l-j) \quad \text{with } \varphi_0 \triangleq 1 \tag{45}$$

Finally, from (37) and the canonical forms it follows that

$$\begin{bmatrix} H^* \\ H^*\Phi^* \\ \vdots \\ H^*\Phi^{*l-1} \end{bmatrix} N = N = \begin{bmatrix} c(0) - r \\ c(1) \\ \vdots \\ c(l-1) \end{bmatrix} \tag{46}$$

The estimates $\hat{r}^i$ and $\hat{N}^i$ are immediately derived by replacing the $c(k)$'s in (45) and (46) by their ith estimates $\hat{c}_i(k)$. The estimates of φ, r and N are asymptotically unbiased, normal and consistent.

(*c*) Only K^* and σ_ε^2 still remain to be estimated. From the theory of the innovations representations (Gevers and Kailath 1973), we know that the state covariance Σ of the steady-state innovations model is the limiting solution of the following Riccati equation :

$$\Sigma_{i+1} = \Phi^* \Sigma_i \Phi^{*\mathrm{T}} + \Phi^*(N - \Sigma_i H^{*\mathrm{T}})(H^* N + r + H^* \Sigma_i H^{*\mathrm{T}})^{-1} \times (N - \Sigma_i H^{*\mathrm{T}})^{\mathrm{T}} \Phi^{*\mathrm{T}} \tag{47 a}$$

$$\Sigma_0 = 0 \tag{47 b}$$

$$\Sigma = \lim_{i \to \infty} \Sigma_i \tag{48}$$

The gain K^* and the variance σ_ε^2 are derived from this steady-state solution :

$$K^* = \Phi^*(N - \Sigma H^{*\mathrm{T}})(H^* N + r - H^* \Sigma H^{*\mathrm{T}})^{-1} \tag{49}$$

$$\sigma_\varepsilon^2 = H^* N + r - H^* \Sigma H^{*\mathrm{T}}. \tag{50}$$

It is worth noting that the computation of the Riccati equation is greatly simplified by the canonical forms of H^* and Φ^*.

The estimates of K^* and σ_ϵ^2 are obtained by replacing Φ^*, N and r by their estimated values in (47)–(50); they can be shown to converge asymptotically to their true values. A recursive estimation scheme for K^* and σ_ϵ^2 is obtained by integrating the Riccati equation in real-time and substituting for ϕ^*, N and r their latest updated values.

When the amount of available data is limited, it may be desirable to use the estimates thus obtained as starting values for more efficient algorithms, such as a maximum likelihood algorithm.

Comment on the order of the system

So far we have assumed that the order n of the input–output system and the order l of the noise model are known. For reasons of space limitations we shall only briefly indicate how these orders can be determined, and we refer to the literature for more details on the various tests that have been proposed (see, e.g. Mehra (1971), Van den Boom and Van den Enden (1973), Chan *et al.* (1973) and the references therein).

To determine the order of the input-output system the singularity of the Hankel matrix (29) can be tested for increasing n.

A computationally efficient method for determining the rank of the Hankel matrix is the factorization scheme proposed by Rissanen (1971) and Rissanen and Kailath (1972); this is an alternative way of obtaining an h, F, g realization from the impulse response elements.

The determination of the order l of the noise dynamics can similarly be based on the rank of the Hankel matrix. Alternatively a test based on the whiteness of the innovation sequence $\epsilon(\cdot)$ can be used (Mehra 1971), or a more sophisticated test based on the analysis of the estimated variance of the innovations (Chan *et al.* 1973).

5. The global model

The matrices h, F, g of the nth input–output model and the matrices H^*, ϕ^*, K^* of the noise model have been identified in different coordinate bases.

In order to combine the two phases of the identification, it is therefore necessary to apply a coordinate transformation to the state X_2 such that in the new coordinate space the first n basis vectors coincide with the basis vectors in which the input–output model (9) has been identified, i.e. such that with $X(i)=X_1(i)+X_2(i)$, one can write, in the new coordinate space,

$$X(i+1)=\begin{bmatrix}\hat{F} & \hat{D}\\ 0 & \hat{\psi}\end{bmatrix}X(i)+\begin{bmatrix}\hat{g}\\ 0\end{bmatrix}u(i)+\begin{bmatrix}\hat{K}_1\\ \hat{K}_2\end{bmatrix}\epsilon(i) \tag{51 a}$$

$$y(i)=[\underbrace{1\ 0\ \dots\ 0}_{n}\ \vdots\ \underbrace{\hat{h}_2}_{m}]X(i)+\epsilon(i) \tag{51 b}$$

with $\hat{F}$ and $\hat{g}$ given by the first part of the identification (see (30) and (33)).

This requires finding a non-singular transformation matrix T of dimension $l\times l$ that obeys the following constraints:

(1)
$$T^{-1}\left[\begin{array}{cc}0 & \\ \vdots & I_{l-1}\\ 0 & \\ \hline \multicolumn{2}{c}{-\hat{\varphi}^{\mathrm{T}}}\end{array}\right]T=\left[\begin{array}{c:c}\hat{F} & \hat{D}\\ \hdashline 0 & \hat{\psi}\end{array}\right] \tag{52}$$

where $\hat{F}$ is fixed, and $\hat{D}$ and $\hat{\psi}$ are arbitrary, and

(2)
$$H^*T\triangleq[1\ 0\ \dots\ 0]\ T=[1\ 0\ \dots\ 0\ \vdots\ \hat{h}_2] \tag{53}$$

where $\hat{h}_2$, of dimension m, is arbitrary.

The general form of T is found by solving the constraints (52) and (53) after partitioning T into four submatrices:

$$T=\begin{bmatrix}T_1 & T_2\\ T_3 & T_4\end{bmatrix} \tag{54}$$

It is found that the general solution has the following form:

$$T_1=I_n \tag{55 a}$$

$$T_3=-\begin{bmatrix}\hat{f}^{\mathrm{T}}\\ \hat{f}^{\mathrm{T}}\hat{F}\\ \vdots\\ \hat{f}^{\mathrm{T}}\hat{F}^{m-1}\end{bmatrix} \tag{55 b}$$

$$T_2 \text{ and } T_4 \text{ arbitrary, but such that } T_4-T_3T_2 \text{ is non-singular} \tag{55 c}$$

Example

For $T_2=0$ and $T_4=I_m$, T has the form

$$T=\left[\begin{array}{c:c}I_n & 0\\ \hdashline T_3 & I_m\end{array}\right]$$

with T_3 given by (55 *b*).

Comments

(1) It is important to notice that T depends only upon the parameters $\hat{f}_1,\dots,\hat{f}_n$ produced by the first phase of the identification. The transformation is such that the upper left-hand corner of the transformed Φ matrix will coincide exactly with the $\hat{F}$ matrix that was identified in the first part, independently of the accuracy with which the parameter vector $\hat{\varphi}$ has been identified. Recall that the first identification is the most accurate since the identification of the noise dynamics depends for its accuracy on how well the input–output model has been identified. It is interesting to notice, therefore, that in the global model the input–output dynamics is the most accurate, while the inaccuracies introduced in the second phase of the identification algorithm are entirely concentrated in the matrices $\hat{D}$, $\hat{\psi}$, $\hat{K}$ and $\hat{h}_2$ that affect only the noise model.

(2) Notice that only n columns of the T matrix are fixed. The remaining p degrees of freedom can be used to choose a coordinate space that is convenient for the ensuing regulator problem. For example, one can choose $\hat{D}=0$, which leads to two decoupled lower-order Riccati equations in the regulator synthesis.

6. Numerical example

A second-order system with a second-order noise process has been simulated on a IBM 370/158 digital computer. The actual values of the parameters defined in (1) are

$$F=\begin{bmatrix} 0 & 1 \\ -0{\cdot}8 & -0{\cdot}8 \end{bmatrix} \quad g=\begin{bmatrix} 1 \\ 0{\cdot}4 \end{bmatrix} \quad D=\begin{bmatrix} 0{\cdot}1 & 0 \\ 0 & 0{\cdot}1 \end{bmatrix}$$

$$h=[1 \;\; 0] \quad \psi=\begin{bmatrix} 0 & 1 \\ -0{\cdot}82 & 1{\cdot}8 \end{bmatrix} \quad \gamma=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

During the identification the system was controlled by a linear state-feedback

$$u(i)=(0{\cdot}7 - 0{\cdot}1)x(i)$$

$w(\cdot)$, $v(\cdot)$ and $s(\cdot)$ were taken as zero-mean jointly independent white Gaussian noises with

$$\sigma_w^2=1, \quad \sigma_v^2=0{\cdot}25, \quad \sigma_s^2=1 \text{ or } 4$$

Notice that the eigenvalues of F are $0{\cdot}4 \pm j0{\cdot}8$, i.e. inside the unit circle.

The two-stage identification algorithm was applied to the simulated data. Actual and estimated values of g, f and r for two different values of the variance of the identification noise $s(\cdot)$ are given in the table. In each case the second stage of the identification was started after 1000 iterations of the first part. CPU time is about 1 min for 2000 iterations.

	Actual values	Estimates based on k iterations		
		$\sigma_s^2=1$	$\sigma_s^2=4$	
		$k=2000$	$k=2000$	$k=7000$
g	1	0·930	0·947	1·000
	0·4	0·480	0·448	0·422
f	0·8	0·702	0·733	0·787
	0·8	0·782	0·790	0·765
ϕ	0·656	0·61	0·639	0·593
	−0·784	−0·822	−0·879	−0·718
	0·180	0·318	0·269	0·177
	−1	−1·055	−0·965	−0·995
r	0·25	0·270	0·286	0·267

7. Concluding remarks

We have given a complete solution to the identification of a feedback controlled single-input single-output linear system perturbed by coloured noise. The proposed two-stage method separately identifies the input–output dynamics for the open-loop system and the noise dynamics. Both are necessary for the synthesis of an optimal regulator, which was the stated purpose of the identification.

The main advantage of the proposed method is that the identification can be performed on an operating system while it is being kept under closed-loop control (by a manual operator, say). The practical applicability of most identification methods proposed so far was severely limited by the requirement that the input sequence had to be independent of previous outputs, i.e. the identification was to be performed while the system was open-loop. The final state-representation obtained by this algorithm is particularly well suited for the synthesis of an optimal regulator because its state is trivially related to the Kalman filter estimate that is required for the regulator (see (2) and (16)). On the other hand, the algorithm has the inherent disadvantage of any two-stage method, in that the convergence of the second part of the identification scheme is dependent upon the convergence of the first part.

This paper has been voluntarily restricted to a presentation of the main ideas of our proposed method. The algorithm has been shown to converge and it has been successfully applied to simulated data. But clearly many of the technical details are still subject to improvement : alternative canonical forms might be chosen and it might actually be possible to identify Φ in a form that does not require a subsequent state-transformation ; the algorithm might be extended to multivariable systems ; the convergence factor μ_k could be optimized as the identification proceeds. These questions are the subject of continuing inquiry.

Acknowledgment

The authors wish to thank Professors R. K. Mehra and Y. C. Ho of Harvard University for a fruitful discussion on an earlier version of this algorithm.

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