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Optimal point-wise discrete control and controllers' allocation strategies for stochastic distributed systems

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The design of point-wise discrete controllers for a class of stochastic distributed-parameter systems is considered. Assuming a fixed set of controllers' positions, the optimal feedback control is derived using a direct approach in which the infinite dimensional space is approximated using a set of orthonormal functions. The resulting optimal cost is minimized again w.r.t. this set of positions, using gradient techniques, to get the optimal locations for the controller. A one-dimensional diffusion process is used to demonstrate the algorithm.

1. Introduction

The necessity of developing a general theory for distributed-parameter systems arises from the fact that the use of point (lumped) models for such systems is no longer valid in many practical applications where local spatial distortions cannot be neglected, e.g. in nuclear reactors (Hsu 1967).

In many cases the physical nature of the system permits only point-wise controls, due to the system structural complexity and impossibility of deployment (the case in many industrial processes control). In addition, when taking into account economic consideration, another interesting problem will arise, namely the choice of the best positions of such controllers.

Previous work dealing with point-wise control of distributed-parameter systems include that of Greenberg (1969), Matsumoto and Ito (1970), Hassan and Solberg (1970), Aidarous and Ghonaimy (1974), and Surendran (1975).

Amouroux and Babary (1973) determined the optimal locations of point-wise controllers, as a function of system modes, through the optimization of a criterion derived from the system's intrinsic properties.

This paper considers discrete-time control for a class of stochastic distributed-parameter systems. The system behaviour at discrete instants is expressed in terms of recursive functional expressions involving the system's Green's function. Considering a quadratic performance index the optimal feedback point-wise control is derived, via a dynamic programming approach, in terms of an auxiliary spatially dependent variable. The approach used is direct in the sense that the infinite dimensional space is approximated, using orthonormal series expansion, by a finite number of modes. This reduces the problem to a computationally efficient Riccati-like recursive equation in the expansion coefficients of the auxiliary variable.

Having the structure of the optimal controller, the second step will be to search for the optimal positions for such controllers that minimize the same quadratic cost. Since the structure of the optimal control law is independent

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of the positions of the controllers (i.e. for a fixed number of controllers the structure of the feedback law does not change when the positions of the controllers are changed), the problem can be decoupled in two steps. First the optimal control is derived with a fixed set of controller positions; next, the resulting optimal cost is minimized w.r.t. this set of positions.

The duality between the sensor and the controller allocation problems is examined making use of the previously achieved results in the sensor's allocation problem (cf. Aidarous *et al.* 1975 a, b). An illustrative example is given to demonstrate the presented algorithm.

2. Problem formulation

Consider a linear distributed-parameter system given by the following discrete integral equation:

$$\begin{aligned} U_{k+1}(x) = & \int_{\Omega} G(\overline{k+1}\tau, k\tau, x, x')U_k(x')\, d\Omega' \\ & + \sum_{i=1}^{m} \int_{k\tau}^{\overline{k+1}\tau} G(\overline{k+1}\tau, t, x, x_i)D_k(x_i)F_k(x_i)\, dt \\ & + \int_{\Omega} \int_{k\tau}^{\overline{k+1}\tau} G(\overline{k+1}\tau, t, x, x')d_t\beta(t, x')\, d\Omega' \end{aligned} \tag{1}$$

where Ω is a simply-connected open subset of a ν-dimensional Euclidean space E^{ν}, with boundary $\partial\Omega$. $U_k(x)$ is an n-dimensional state vector at time $k\tau$. $F_k(x_i)$ is the p-dimensional control vector implemented at the points x_i, $i = 1, \ldots, m$ during the period $(k\tau, \overline{k+1}\tau)$. $D_k(x_i)$ is the $n \times p$ input matrix at point x_i. $d_t\beta(t, x)$ is the incremental Wiener process with zero mean and incremental covariance

$$\operatorname{cov}\{d\beta(t, x), d\beta(t', x')\} = \Xi(t, x, x')\, dt \tag{2}$$

$G(t, t', x, x')$, defined for all $t \geqslant t'$, is an $n \times n$ fundamental matrix which is known as the system's Green's function, and is determined from the corresponding autonomous partial differential equation together with the imposed boundary conditions and satisfying

$$G(t, t', x, x') = G^{\mathrm{T}}(t, t', x', x), \quad t \geqslant t' \tag{3}$$

$$G(0, 0, x, x') = I\delta_{\mathrm{D}}(x - x') \tag{4}$$

where δ_{D} is the Dirac delta function and I is the $n \times n$ identity matrix.

The system output is observed at q points and is given by

$$Y_k(x_j) = M_k(x_j)U_k(x_j) + \theta_k(x_j), \quad i = 1, \ldots, q \tag{5}$$

where $M_k(x_j)$ is an $s \times n$ output matrix. $Y_k(x_j)$ is an s-dimensional observation vector. The measurement noise $\theta_k(x_j)$ is white Gaussian with zero mean and covariance

$$\operatorname{cov}\{\theta_k(x_i), \theta_l(x_j)\} = \Theta_k(x_i)\delta_{\mathrm{K}}(i, j)\delta_{\mathrm{K}}(k, l) \tag{6}$$

where δ_{K} is the Kronecker delta. Furthermore, it is independent of the noise process $d\beta(t, x)$.

The above white-in-space assumption is valid in all cases where the measurement noise covariance is only related to the sensor characteristics and not to the measured process.

The initial state of the system $U_0(x)$ is a Gaussian random variable independent of $d\beta(t, x)$ and $\theta_k(x_j)$, with

$$E\{U_0(x)\} = 0 \tag{7}$$

$$\text{cov}\,\{U_0(x), U_0(x')\} = P_0(x, x') \tag{8}$$

where $P_0(x, x')$ is an $n \times n$ positive definite symmetric matrix.

Define by Y^k the σ-field induced by the past and present measurements :

$$Y^k = \sigma\{Y_l(x_i), \quad i = 1, \ldots, q\,; \;\; l = 0, 1, \ldots, k\} \tag{9}$$

The problem now can be formulated as follows.

Find the feedback control F_k, with

$$F_k = \begin{bmatrix} F_k(x_1) \\ \vdots \\ F_k(x_m) \end{bmatrix} \tag{10}$$

that minimizes the following error plus control energy measure of performance :

$$J_N = \sum_{k=1}^{N} E\{E\{\iint_{\Omega} U_k^{\mathrm{T}}(x) Q(x, x') U_k(x')\, d\Omega\, d\Omega' / Y^{k-1}\} + E\{F_{k-1}^{\mathrm{T}} R(X_c) F_{k-1} / Y^{k-1}\}\} \tag{11}$$

where R is an $mp \times mp$ positive definite block diagonal matrix representing the control cost at positions $X_c = \text{vec}\,[x_1, \ldots, x_m]$. Q is an $n \times n$ non-negative matrix. The outer expectation is taken over all random quatities $U_0(x)$, $d\beta(t, x)$ and $\theta_k(x_i)$.

It must be noticed that the first term on the right-hand side of (11) implies that the state should be kept as close as possible to zero during the entire trajectory. If it is required that the state trajectory be tracking another state trajectory, then $U_k(x)$ in (11) would be replaced by $[U_k(x) - U_k^d(x)]$. As a matter of convenience the desired state will be chosen equal to zero.

3. Controller synthesis

Define J_N^* to be the minimum value of the performance index, i.e.

$$J_N^* = \min_{F_0, \ldots, F_{N-1}} J_N \tag{12}$$

Now the problem is an N-stage decision problem in which the N decisions $F_0, \ldots, F_{N-1}$ are to be taken such that the given quadratic form (11) is minimized under the constraint (1).

Rather than attempting to make the N decisions simultaneously, Bellman's principle of optimality will be used to reduce the N-stage problem to N one-stage problems.

For the last $N-k$ stages the optimum performance index can be written in the following form :

$$J_{N-k}^* = \min_{F_k, \ldots, F_{N-1}} E\left\{ \sum_{l=k}^{N-1} E\{ \iint_{\Omega} U_{l+1}^{\mathrm{T}}(x) Q(x, x') \times U_{l+1}(x')\, d\Omega\, d\Omega' / Y^l\} + E\{F_l^{\mathrm{T}} R(X_c) F_l / Y^l\} \right\}$$
$$= \min_{F_k} E\{E\{ \iint_{\Omega} U_{k+1}^{\mathrm{T}}(x) Q(x, x') U_{k+1}(x')\, d\Omega\, d\Omega' / Y^k\} + E\{F_k^{\mathrm{T}} R(X_c) F_k / Y^k\} + J_{N-(k+1)}^* \tag{13}$$

By the stochastic optimal control theory we know that J_{N-k}^* will be a quadratic function of the initial state $U_k(x)$ plus a non-negative definite term representing the cost due to the process noise and the estimation errors ; this last term being independent of the state $U_k(x)$. Therefore, the optimum performance index could be written in the form (cf. Hassan and Solberg 1970, and Meier *et al.* 1971),

$$J_{N-k}^* = E\{E\{ \iint_{\Omega} U_k^{\mathrm{T}}(x) \pi_{N-k}(x, x') U_k(x')\, d\Omega\, d\Omega' / Y^k\}\} + \eta_{N-k} \tag{14}$$

Substituting from (14) into (13),

$$J_{N-k}^* = \min_{F_k} E\{E\{ \iint_{\Omega} U_{k+1}^{\mathrm{T}}(x) Q(x, x') U_{k+1}(x')\, d\Omega\, d\Omega' / Y^k\} + E\{ \iint_{\Omega} U_{k+1}^{\mathrm{T}}(x) \pi_{N-(k+1)}(x, x') U_{k+1}(x')\, d\Omega\, d\Omega' / Y^k\} + E\{F_k^{\mathrm{T}} R(X_c) F_k / Y^k\}\} + \eta_{N-(k+1)}$$
$$= \min_{F_k} E\{E\{ \iint_{\Omega} U_{k+1}^{\mathrm{T}}(x) S_{N-(k+1)}(x, x') U_{k+1}(x')\, d\Omega\, d\Omega' / Y^k\} + E\{F_k^{\mathrm{T}} R(X_c) F_k / Y^k\}\} + \eta_{N-(k+1)} \tag{15}$$

where

$$S_k(x, x') = \pi_k(x, x') + Q(x, x') \tag{16}$$

Define

$$G_k(x, x') = G(\overline{k+1}, k, x, x') \tag{17}$$

$$H_k(x, x_i) = \int_{k\tau}^{\overline{k+1}\tau} G(\overline{k+1}\tau, t, x, x_i)\, dt\, D_k(x_i), \quad i = 1, \ldots, m \tag{18}$$

$$L_k(x) = [H_k(x, x_1) \ldots H_k(x, x_m)] \tag{19}$$

Then eqn. (1) can be written in the form

$$U_{k+1}(x) = \int_{\Omega} G_k(x, x') U_k(x')\, d\Omega' + L_k(x) F_k + \int_{\Omega} \int_{k\tau}^{\overline{k+1}\tau} G(\overline{k+1}\tau, t, x, x') d_t \beta(t, x')\, d\Omega' \tag{20}$$

Substituting from (20) into (15) and taking into consideration the independence between the incremental Wiener process $d\beta(t, x)$ and both the control

F_k and the state $U_k(x)$ for $k\tau \leqslant t \leqslant \overline{k+1}\tau$, eqn. (15) can be written in the following form :

$$\begin{aligned} J_{N-k}{}^* = \min_{F_k} E \Bigg\{ & E\{ \iiiint_\Omega U_k{}^{\mathrm{T}}(x'')G_k{}^{\mathrm{T}}(x, x'')S_{N-(k+1)}(x, x') \\ & \times G_k(x', x''')U_k(x''')\, d\Omega\, d\Omega'\, d\Omega''\, d\Omega'''/Y^k\} \\ & + E\Bigg\{ \iiiint_\Omega \iint_{k\tau}^{\overline{k+1}\tau} d_t\beta(t, x'')^{\mathrm{T}} G^{\mathrm{T}}(\overline{k+1}\tau, t, x, x'')S_{N-(k+1)}(x, x') \\ & \times G(\overline{k+1}\tau, t', x', x''')d_{t'}\beta(t', x''')\, d\Omega\, d\Omega'\, d\Omega''\, d\Omega'''/Y^k \Bigg\} \\ & + E\{ \iiint_\Omega U_k{}^{\mathrm{T}}(x'')G_k{}^{\mathrm{T}}(x, x'')S_{N-(k+1)}(x, x') \\ & \times L_k(x')\, d\Omega\, d\Omega'\, d\Omega''\; F_k/Y^k\} \\ & + E\{F_k{}^{\mathrm{T}} \iiint_\Omega L_k{}^{\mathrm{T}}(x)S_{N-(k+1)}(x, x')G_k(x, x') \\ & \times U_k(x'')\, d\Omega\, d\Omega'\, d\Omega''/Y^k\} \\ & + E\{F_k{}^{\mathrm{T}} \iint_\Omega L_k{}^{\mathrm{T}}(x)S_{N-(k+1)}(x, x')L_k(x')\, d\Omega\, d\Omega'\; F_k/Y^k\} \\ & + E\{F_k{}^{\mathrm{T}}R(X_c)F_k/Y^k\} \Bigg\} + \eta_{N-(k+1)} \end{aligned} \tag{21}$$

In order to solve the above equation, the original infinite dimensional space will be projected on a new finite dimensional one.

Assume a complete set of orthonormal basis $\{z_i(x),\ i=1, \ldots, \infty\ ;\ x\in\Omega\}$, where

$$\int_\Omega z_i(x)z_j(x)\, d\Omega = \delta_K(i, j) \tag{22}$$

Consider r of these functions, and assume that we can write, approximately,

$$G_k(x, x') = Z^{\mathrm{T}}(x)A_kZ(x') \tag{23}$$

$$\Xi(t, x, x') = Z^{\mathrm{T}}(x)C(t)Z(x') \tag{24}$$

$$S_k(x, x') = Z^{\mathrm{T}}(x)\,V_kZ(x') \tag{25}$$

$$Q(x, x') = Z^{\mathrm{T}}(x)BZ(x') \tag{26}$$

Let $Z^{\mathrm{T}}(x)$ be an $n \times nr$ matrix of orthonormal functions given by

$$Z^{\mathrm{T}}(x) = \begin{bmatrix} z_1(x) & \ldots & z_r(x) & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & \ldots & 0 & z_1(x) & \ldots & z_r(x) & \ddots & \vdots & & \vdots \\ \vdots & & \vdots & 0 & \ldots & 0 & 0 & \ldots & 0 \\ & & & \vdots & & \vdots & & & \\ 0 & \ldots & 0 & 0 & \ldots & 0 & z_1(x) & \ldots & z_r(x) \end{bmatrix} \tag{27}$$

A_k, $C(t)$, V_k and B are $nr \times nr$ matrices of coefficients, all having the same configuration. For example,

$$A_k = \begin{bmatrix} \Lambda_k{}^{1,1} & \ldots & \Lambda_k{}^{1,n} \\ \vdots & & \vdots \\ \Lambda_k{}^{n,1} & \ldots & \Lambda_k{}^{n,n} \end{bmatrix} \tag{28}$$

with

$$\Lambda_k^{i,j} = \begin{bmatrix} a_k^{i,j}(1,1) & \dots & a_k^{i,j}(1,r) \\ \vdots & & \vdots \\ a_k^{i,j}(r,1) & \dots & a_k^{i,j}(r,r) \end{bmatrix} \tag{29}$$

Following the same notations,

$$H_k(x, x_i) = Z^{\mathrm{T}}(z)\mathscr{A}_k Z(x_i) D_k(x_i) \tag{30}$$

$$\begin{aligned} L_k(x) &= Z^{\mathrm{T}}(x)\mathscr{A}_k[Z(x_1)D_k(x_1) \dots Z(x_m)D_k(x_m)] \\ &= Z^{\mathrm{T}}(x)\mathscr{A}_k\mathscr{Z}(X_c) \end{aligned} \tag{31}$$

where $\mathscr{Z}(X_c)$ is an $nr \times pm$ matrix.

Substituting in (21) and making use of the orthonormality properties of $Z(x)$, yields

$$\begin{aligned} J_{N-k}^* = \min_{F_k} E \Bigg\{ & E\{ \iint_\Omega U_k^{\mathrm{T}}(x)Z^{\mathrm{T}}(x)A_k^{\mathrm{T}}V_{N-(k+1)}A_k \\ & \qquad\qquad \times Z(x')U_k(x')\, d\Omega\, d\Omega'/Y^k\} \\ & + \int_{k\tau}^{\overline{k+1}\tau} \operatorname{tr}[C(t)a^{\mathrm{T}}(\overline{k+1}\tau, t)V_{N-(k+1)}a(\overline{k+1}\tau, t)]\, dt \\ & + E\{ \int_\Omega U_k^{\mathrm{T}}(x)Z^{\mathrm{T}}(x)\, d\Omega\; A_k^{\mathrm{T}}V_{N-(k+1)}\mathscr{A}_k\mathscr{Z}(X_c)F_k/Y^k\} \\ & + E\{F_k^{\mathrm{T}}\mathscr{Z}^{\mathrm{T}}(X_c)\mathscr{A}_k^{\mathrm{T}}V_{N-(k+1)}A_k \int_\Omega Z(x)U_k(x)\, d\Omega/Y^k\} \\ & + E\{F_k^{\mathrm{T}}[\mathscr{Z}^{\mathrm{T}}(X_c)\mathscr{A}_k^{\mathrm{T}}V_{N-(k+1)}\mathscr{A}_k\mathscr{Z}(X_c) \\ & \qquad\qquad + R(X_c)]F_k/Y^k\} \Bigg\} + \eta_{N-(k+1)} \end{aligned} \tag{32}$$

where

$$G(\overline{k+1}\tau, t, x, x') = Z^{\mathrm{T}}(x)a(\overline{k+1}, t)Z(x') \tag{33}$$

and the derivations of the second term on the right-hand side of (32) are given in the Appendix.

Define the conditional mean and covariance of the state

$$\hat{U}_k(x) = E\{U_k(x)/Y^k\} \tag{34}$$

$$P_k(x, x') = E\{\tilde{U}_k(x)\tilde{U}_k^{\mathrm{T}}(x')/Y^k\} \tag{35}$$

where

$$\tilde{U}_k(x) = U_k(x) - \hat{U}_k(x) \tag{36}$$

Differentiating the right-hand side of (32) w.r.t. F_k, equating to zero and replacing each state by its conditional mean, gives

$$\begin{aligned} F_k^* &= -[\mathscr{Z}^{\mathrm{T}}(X_c)\mathscr{A}_k^{\mathrm{T}}V_{N-(k+1)}\mathscr{A}_k\mathscr{Z}(X_c) + R(X_c)]^{-1} \\ & \qquad \times \mathscr{Z}^{\mathrm{T}}(X_c)\mathscr{A}_k^{\mathrm{T}}V_{N-(k+1)}A_k \int_\Omega Z(x)\hat{U}_k(x)\, d\Omega \\ &= \int_\Omega \mathscr{K}_k(x, X_c)\hat{U}_k(x)\, d\Omega \end{aligned} \tag{37}$$

Hence, the optimal control law is determined as a linear feedback of the state conditional mean, which is known as the separation principle (Aström 1970 and Bensoussan 1971).

To evaluate the feedback gain we need to compute the auxiliary variable V_k during the control period. This can be done by substituting the optimal value of F_k in (32) :

$$\begin{aligned} J_{N-k}^* = E\{E\{ \iint_\Omega U_k^{\mathrm{T}}(x) Z^{\mathrm{T}}(x) A_k^{\mathrm{T}} V_{N-(k+1)} A_k Z(x') \\ \times U_k(x')\, d\Omega\, d\Omega' / Y^k\} \\ - 2E\{ \iint_\Omega U_k^{\mathrm{T}}(x) Z^{\mathrm{T}}(x) A_k^{\mathrm{T}} V_{N-(k+1)} \mathscr{A}_k \Phi_{N-(k+1)}(X_c) \mathscr{A}_k^{\mathrm{T}} \\ \times V_{N-(k+1)} A_k Z(x') \hat{U}_k(x')\, d\Omega\, d\Omega' / Y^k\} \\ + E\{ \iint_\Omega \hat{U}_k^{\mathrm{T}}(x) Z^{\mathrm{T}}(x) A_k^{\mathrm{T}} V_{N-(k+1)} \mathscr{A}_k \Phi_{N-(k+1)}(X_c) \mathscr{A}_k^{\mathrm{T}} \\ \times V_{N-(k+1)} A_k Z(x') \hat{U}_k(x')\, d\Omega\, d\Omega' / Y^k\}\} \\ + \int_{k\tau}^{\overline{k+1}\tau} \operatorname{tr}\, [C(t) a^{\mathrm{T}}(\overline{k+1}\tau, t) V_{N-(k+1)} a(\overline{k+1}\tau, t)]\, dt + \eta_{N-(k+1)} \end{aligned} \tag{38}$$

where

$$\begin{aligned} \Phi_{N-(k+1)}(X_c) = \mathscr{Z}(X_c)[\mathscr{Z}^{\mathrm{T}}(X_c) \mathscr{A}_k^{\mathrm{T}} V_{N-(k+1)} \mathscr{A}_k \mathscr{Z}(X_c) \\ + R(X_c)]^{-1} \mathscr{Z}^{\mathrm{T}}(X_c) \end{aligned} \tag{39}$$

The second and third terms on the right-hand side of (38) can be expressed as (Meditch 1969)

$$\begin{aligned} E\{ \iint_\Omega \tilde{U}_k^{\mathrm{T}}(x) Z^{\mathrm{T}}(x) A_k^{\mathrm{T}} V_{N-(k+1)} \mathscr{A}_k \Phi_{N-(k+1)}(X_c) \mathscr{A}_k^{\mathrm{T}} \\ \times V_{N-(k+1)} A_k Z(x') \tilde{U}_k(x')\, d\Omega\, d\Omega' / Y^k\} \\ - E\{ \iint_\Omega U_k^{\mathrm{T}}(x) Z^{\mathrm{T}}(x) A^{\mathrm{T}} V_{N-(k+1)} \mathscr{A}_k \Phi_{N-(k+1)}(X_c) \mathscr{A}_k^{\mathrm{T}} \\ \times V_{N-(k+1)} A_k Z(x') U_k(x')\, d\Omega\, d\Omega' / Y^k\} \end{aligned}$$

Hence, eqn. (38) can be written in the form

$$\begin{aligned} J_{N-k}^* = E\{E\{ \iint_\Omega U_k^{\mathrm{T}}(x) Z^{\mathrm{T}}(x) A_k^{\mathrm{T}} [V_{N-(k+1)} \\ - V_{N-(k+1)} \mathscr{A}_k \Phi_{N-(k+1)}(X_c) \mathscr{A}_k^{\mathrm{T}} V_{N-(k+1)}] A_k \\ \times Z(x') U_k(x')\, d\Omega\, d\Omega' / Y^k\}\} \\ + \operatorname{tr}\, [W_k A_k^{\mathrm{T}} V_{N-(k+1)} \mathscr{A}_k \Phi_{N-(k+1)}(X_c) \mathscr{A}_k^{\mathrm{T}} V_{N-(k+1)} A_k] \\ + \int_{k\tau}^{\overline{k+1}\tau} \operatorname{tr}\, [C(t) a^{\mathrm{T}}(\overline{k+1}\tau, t) V_{N-(k+1)} a(\overline{k+1}\tau, t)]\, dt + \eta_{N-(+1)} \end{aligned} \tag{40}$$

where W_k is the matrix of expansion coefficients of the state-error covariance in terms of the orthonormal basis (Aidarous 1975)

$$P_k(x, x') = Z^{\mathrm{T}}(x) W_k Z(x') \tag{41}$$

Now comparing terms in (14) and (40), the following two equations are easily obtained :

$$\begin{aligned} V_{N-k} = B + A_k^{\mathrm{T}} V_{N-(k+1)} A_k - A_k^{\mathrm{T}} V_{N-(k+1)} \mathscr{A}_k \Phi_{N-(k+1)}(X_c) \mathscr{A}_k^{\mathrm{T}} \\ \times V_{N-(k+1)} A_k \end{aligned} \tag{42}$$

$$\eta_{N-k} = \eta_{N-(k+1)} + \int_{k\tau}^{\overline{k+1}\tau} \mathrm{tr}\,[C(t)\,\sigma^{\mathrm{T}}(\overline{k+1}\,\tau, t)\,V_{N-(k+1)}\sigma(\overline{k+1}\,\tau, t)]\,dt$$
$$+ \mathrm{tr}\,[W_k A_k^{\mathrm{T}} V_{N-(k+1)} \mathscr{A}_k \Phi_{N-(k+1)}(X_c) \mathscr{A}_k^{\mathrm{T}} V_{N-(k+1)} A_k] \tag{43}$$

where

$$V_0 = B \tag{44}$$

$$\eta_0 = 0 \tag{45}$$

The matrix sequence V_k must be determined prior to system operation, i.e. it must be computed and stored for the computation of the optimum gains (cf. eqn. (37)), while the computation of η_k is necessary only to evaluate the optimum measure of performance, and need not be carried out in the computation of the optimal control.

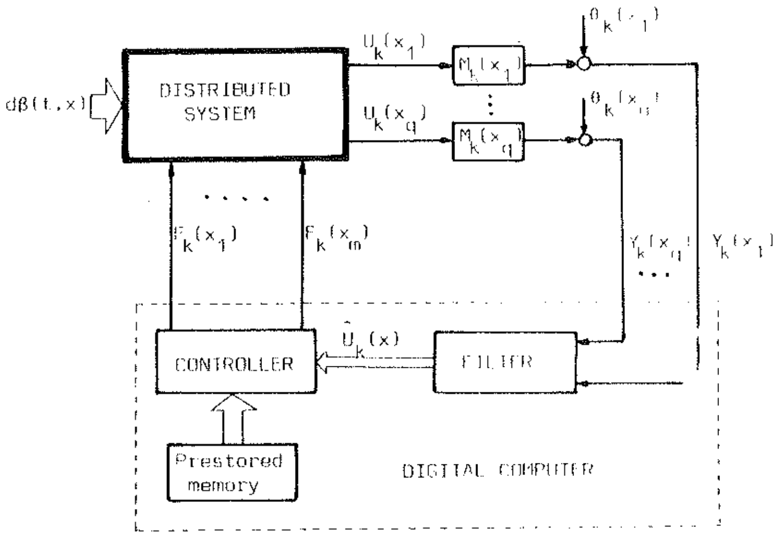

Figure 1. Optimum feedback controller.

The block diagram of the optimal feedback controller is depicted in Fig. 1. The distributed Kalman filter algorithm for the computation of $\hat{U}_k(x)$ and $P_k(x, x')$ using orthonormal expansions has been described elsewhere (cf. Aidarous 1975).

4. Controller optimal allocation (duality with sensor problem)

In this section the duality concept of Kalman (1960) will be applied to derive an algorithm for the optimal allocation of controllers. Assuming that there is no measurement noise and no stochastic inputs for the system, then the optimal control law that minimizes the performance index (11) is given by

$$F_k = \int_{\Omega} \mathscr{K}_k(x, X_c) U_k(x)\, d\Omega \tag{46}$$

where,

$$\mathscr{K}_k(x, X_c) = -[\mathscr{Z}^{\mathrm{T}}(X_c) \mathscr{A}_k^{\mathrm{T}} V_{N-(k+1)} \mathscr{A}_k \mathscr{Z}(X_c) + R(X_c)]^{-1}$$
$$\times \mathscr{Z}^{\mathrm{T}}(X_c) \mathscr{A}_k^{\mathrm{T}} V_{N-(k+1)} \mathscr{A}_k Z(x) \tag{47}$$

and V_k can be calculated backward in time from (42) together with the initial condition (44). In this deterministic case the optimal measure of performance for the last $N-k$ stages will be (cf. eqn. (14))

$$J_{N-k}^* = \iint_{\Omega} U_k^{\mathrm{T}}(x) Z^{\mathrm{T}}(x) [V_{N-k} - B] Z(x') U_k(x')\, d\Omega\, d\Omega' \tag{48}$$

Two possible formulations for the allocation problems can be derived.

(i) *Control allocation strategy (finite-time regulator)*, where at each stage of the control interval it is required to minimize the above cost w.r.t. the controller positions X_c. Following the same steps as in the sensor allocation problem (cf. Aidarous *et al.* 1975), the problem can be formulated as follows :

$$\min_{X_c} [J_{N-k}^* / V_{N-(k+1)}] \tag{49}$$

From (42) this is equivalent to

$$\min_{X_c} \mathrm{tr}\, [A_k^{\mathrm{T}} V_{N-(k+1)} A_k - A_k^{\mathrm{T}} V_{N-(k+1)} \mathscr{A}_k \Phi_{N-(k+1)}(X_c) \mathscr{A}_k^{\mathrm{T}} \times V_{N-(k+1)} A_k] \tag{50}$$

which can be replaced by

$$\max_{X_c} \mathrm{tr}\, [A_k^{\mathrm{T}} V_{N-(k+1)} \mathscr{A}_k \Phi_{N-(k+1)}(X_c) \mathscr{A}_k^{\mathrm{T}} V_{N-(k+1)} A_k] \tag{51}$$

Assuming that the system is time-invariant and replacing k, for simplicity of notations, by $N-k$, then (51) can be written as

$$\max_{X_c} \mathrm{tr}\, [A^{\mathrm{T}} V_{k-1} A\, \Phi_{k-1}(X_c) A^{\mathrm{T}} V_{k-1} A] \tag{52}$$

It must be noticed that, for the time-invariant system considered, the matrix of coefficients $\mathscr{A}$ is replaced by τA and consequently the matrix $\Phi_k(X_c)$ will have the simplified form

$$\Phi_k(X_c) = \mathscr{Z}(X_c)[\tau^2 \mathscr{Z}^{\mathrm{T}}(X_c) A^{\mathrm{T}} V_k A\, \mathscr{Z}(X_c) + R(X_c)]^{-1} \mathscr{Z}^{\mathrm{T}}(X_c) \tag{53}$$

The controller allocation problem, as it appears in (52), is the dual of the sensor allocation problem that has been solved by Aidarous *et al.* (1975 a, b). This duality appears quite clearly from Table 1, that has been constructed for the one-controller against the one-sensor case. From this table a controller allocation algorithm can be derived from the sensor allocation algorithm, by establishing the following duality relations :

1. The measurement error covariance $Q(x_m)$ must be replaced by the weighted controller position cost $R(x_c)/\tau^2$.
2. The auxiliary estimation variable

$$E_k = A[W_k + \tau H]A \tag{54}$$

must be replaced by the auxiliary control variable V_k. Notice that

$$V_k = \Gamma_k + B \tag{55}$$

where Γ_k is the matrix of expansion coefficients of $\pi_k(x, x')$ in eqn. (14).
3. The state error covariance matrix W_k is the dual of the state weighting matrix Γ_k (for optimal cost evaluation).

	Sensor allocation	Controller allocation
Message model	$\frac{\partial u}{\partial t} = \mathscr{L}_x u + f(t, x)$ $f(t, x)$ distributed white Gaussian process	$\frac{\partial u}{\partial t} = \mathscr{L}_x u + f(t, x_c)\delta(x - x_c)$ $f(t, x_c)$ point-wise control at point x_c
Observation model	$y_k(x_m) = u_k(x_m) + v_k(x_m)$ $v_k(x_m)$ white Gaussian measurement noise	$y_k(x) = u_k(x)$
Measure of performance	$\min_{\substack{\text{filter,} \\ x_m}} E\{\tilde{u}_k(x)\tilde{u}_k(x)/Y^k\}$	$\min_{\substack{\text{controller,} \\ x_c}} \sum_{k=1}^{N} \{\iint_{\Omega} u_k(x)Q(x, x')u_k(x')\, dx\, dx' + f_{k-1}(x_c)R(x_c)f_{k-1}(x_c)\}$
Gain algorithm	$\hat{u}_{k/k}(x) = \hat{u}_{k/k-1}(x) + K_k(x, x_c)[y_k(x_m) - \hat{u}_{k/k-1}(x_m)]$ $\hat{u}_{k/k-1}(x) = \int_{\Omega} G_k(x, x')\hat{u}_{k-1/k-1}(x')\, dx'$ $K_k(x, x_m) = Z^T(x)E_{k-1}Z(x_m)[Z^T(x_m)E_{k-1}Z(x_m) + Q(x_m)]^{-1}$	$f_k(x_c) = \int_{\Omega} K_k(x, x_c)u_k(x)\, dx$ $K_k(x, x_c) = -[\tau^2 \mathscr{Z}^T(X_c)A^T V_{N-(k+1)} A \mathscr{Z}(X_c) + R(X_c)]^{-1} \tau \mathscr{Z}(X_c) A^T V_{N-(k+1)} A Z(x)$
Riccati equation	$E_k = \tau A^T H A + A^T E_{k-1} A - A^T E_{k-1} Z(x_m)[Z^T(x_m)E_{k-1}Z(x_m) + Q(x_m)]^{-1} Z^T(x_m) E_{k-1} A$ $E_k = A^T[W_k + \tau H]A$ where $C(x, x') = Z^T(x) H Z(x')$ $P_k(x, x') = Z^T(x) W_k Z(x')$	$V_k = B + A^T V_{k-1} A - A^T V_{k-1} A \mathscr{Z}(x_c) \left[\mathscr{Z}^T(x_c) A^T V_{k-1} A \mathscr{Z}(x_c) + \frac{R(x_c)}{\tau^2} \right]^{-1} \mathscr{Z}(x_c) A^T V_{k-1} A$ $V_k = \Gamma_k + B$ where $Q(x, x') = Z^T(x) B Z(x')$ $\pi_k(x, x') = Z^T(x) \Gamma_k Z(x')$
Allocation algorithm	$\max_{x_m} \mathrm{tr}\, [E_{k-1} Z(x_m) \Psi_{k-1}^{-1}(x_m) Z^T(x_m) E_{k-1}]$ where $\Psi_k(x_m) = Z^T(x_m) E_k Z(x_m) + Q(x_m)$	$\max_{x_c} \mathrm{tr}\, [A^T V_{k-1} A \mathscr{Z}(x_c) \Psi_{k-1}^{-1}(x_c) \mathscr{Z}^T(x_c) A^T V_{k-1} A]$ where $\Psi_k(x_c) = \mathscr{Z}^T(x_c) A^T V_k A \mathscr{Z}(x_c) + \frac{R(x_c)}{\tau^2}$

Table 1. Duality between the sensor and controller allocation problems.

4. The process noise covariance H will be replaced by the state deviation weighting matrix B/τ.

(ii) *Optimal position of controllers (infinite-time regulators)*, where it is required to find the steady-state control law as well as to determine the steady-state positions of the controller to minimize (48) for $N \to \infty$.

Here the control law will be

$$F_k = -\int_{\Omega} \mathscr{K}^0(x, X_c) U_k(x)\, dx \tag{56}$$

where the steady-state feedback gain is constant and given by

$$\mathscr{K}^0(x, X_c) = -[\mathscr{Z}^{\mathrm{T}}(X_c)\mathscr{A}^{\mathrm{T}} V^0 \mathscr{A} \mathscr{Z}(X_c) + R(X_c)]^{-1} \times \mathscr{Z}^{\mathrm{T}}(X_c)\mathscr{A} V^0 \mathscr{A} Z(x) \tag{57}$$

and V^0 is the solution of the algebraic Riccati equation

$$V^0 = B + A V^0 A - A V^0 \mathscr{A} \Phi^0(X_c) \mathscr{A} V^0 A \tag{58}$$

with

$$\Phi^0(X_c) = \mathscr{Z}(X_c)[\mathscr{Z}^{\mathrm{T}}(X_c)\mathscr{A}^{\mathrm{T}} V^0 \mathscr{A} \mathscr{Z}(X_c) + R(X_c)]^{-1} \mathscr{Z}^{\mathrm{T}}(X_c) \tag{59}$$

The controller allocation problem can be casted in the following minimization form :

$$\min_{X_c} [J_{ss}^*] \tag{60}$$

which can be replaced by

$$\max_{X_c} \operatorname{tr} [A V^0 A \Phi^0(X_c) A V^0 A] \tag{61}$$

It is obvious that the same algorithm used in the calculation of the sensor's allocation problem (cf. Aidarous *et al.* 1975 a, b) can be applied in a straightforward way to determine the optimal controller positions.

5. Illustrative example

Consider the normalized diffusion process given by

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial 2u}{\partial t^2} + f(t, x_c)\delta(x - x_c), \quad x \in [0, 1]$$

with boundary conditions,

$$u(t, 0) = u(t, 1) = 0$$

and the diffusion coefficient $\alpha^2 = 0{\cdot}0033$.

It is required to determine the optimal position x_c of the point-wise controller which minimizes the following quadratic cost :

$$J = \lim_{N \to \infty} \frac{1}{N} \sum_{k=1}^{N} \left\{ \int_0^1 \int_0^1 u_k(x) Q(x, x') u(x')\, dx\, dx' + R(x_c) f_{k-1}^2(x_c) \right\}$$

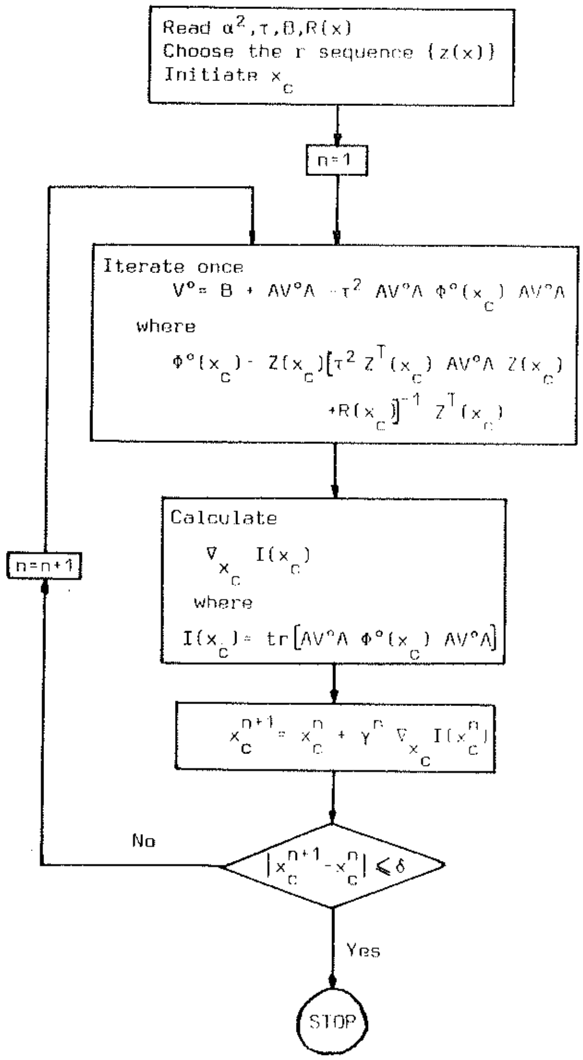

Figure 2. Flow chart of the controller allocation algorithm.

The discretization period is taken as $\tau = 0{\cdot}1$ sec. The set of orthonormal basis functions is chosen as $\{\sqrt{2}\sin i\pi x,\ i = 1, \ldots, r\}$. The $Z(x)$ will be an r-dimensional vector given by

$$Z(x) = \begin{bmatrix} \sqrt{2}\sin \pi x \\ \vdots \\ \sqrt{2}\sin r\pi x \end{bmatrix}$$

Since the problem is scalar, $\mathscr{Z}(x_c)$ is simply defined by replacing x in $Z(x)$ by x_c.

A and $\mathscr{A}$ are both constant diagonal matrices given by

$$A = \mathrm{diag}\,[a_1 \ldots a_r]$$

$$\mathscr{A} = \tau A$$

where

$$a_i = \exp\left(-\pi^2 \alpha^2 i^2 \tau\right)$$

The computational scheme for this example is illustrated by the flow chart given in Fig. 2.

6. Numerical results

The following numerical results are obtained using five elements ($r = 5$) from the chosen set of orthonormal basis. For the sake of comparison, the optimum position as well as the optimum cost have been computed for

r	$Q(x, x') = 0{\cdot}05$ $R(x_c) = 0{\cdot}1$		$Q(x, x') = 0{\cdot}05$ $R(x_c) = 0{\cdot}5 + 0{\cdot}3 \sin 2\pi x_c$	
	x_c^*	J^*	x_c^*	J^*
2	0·5	0·6009	0·6355	1·155
3	0·5	0·7347	0·6300	1·267
4	0·5	0·7788	0·6270	1·308
5	0·5	0·8031	0·6255	1·320
6	0·5	0·8134	0·6250	1·322

Table 2. Convergence of the optimum position and cost.

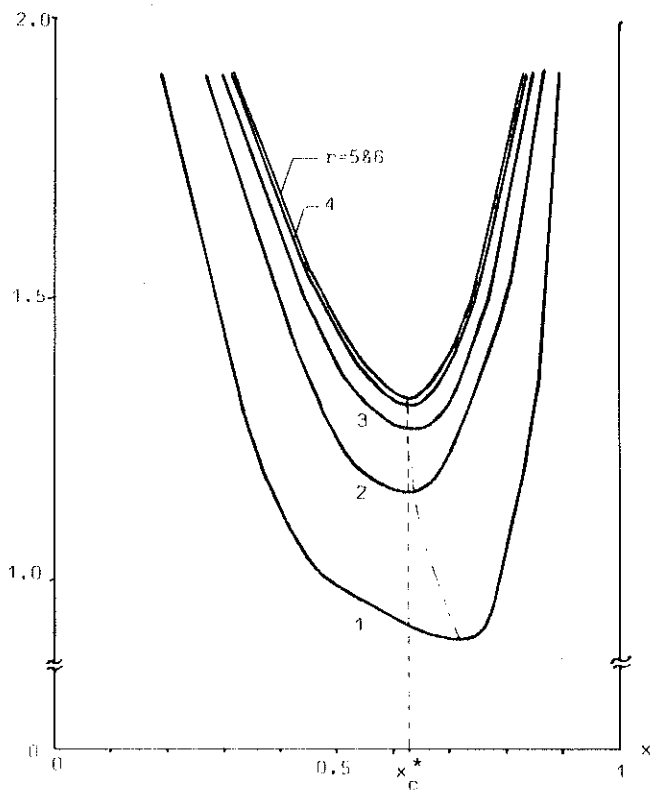

Figure 3. Cost profile for different r, where $Q(x, x') = 0{\cdot}05$ and $R(x_c) = 0{\cdot}5 + 0{\cdot}3 \sin 2\pi x_c$.

different values of r, as shown in Table 2. For small r the cost is poorly approximated, since the number of orthonormal functions are not enough. However, very slight differences in the computation of the optimum position and cost for values of r beyond five. The same holds for the cost profile as shown in Fig. 3, where the cost profile is represented for different values of r.

The values of error and control costs are taken as

$$Q(x, x') = 0{\cdot}05$$

$$R(x_c) = 0{\cdot}5 + 0{\cdot}3 \sin 2\pi x_c$$

Next the influence of the two terms of the performance index has been examined. Two forms of the control energy cost have been considered; first a constant cost, and, secondly, a cost that is a function of the space. This second case takes into consideration the case where the implementation cost of the controller depends upon the structure of the controlled process, which has an important significance in practical applications.

(*a*) For a spatially independent control cost the optimal position lies in an area in which the state-error weighting function reaches its maximum value (see Table 3). This is to be expected, for this example, since in this area a higher accuracy is required.

(*b*) For a spatially dependent control cost and a constant error weighting function the optimal position lies in an area of minimum control cost. This appears in Table 4 in which the optimum controller position was calculated using different control cost functions.

Error weight $Q(x)$	$x_c{}^*$
$0{\cdot}05$	$0{\cdot}5$
$0{\cdot}05 + 0{\cdot}03 \sin \pi x$	$0{\cdot}5$
$0{\cdot}05 - 0{\cdot}03 \sin 2\pi x$	$0{\cdot}61$
$0{\cdot}05 - 0{\cdot}03 \sin 3\pi x$	$0{\cdot}5$

Table 3. Optimum position for constant controller cost, $R(x_c) = 0{\cdot}05$.

Control cost $R(x_c)$	$X_c{}^*$
$0{\cdot}05 + 0{\cdot}03 \sin 2\pi x_c$	$0{\cdot}58$
$0{\cdot}5 \ + 0{\cdot}3 \ \sin 2\pi x_c$	$0{\cdot}63$
$0{\cdot}5 \ - 0{\cdot}3 \ \sin 3\pi x_c$	$0{\cdot}275$ and $0{\cdot}725$

Table 4. Optimum position for variable controller cost, $Q(x) = 0{\cdot}05$.

7. Conclusions

The discrete-time control of a class of linear stochastic distributed-parameter systems is considered. A direct approach, in which the infinite dimensional space is approximated by a finite number or orthonormal basis functions, is used. The optimal feedback point-wise control is derived, via dynamic programming, in terms of an auxiliary variable obeying a computationally efficient Riccati-like recursive equation. The optimal positions of the point-wise controllers are found using a gradient algorithm that is dual to the algorithm derived earlier for the search of the optimal sensor positions in the estimation problem. The algorithm has been used to find the optimal position for the controller of a diffusion process.

Appendix

Using (23) and (25) the second term in (21) can be written as

$$E\left\{\iint_{\Omega}\iint_{k\tau}^{\overline{k+1}\tau}[d_t\beta(t,x)]^{\mathrm{T}}Z^{\mathrm{T}}(x)a^{\mathrm{T}}(\overline{k+1}\tau,t)V_{N-(k+1)}a(\overline{k+1}\tau,t')\right.$$
$$\left.\times Z(x')d_{t'}\beta(t',x')\,d\Omega\,d\Omega'\,dt\,dt'/Y^k\right\} \qquad (\mathrm{A}\ 1)$$

Let us write (A 1) in the compact form

$$E\left\{\iint_{\Omega}\iint_{k\tau}^{\overline{k+1}\tau}[d_t\beta(t,x)]^{\mathrm{T}}Z^{\mathrm{T}}(x)\Psi_k(t,t')Z(x')\right.$$
$$\left.\times d_{t'}\beta(t',x')\,d\Omega\,d\Omega'\,dt\,dt'/Y^k\right\} \qquad (\mathrm{A}\ 2)$$

Dropping for the moment the subscript indicating time, this would not affect the result but it will simplify the notations, then the above expression could be written in the scalar form

$$E\left\{\iint_{\Omega}\iint_{k\tau}^{\overline{k+1}\tau}\sum_{s,l=1}^{n}\sum_{p=(s-1)r+1}^{sr}\sum_{q=(l-1)r+1}^{lr}\psi(p,q)d\beta(s)d\beta(l)\right.$$
$$\left.z_i(x)z_j(x')\,d\Omega\,d\Omega'\,dt\,dt'/Y^k\right\}$$
$$=\iint_{\Omega}\int_{k\tau}^{\overline{k+1}\tau}\sum_{s,l=1}^{n}\sum_{p=(s-1)r+1}^{sr}\sum_{q=(l-1)r+1}^{lr}\psi(p,q)\Xi(s,l)$$
$$\times z_i(x)z_j(x')\,d\Omega\,d\Omega'\,dt\,;\quad i=p-(s-1)r,\ j=q-(l-1)r \qquad (\mathrm{A}\ 3)$$

where $\psi(p,q)$ and $\Xi(s,l)$ stands for the pqth and slth elements of $\Psi_k(t,t')$ and $\Xi(t,x,x')$ respectively, and $d\beta(s)$ stands for the sth element of $d\beta(t,x)$. From (24), the slth element of Ξ can be written also in the following scalar form:

$$\Xi(s,l)=\sum_{a=(s-1)r+1}^{sr}\sum_{b=(l-1)r+1}^{lr}c(a,b)z_i(x)z_j(x')\,;$$
$$i=a-(s-1)r,\ j=b-(l-1)r \qquad (\mathrm{A}\ 4)$$

Substituting from (A 4) into (A 3) and making use of the orthonormality properties of $z_i(x)$

$$\int_{k\tau}^{\overline{k+1}\tau}\sum_{s,l=1}^{n}\sum_{a=(s-1)r+1}^{sr}\sum_{b=(l-1)r+1}^{lr}\psi(a,b)c(a,b)\,dt$$
$$=\int_{k\tau}^{\overline{k+1}\tau}\mathrm{tr}\,[C(t)\Psi_k(t,t')]\,dt \qquad (\mathrm{A}\ 5)$$

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